Condensate with a finite momentum in a moving medium

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We study the possibility for condensation of quasiparticles with a finite momentum $k=k_0\neq 0$ in a moving medium. We investigate the general conditions for the occurrence of a condensate with a finite momentum k_0 and a frequency corresponding to a well defined value $\omega(k_0)$ on a low-lying branch of the Bose-excitation spectrum $\omega(k)$ when the medium moves uniformly along a straight line with an arbitrary velocity v. We show that the condensate of the excitations appears for the first time when the velocity v of the medium becomes larger than some critical value v_c. As a model for describing the field of the condensate we chose a two-component scalar field defined by giving the appropriate effective Lagrangian. We study both the case when we are just above criticality, $|v-v_c|\ll v_c$ for which a rather weak condensate field appears, and the case of a well developed condensate field. We consider in detail various possibilities, when a condensate of excitations with $\omega(k_0)<0$ appears, when a condensate with $\omega(k_0)<0$ is already present in a medium at rest, and when there is a static condensate in a medium with nonrelativistic interactions between the quasiparticles. We investigate the condensation of excitations in a rotating medium. We consider the rotation of a cylinder and of a spheroid. We find the critical values of the rotational frequencies for which a condensate appears corresponding to the branch with $\omega(k_0)>0$ and when a condensate appears corresponding to the branch with $\omega(k_0)<0$. We show that under rotation a liquid drop can change its shape and volume because a condensate forms in it. We discuss the possibility of the appearance of a metastable state which is not present in a rotating drop when there is no condensate. We list possible consequences of the effects considered which can be observed experimentally in such systems as condensed ^{4}He, neutron stars, a nucleus with a large angular momentum, or the expanding hadron gas during a nucleus-nucleus collision.

1. INTRODUCTION

We consider a condensed medium in which there are various branches of collective Bose excitations determined by the appropriate dispersion equations. It is well known that there exist a whole class of such excitations. For instance, they can be the excitations of the phonon-roton branch of condensed ^{4}He (the dispersion law $\omega(k)$ is shown in Fig. 1a), the zero-sound excitations of a Fermi liquid, in particular the spin-isospin-sound excitations in nuclear matter (Fig. 1b), the gap excitations of paired particles (electrons in superconducting metals or alloys, nucleons in the nuclear Fermi liquid at temperatures $T<T_c$, and so on), spin waves, ordinary sound waves, electromagnetic excitations in a medium, phonons in a solid, and so on. Different kinds of Bose excitations are described by the corresponding quantum fields: scalar, pseudoscalar, vector, and tensor fields, and so on. These fields can be defined by effective Lagrangians which in some of the simplest cases can be introduced starting from first principles, but in most cases are given phenomenologically.

Let the condensed medium in question move as a whole uniformly along a straight line with a velocity v relative to fixed massive walls which play the role of an excitation generator. If the medium is a quantum Bose liquid, at a temperature $T<T_c$ and for a sufficiently low flow velocity $v<v_{c1}$, one can regard it as consisting of two components, a normal and a superfluid one. The thermal Bose excitations corresponding to a branch of the spectrum which behaves like $\omega=v_s k$, $v_s=\text{const.}$, or $\omega\to\Delta=\text{const.}$, and also the impurity particles with a quadratic dispersion law (e.g., ^{3}He atoms in ^{4}He) form the normal component. For $v<v_{c1}$ the superfluid component interacts neither with the normal component nor with the walls and it has zero viscosity. The normal component has a nonvanishing viscosity. In the velocity range $v_{c1}<v<v_c$ the deceleration of the superfluid component is caused by the creation of vortices which guarantee an effective viscosity between the two components of the liquid. According to the well known Landau criterion for $v>v_c=v_c^r$ it becomes possible to populate the roton section of the elementary excitation spectrum and for $v > v_c^{\text{ph}}$ the phonon section of the spectrum will also become populated. The elementary Bose excitations which are generated contribute to the viscosity (η^{exc}); this leads to an effective deceleration of the whole system.

Until recently it was assumed that for $v>v_c$ the superfluid component of the liquid vanished completely as a consequence of the generation of elementary excitations and the liquid becomes a normal one. However, it was demonstrated in Ref. 1 that when ^{4}He moves in a capillary a condensate of rotons with a wavevector $k=k_0\neq 0$ can appear, and its appearance prevents the further generation of excitations and destruction of the superfluid component.

The main point of the effect is that when the medium

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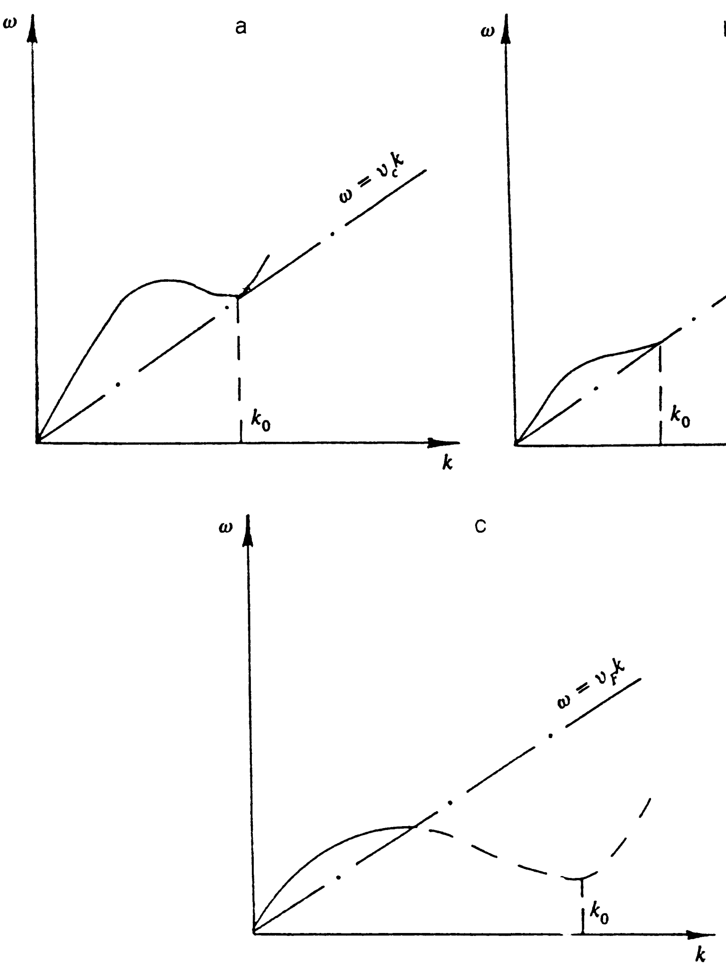

FIG. 1. Dispersion law of low-lying collective Bose excitations a) for ^{4}He, b) for a Fermi liquid; c) the dashed line shows the diffusive extension of the zero-sound branch for a Fermi liquid for a density satisfying $\rho_c > \rho > \rho_{c1}$.

moves with a velocity $v > v_c$ it becomes energetically advantageous to transfer part of the momentum from the particles in the medium to the condensate of excitations which may possess a large momentum and small energy. The interaction of the medium with the walls guarantees that excitations can be generated and subsequently undergo a transition to a coherent condensate state. These conditions can be satisfied not only for flow of a superfluid liquid in a capillary. It is therefore of interest to investigate the possibility of the condensation of excitations in other systems: not only in superfluid media but also in liquids with a small, but finite viscosity, in media in which the dispersion law does not have a roton minimum, not only in capillaries, for relativistic motion of the medium, or for nonuniform motion, in particular, for rotation. There exist media in which there are condensates with $k \neq 0$ even in the ground state, i.e., for $v=0$. It is of interest also to consider the possibility that part of the momentum can be transferred from the particles which make up the medium to the condensate subsystem when such a medium moves. Condensation is accompanied by a lowering of the energy of the system and the conditions for its stability are therefore changed. In particular, the appearance of a condensate when the system rotates may guarantee a metastable state which is not present when there is no condensate. The shape of the system may also change at the same time.

All these problems are discussed in the present paper. The paper is organized as follows. In the next section we consider the general properties of Bose excitation condensation. We discuss the possibility for excitations to condense in normal liquids with a low viscosity. We study the general case of relativistic uniform rectilinear motion. We formulate the model to be used in Sec. 3. Using the example of an effective Lagrangian of a two-component scalar field we study the possibility of condensation of excitations. In Sec. 4 we consider the case just above criticality when a rather weak field is generated, while in Sec. 5 we investigate a fully developed condensate field. We consider in detail various possibilities when there is no condensate in the medium at rest and when there is a condensate with $k \neq 0$ even for $v=0$. We investigate in Sec. 6 the case of a rotating medium which is of practical interest. We study in Sec. 7 the possibility of a change in shape and in volume of a rotating drop as a consequence of condensation. We consider the possibility of the appearance, as a consequence of condensation, of a metastable state which is not present in a fast rotating medium if we neglect condensation. In Sec. 8 we discuss some physical consequences of the phenomenon considered here.

2. LANDAU CRITERION

Consider a condensed medium which moves as a whole uniformly along a straight line with a velocity **v** relative to fixed massive walls which play the role of a generator of excitations. It is convenient, using the Lorentz invariance condition, to change to a frame of reference in which the walls move with a velocity $-\mathbf{v}$ while the medium is at rest.

Let the momentum of the set of excitations be $\mathbf{P}_{exc}$ and its energy be E_{exc}. From the momentum conservation law we then have

$$-\frac{M\mathbf{v}}{\sqrt{1-v^2/c^2}}=-\frac{M\mathbf{v}_1}{\sqrt{1-v_1^2/c^2}}+P_{exc}, \tag{2.1}$$

where M is the mass of the walls,—$\mathbf{v}$ their velocity before the generation of the excitations, and—$\mathbf{v}_1$ their velocity after some excitations have been generated.

The energy of the medium before the generation of the excitations $E_{tot}^{(1)}$ is the sum of the internal energy E_{med} of the medium and the energy of the moving walls (in the frame fixed to the liquid). As a result we have

$$E_{tot}^{(1)}=E_{med}+\frac{Mc^2}{\sqrt{1-v^2/c^2}}. \tag{2.2}$$

When there are Bose excitations present the energy will be

$$E_{tot}^{(2)}=E_{med}+\frac{Mc^2}{\sqrt{1-v_1^2/c^2}}+E_{exc}. \tag{2.3}$$

Using (2.1) to express $\mathbf{v}_1$ in terms of $\mathbf{v}$ and $\mathbf{P}_{exc}$ we obtain

$$v_1^2=A^2/(1+A^2/c^2),\quad A^2=\left(\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}+\frac{P_{exc}}{M}\right)^2. \tag{2.4}$$

Substituting the expressions obtained into (2.3) we have

$$E_{tot}^{(2)}=Mc^2\left[1+\frac{1}{c^2}\left(\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}+\frac{P_{exc}}{M}\right)^2\right]^{1/2}+E_{exc}. \tag{2.5}$$

Hence we can get the Landau criterion for the appearance of a single elementary excitation with momentum $P_{exc}=k$ and energy $E_{exc}=\omega(k)$. Expanding (2.5) in powers of the quantity P_{exc}/M we find

$$E_{tot}^{(2)}=\frac{Mc^2}{\sqrt{1-v^2/c^2}}-P_{exc}\mathbf{v}+E_{exc}+O(1/M). \tag{2.6}$$

From the enery conservation law, $E_{tot}^{(1)}=E_{tot}^{(2)}$, we get the condition

$$\mathbf{k}\mathbf{v}=\omega(k). \tag{2.7}$$

The dispersion law of the elementary Bose excitations for condensed ^{4}He has the form shown in Fig. 1a. One can see from that figure and the condition (2.7) that rotons begin to be generated in ^{4}He when

$$v>v_c=\omega(k_0)/k_0. \tag{2.8}$$

For the linear portion of the spectrum, $\omega(k)=v^{ph}k, k\to 0$, the criterion (2.7) can be rewritten in the form

$$\cos\theta=v^{ph}/v, \tag{2.9}$$

where θ is the angle between the directions of $\mathbf{v}$ and $\mathbf{v}^{ph}$.

Equation (2.9) determines the well known Mach angle in the case of sound waves[2] ($v^{ph}=c_s$ where c_s is the sound velocity in the medium considered) and the Cherenkov cone for charged particles uniformly moving in a medium with a velocity v greater than the light speed in the medium ($v^{ph}=c_{med}$).[3]

For a supersonic flow propagating in the x-direction with a velocity v Eq. (2.9) together with the equations of geometric acoustics[2]

$$\dot{\pi}=\frac{\partial\omega}{\partial k} \tag{2.10}$$

determines two rays

$$\sin-\sin\alpha=\pm c_s/v, \tag{2.11}$$

where α is the angle between the direction of the ray and the x-axis.

One can raise the problem of what happens to the Bose excitations considered here at later times. The answer is essentially determined by the properties of the Bose excitations considered and of the medium. In the case of Cherenkov radiation only a small number of light quanta is generated, which leads to some loss of energy of the moving particle. In supersonic motion shock waves may occur, transporting the excess matter density. There is also another possibility. When a significant number of excitations are generated they can, as a result of collisions, give off the excess energy and change into an energetically more favorable coherent state described by a nonlinear wave or, put differently, by a classical field of a condensate with a finite momentum $k_0\neq 0$. The excess energy is then spent on heating the medium and on radiation.

In a quantum Bose liquid there are no single-quasiparticle excitations with a quadratic dispersion law, $\omega=k^2/2m^*$ for $k\to 0$ with m^* the effective mass of the constituent particles of the liquid. In a normal liquid such excitations are present. According to Landau's criterion (2.8) they can be generated even for arbitrarily low velocities of the liquid. The interaction between the particles excited by the walls and the bulk of the liquid effectively decelerates the latter. The energy dissipated in the liquid is determined by the expression[2]

$$\dot{E}=-\eta\int\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2 dV, \tag{2.12}$$

from which one can estimate the characteristic time, $t_\eta\sim m^*l^2/\eta$ (l is the distance of the layer considered to the closest wall) after which a given layer of the liquid is decelerated. In addition to excitations with a quadratic dispersion law, we assume as before that there is a branch of Bose excitations with a dispersion law $\omega(k)$ of the form shown in Fig. 1. If the characteristic time t_η (which depends on l) for the bulk of the liquid is much longer than the time t_{cond} (i.e., the time for the Bose excitations to be generated and produce a condensate) Eqs. (2.1) to (2.8) can be approximately satisfied and the condensation effect considered is thus possible for times $t\ll t_\eta$. For $t\gtrsim t_\eta$, of course, the condensate disappears together with the disappearance of the relative momentum of the wall and the bulk of the liquid.

Moreover, one can change the statement of the problem somewhat and consider stationary flow at a given external pressure. A certain velocity profile $v(l)$ is then established and the condensation effect considered here is then possible in principle in regions with $v(l)>v_c$. (Of course, we assume here, as in all previous discussions, the presence of a branch of Bose excitations of the appropriate kind.) For a given external pressure the effect considered here could, in principle, arise also in the case of solid matter. The excitations then appear due to the presence of friction at the wall.

Note that all considerations expressed above only point out the possibility in principle of the phenomenon of Bose condensation of excitations in various systems. In the case of an actual system, however, the problem may, of course, turn out to be considerably more complicated.

3. FORMULATION OF THE MODEL

For definiteness we shall assume that the low-lying Bose excitations are described by a two-component scalar field $\varphi=\varphi_1+i\varphi_2$ determined directly by giving its effective Lagrangian. A single-component field can be considered to be a special case of a two-component one when the magnitude of one of the field components tends to zero. In principle one can formulate in a similar way the description of fields with a different symmetry.

A two-component field of spin-0 particles can be defined using the effective Lagrangian[4]

$$\mathscr{L}_{\text{exc}}=\int dV\{|\partial_\mu\varphi|^2-m^2|\varphi|^2 -\varphi^*\overleftrightarrow{\Pi}(i\partial_t,i\nabla)\varphi-\lambda|\varphi|^4/2\}, \tag{3.1}$$

where m is the mass, $\overleftrightarrow{\Pi}(i\partial_t,i\nabla)$ is the boson polarization operator in the coordinate representation in a symmetric formulation with respect to the action of the derivatives with respect to φ and φ^* which occur in it, and the quantity λ corresponds to the effective boson-boson interaction in the medium.

Such a Lagrangian describes, for instance, the field of $\pi^\pm$-mesons in a nuclear medium. In the case of a charged field φ one must in (3.1) introduce also its interaction with the electromagnetic field A_μ which is realized through the gauge transformation $\partial_\mu\to\partial_\mu-ieA_\mu$. In what follows for simplicity we shall neglect the possible interaction of the field φ with the electromagnetic field A_μ. This can be done, firstly, if the field φ is uncharged (as in the case of ^{4}He), and for a charged field φ if the system considered occupies a sufficiently small volume so that one can neglect Coulomb effects.[1)] For a charged system in a large volume one must additionally take into account the condition that the system is electrically neutral.

The energy of the boson subsystem is found from the formula

$$E_{\text{exc}}=\dot\varphi\frac{\partial\mathscr{L}_{\text{exc}}}{\partial\dot\varphi}+\dot\varphi^*\frac{\partial\mathscr{L}_{\text{exc}}}{\partial\dot\varphi^*}-\mathscr{L}_{\text{exc}}. \tag{3.2}$$

Changing to the Fourier transform

$$\varphi=\sum_k\varphi_k\exp(ikr-i\omega t), \tag{3.3}$$

we have

$$\mathscr{L}_{\text{exc}}=V\sum_k(\omega^2-m^2-k^2-\Pi(\omega,k))\varphi_k\varphi_{-k} -\frac{1}{2}V\sum_{kk_1k_2}\lambda\varphi_k\varphi_{k_1}\varphi_{-k_2}\varphi_{k_2-k_1-k}, \tag{3.4}$$

$$E_{\text{exc}}=V\sum_k\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\omega\varphi_k\varphi_{-k} +\frac{1}{2}V\sum_{kk_1k_2}\lambda\varphi_k\varphi_{k_1}\varphi_{-k_2}\varphi_{k_2-k_1-k}-V\sum_k \times(\omega^2-m^2-k^2-\Pi(\omega,k))\varphi_k\varphi_{-k}, \tag{3.5}$$

where V is the volume of the system.

In the momentum representation the quantity λ is a complicated function of the frequencies and momenta of the particles which occur in the diagrams defining it. For simplicity we take $\lambda=\text{const}$ in what follows. The generalization is quite obvious.

The total momentum of the boson subsystem is determined in terms of the corresponding components of the energy-momentum tensor ($T_{\alpha0},\alpha=1,2,3$):

$$P^\alpha_{\text{exc}}=T^\alpha_0=\sum_k\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)k_\alpha\varphi_k\varphi_{-k}V=\sum_k\varphi_kk_\alpha V, \tag{3.6}$$

where $\rho_{\mathbf{k}}$ is the density of excitations with the given momentum $\mathbf{k}$ and frequency ω.

The spatial dependence of the field φ is found from the equation of motion $\delta\mathscr{L}=0$ which has a whole class of solutions corresponding to different spatial structures. We further put

$$\varphi=\varphi_0\exp(ik_0r-i\omega t), \tag{3.7}$$

where $\varphi_0=\text{const}$ is the amplitude of the condensate field, $\mathbf{k}_0$ the wavevector, and ω the frequency. Such a choice for the structure of the field corresponds to leaving just one harmonic $\varphi_{\mathbf{k}}=\varphi_0$ with $\mathbf{k}=\mathbf{k}_0$ in Eqs. (3.4) to (3.6). The field parameters φ_0 and $\mathbf{k}_0$ are determined by minimizing the energy. The frequency ω is found from the dispersion law of the Bose excitations

$$\omega^2-m^2-\mathbf{k}^2-\Pi(\omega,\mathbf{k})=0. \tag{3.8}$$

For solutions which have a physical meaning we then have[5]

$$\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega=\omega(k)}>0. \tag{3.9}$$

This factor arises as a consequence of the normalization of the wavefunction to a single quasiparticle. If for some branch of the spectrum $\omega=\omega^-(k)$ we have

$$\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega^-(k)}<0, \tag{3.10}$$

such a branch must be interpreted as an antiparticle branch after the substitution $\omega=-\omega^{-}(k)$, as a result of which condition (3.9) is now satisfied.

The excitation spectrum can thus have branches with $\omega(k)>0$ and with $\omega(k)<0$, but only solutions satisfying condition (3.9) have a physical meaning. In some cases the excitations have a diffusive nature. For instance, let us say, the spectrum of the low-lying excitations with pion quantum numbers in a nuclear medium with the same number of neutrons and protons ($N=Z$) for a nuclear matter density $\rho \gg \rho_{c1}=(0.5$ to $0.7)\rho_0$ where $\rho_0 \simeq 0.17$ fm^{-3} is the density of an atomic nucleus, has the form

$$i\beta\omega \simeq \tilde{\omega}^2(k), \tag{3.11}$$

where $\beta \propto k$ is a positive quantity while we have

$$\tilde{\omega}^2(k)=m_\pi^2+k^2+\Pi(0,k), \tag{3.11a}$$

with $\tilde{\omega}^2(k)<m_\pi^2$ for values of the wavenumbers k close to the value k_0 corresponding to the minimum of the quantity $\tilde{\omega}^2(k)$. For densities $\rho>\rho_c>\rho_0$ the quantity $\tilde{\omega}^2(k_0)$ becomes negative, which corresponds to the appearance of a static condensate of virtual pion quasiparticles. The factor $(2\omega-\partial \mathrm{Re}\Pi/\partial\omega)|_{\omega=0}$ vanishes and, hence, for the total momentum we have $P_{\rm exc}=0$, i.e., the effect considered in the present paper does not occur.

In most cases condensed media consist of nonrelativistic quasiparticles and the dispersion law for the low-lying excitations is determined by nonrelativistic energies and momenta. For brevity we shall call such media nonrelativistic in what follows although they can, as before, move as a whole with a relativistic velocity v. For nonrelativistic media it is convenient to introduce a new definition of the frequency: $\varepsilon=\omega-m$, calculated relative to the mass. In that case we have $|\varepsilon| \ll m$. Introducing obvious transformations we have instead of (3.4)

$$\mathscr{L}_{\rm exc}=V\sum_{\mathbf{k}}\left[\varepsilon-\frac{k^2}{2m}-\Pi(\varepsilon,k)\right]\psi_{\mathbf{k}}\psi_{-\mathbf{k}}$$
$$-\frac{1}{2}V\sum_{\mathbf{k}}\Lambda\psi_{\mathbf{k}}\psi_{\mathbf{k}_1}\psi_{-\mathbf{k}_2}\psi_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}. \tag{3.12}$$

The energy of the system, calculated relative to the rest mass, is given by the expression

$$E_{\rm exc}=\varepsilon\frac{\partial\mathscr{L}_{\rm exc}}{\partial\varepsilon}-\mathscr{L}_{\rm exc}, \tag{3.13}$$

and the momentum of the boson subsystem by

$$P^\alpha_{\rm exc}=\sum_{\mathbf{k}}\left(1-\frac{\partial\Pi}{\partial\varepsilon}\right)k_\alpha\psi_{\mathbf{k}}\psi_{-\mathbf{k}}V=\sum_{\mathbf{k}}\rho_{\mathbf{k}}k_\alpha V, \tag{3.13a}$$

where $\rho_{\mathbf{k}}$ is the density of excitations with a given momentum $\mathbf{k}$ and frequency ε. The function $\varepsilon(k)$ can be found from the appropriate Dyson equation

$$\varepsilon-\frac{k^2}{2m}-\Pi(\varepsilon,k)=0. \tag{3.14}$$

In the case of ^{3}He, as for π-mesons in a medium with $N=Z$, the excitation spectrum is diffusive in character and the factor $(1-\partial\Pi/\partial\varepsilon)|_{\varepsilon=0}$ vanishes. In that case we find according to (3.13a) that $P_{\rm exc}=0$. In the case of ^{4}He the spectrum $\varepsilon(k)$, as we have already mentioned, has the shape shown in Fig. 1a, and the effect considered here is possible.

4. APPROXIMATION OF WEAK EXCITATION

Furthermore, assuming that the total momentum of the Bose excitations is much smaller than the momentum of the walls (or, put differently, the momentum of the whole medium in the wall frame) and expanding (2.5) up to terms quadratic in $P_{\rm exc}$ after substituting there (3.5) and (3.6) and using (2.2), we have

$$\delta E=E^{(2)}_{\rm tot}-E^{(1)}_{\rm tot}$$
$$=V\sum_{\mathbf{k}}\left\{\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)(\mathbf{vk}+\omega)\right.$$
$$\left.-(\omega^2-m^2-k^2-\Pi(\omega,k))\right\}\varphi_{\mathbf{k}}\varphi_{-\mathbf{k}}$$
$$+\frac{\lambda}{2}V\sum_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\varphi_{\mathbf{k}}\varphi_{\mathbf{k}_1}\varphi_{-\mathbf{k}_2}\varphi_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}$$
$$+\frac{V^2}{2M}\sqrt{1-\frac{v^2}{c^2}}\sum_{\mathbf{k}}\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)^2\left(k^2-\frac{(\mathbf{vk})^2}{c^2}\right)$$
$$\times(\varphi_{\mathbf{k}}\varphi_{-\mathbf{k}})^2. \tag{4.1}$$

Substituting the field φ in the form (3.7) into (4.1) we get

$$\delta E=\left\{\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)(\mathbf{vk}_0+\omega)\right.$$
$$\left.-(\omega^2-m^2-k_0^2-\Pi(\omega,k_0))\right\}\varphi_0^2V+\frac{\tilde{\lambda}}{2}\varphi_0^4V, \tag{4.2}$$

where we have written

$$\tilde{\lambda}=\lambda\left\{1+\frac{\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)^2\left(k_0^2-\frac{(\mathbf{vk}_0)^2}{c^2}\right)}{M\lambda}\sqrt{1-v^2/c^2}V\right\}. \tag{4.2a}$$

The choice of the direction of $\mathbf{k}_0$ is determined by minimizing expression (4.2), whence it follows that the vector $\mathbf{k}_0$ is antiparallel to the wall velocity (parallel to the velocity of the liquid in the wall frame). Using this and substituting $\omega=\omega(k_0)$ into (4.2) we have

$$\delta E=-\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\bigg|_{\omega(k_0)}[vk_0-\omega(k_0)]\varphi_0^2V+\frac{\tilde{\lambda}}{2}\varphi_0^4V. \tag{4.3}$$

a) Assume that there is a low-lying branch of the spectrum with $\omega(k_0)>0$. It is clear from Eqs. (4.3) and (3.9) that in that case the quantity δE is negative when the Landau criterion is satisfied.

For the spectrum shown in Fig. 1a the value of k_0 is determined by the roton minimum. The magnitude of the critical velocity $v_c = v_c^r = \omega(k_0)/k_0$ is then less than the corresponding quantity $v_c^{\rm ph}$ for the phonon section of the spectrum. For the Fermi liquid spectrum shown in Fig. 1b the value of k_0 corresponds to the intersection of the $\omega(k)$ curve with the line $\omega = k v_F$, where v_F is the quasiparticle velocity on the Fermi surface. In the general case the quantity k_0 is determined directly by minimizing the energy.

Minimizing (4.3) with respect to φ_0^2 we have

$$\varphi_0^2 = \frac{\left(2\omega - \frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)} [v k_0 - \omega(k_0)]}{\tilde{\lambda}} \theta[v k_0 - \omega(k_0)], \tag{4.4}$$

where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$. Resubstituting the expression obtained into (4.3) we obtain

$$\delta E = -\frac{[v k_0 - \omega(k_0)]^2 \left[2\omega(k_0) - \frac{\partial\Pi}{\partial\omega}\Big|_{\omega(k_0)}\right]^2}{2\tilde{\lambda}} \times V\theta[v k_0 - \omega(k_0)]. \tag{4.5}$$

It is clear from this that for $v > v_c$ the buildup of a condensate field with the structure (3.7) becomes energetically favorable. It is convenient to rewrite Eq. (4.5) in the form

$$\delta E = -\frac{(v-v_c)^2}{2} \frac{d^2\delta E}{dv^2}\Big|_{v_c} \theta(v-v_c), \quad v - v_c \ll v_c, \tag{4.5a}$$

where we have written

$$\frac{d^2\delta E}{dv^2} = \frac{k_0^2}{\tilde{\lambda}} \left[2\omega(k_0) - \frac{\partial\Pi}{\partial\omega}\Big|_{\omega(k_0)}\right]^2 V. \tag{4.5b}$$

It is clear from (4.5a) that the condensate of excitations occurs through a second order phase transition, since $(d\delta E/dv)|_{v=v_c} = 0$ while $(d^2\delta E/dv^2)|_{v=v_c} \neq 0$.

In the limit $\lambda \to 0$ we get from (4.5a), using (4.5b) and (4.2a),

$$\delta E = -\frac{(v-v_c)^2 M}{2\left(1 - \frac{v^2}{c^2}\right)^{3/2}}. \tag{4.6}$$

For $v > v_c$ the quantity δE is thus finite, even when the interactions between the elementary excitations are neglected.

We note that $\partial\Pi/\partial\omega$ and $\tilde{\lambda}(\omega,k)$ are, generally speaking, complicated functions of the momentum k and the frequency ω. The procedure of minimizing the energy, which determines these quantities and also the structure of the field φ, is thus rather laborious in the general case.

In the case of a nonrelativistic medium just above criticality we get instead of (4.1)

$$\delta E = V \sum_{\mathbf{k}} \left\{\left(1 - \frac{\partial\Pi}{\partial\varepsilon}\right)(\mathbf{vk} + \varepsilon) - \left(\varepsilon - \frac{k^2}{2m} - \Pi(\varepsilon,k)\right)\right\} \times \psi_{\mathbf{k}}\psi_{-\mathbf{k}} + \frac{\Lambda}{2} V \sum_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \psi_{\mathbf{k}}\psi_{\mathbf{k}_1}\psi_{-\mathbf{k}_2}\psi_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}} + \sum_{\mathbf{k}} \times \left(1 - \frac{\partial\Pi}{\partial\varepsilon}\right)^2 \left(k^2 - \frac{(\mathbf{vk})^2}{c^2}\right)(\psi_{\mathbf{k}}\psi_{-\mathbf{k}})^2 \frac{V^2\sqrt{1 - \frac{v^2}{c^2}}}{2M}. \tag{4.7}$$

Assuming that the field ψ has the form

$$\psi = \psi_0 \exp(i\mathbf{k}_0\mathbf{r} - i\varepsilon t) \tag{4.8}$$

and substituting its Fourier transform $\psi_{\mathbf{k}} = \psi$ into (4.7) we have

$$\delta E = -\frac{2[v k_0 - \varepsilon(k_0)]^2 \left(1 - \frac{\partial\Pi}{\partial\varepsilon}\Big|_{\varepsilon(k_0)}\right)^2}{\tilde{\Lambda}} \times V\theta[v k_0 - \varepsilon(k_0)], \tag{4.9}$$

where we have written

$$\tilde{\Lambda} = \Lambda\left\{1 + \frac{4\left(1 - \frac{\partial\Pi}{\partial\varepsilon}\Big|_{\varepsilon(k_0)}\right)^2 k_0^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{MA} V\right\}, \tag{4.9a}$$

while the quantity $\varepsilon(k_0)$ is determined from the appropriate Dyson equation (3.14).

As $\Lambda \to 0$ we get from (4.9)

$$\delta E = -\frac{M(v-v_c)^2}{2(1-v^2/c^2)^{3/2}}, \quad v_c = \varepsilon(k_0)/k_0 > 0. \tag{4.10}$$

For ^{4}He we have $M \simeq m_{\rm He}^* \rho V$ where ρ is the particle density and $m_{\rm He}^*$ the effective mass of the ^{4}He atoms.

We consider the simplest complex solution of the equation of motion (3.7). The density of the condensate, $\rho = (2\omega - \partial\Pi/\partial\omega)|\varphi|^2$, is then a constant quantity. For a nonrelativistic medium we have $\rho = (1 - \partial\Pi/\partial\varepsilon)|\psi|^2$, which is also a constant quantity for the corresponding field of the form (4.8).

We note, however, that the matter density can have a modulated component $\delta\rho \sim \varphi_0 \cos(\mathbf{k}_0\mathbf{r} - \omega t)$ (or $\delta\rho \sim \psi_0 \cos(\mathbf{k}_0\mathbf{r} - \omega t)$ in the case of a nonrelativistic medium) even for a field of the form (3.7) [or, respectively, (4.8)]. This occurs if only part of the particles in the substance form a condensate phase and the total field φ is in the form $\langle\varphi\rangle + \varphi'$ (or $\psi = \langle\psi\rangle + \psi'$) where the density component ρ_0 is connected with the quantity $\langle\varphi\rangle$ ($\langle\psi\rangle$). For instance, in the case of the excitations in condensed ^{4}He the macroscopic wavefunction can be written in the form $\psi = \langle\psi\rangle + \psi_0 \exp(i\mathbf{kr} - i\varepsilon t)$, where $\langle\psi\rangle$ is the macroscopic wavefunction of the condensate with $k=0$ and the second term describes the condensate of excitations with $\mathbf{k} = \mathbf{k}_0$, $\varepsilon = \varepsilon(k_0)$. As a result the total density of the medium, $\rho = |\psi|^2$, contains a correction

$\delta\rho\sim\psi_0\cos(\mathbf{k}_0\mathbf{r}-\varepsilon t)$. A similar correction, $\delta\rho\sim\psi_0\cos(\mathbf{k}_0\mathbf{r}-\varepsilon t)$ occurs in the case of liquid crystal substances.

One can consider also other solutions, in particular, those which are defined by real functions. The simplest of such solutions are

$$\varphi=\varphi_0\cos(\omega t-\mathbf{k}_0\mathbf{r}) \tag{4.11}$$

in the relativistic case and

$$\psi=\psi_0\cos(\varepsilon t-\mathbf{k}_0\mathbf{r}) \tag{4.12}$$

for a nonrelativistic medium.

Substituting (4.11) into (3.5) we get, after averaging over space,

$$\delta E\simeq-\left\{\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)(vk_0+\omega)-(\omega^2-m^2-k_0^2-\Pi(\omega,k_0))\right\}\frac{\varphi_0^2}{2}V+\frac{3}{8}\tilde{\lambda}\varphi_0^4V. \tag{4.13}$$

Here we have taken into account only the volume contribution to the energy while the surface terms, which are $\sim\mathcal{O}(1/r_0k_0)$ where r_0 is a characteristic transverse dimension of the medium, have been dropped. Putting $\omega=\omega(k_0)$ and minimizing δE with respect to φ_0^2 we get

$$\delta E=-\frac{[vk_0-\omega(k_0)]^2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{3\tilde{\lambda}}\times V\theta[vk_0-\omega(k_0)]. \tag{4.14}$$

It is clear from comparing (4.4) with (4.14) that solution (3.7) rather than (4.11) is realized for $\lambda=\text{const}$, since in the first case the energy is lower. In a more realistic statement of the problem the quantity λ depends on the structure of the field, and this fact must be taken into account when comparing energies corresponding to different solutions.

In the case of a nonrelativistic medium we have for a field of the form (4.12)

$$\delta E=-\frac{4[vk_0-\varepsilon(k_0)]^2\left(1-\frac{\partial\Pi}{\partial\varepsilon}\Big|_{\varepsilon(k_0)}\right)^2}{3\tilde{\Lambda}}\times V\theta[vk_0-\varepsilon(k_0)]. \tag{4.15}$$

One should note that the structures (4.11) and (4.12) correspond to a modulated density, $\rho=(2\omega-\partial\Pi/\partial\omega)|\varphi|^2$ and $\rho=(1-\partial\Pi/\partial\varepsilon)|\psi|^2$, respectively.

If structures are realized which modulate the density they can be more easily observed than structures corresponding to a constant density.

If the field φ (or ψ) considered is a single-component one, in that case the solution has the form (4.11) (or, respectively, (4.12)).

It is clear from Eqs. (4.4), (4.14) and (4.9), (4.15) given above that the smaller the magnitude of $\omega(k_0)$ [or $\varepsilon(k_0)$] the lower the critical velocity for which the corresponding branch of the excitation spectrum starts to be populated.

b) Let there now be a branch with $\omega(k_0)<0$ in the excitation spectrum. Such a branch occurs, for instance, in the $\pi^{\pm}$-meson spectrum in a neutron medium with density $\rho>\rho_c^+$ $(\rho_c^+<\rho_0)$. From Eq. (4.3) which is clearly valid also for $\omega(k_0)<0$ we have, using (2.3),

$$E_{\text{tot}}^{(2)}=-\frac{[vk_0+|\omega(k_0)|]^2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{2\tilde{\lambda}}\times V+\frac{Mc^2}{\sqrt{1-v^2/c^2}}+E_{\text{med}}. \tag{4.16}$$

In this case there is even a condensate for $v=0$. The initial energy of the condensate subsystem (for $v=0$) is determined by substituting a field of the form (3.7) into (3.5) and subsequently minimizing with respect to the amplitude φ_0 of the field. As a result we have for the condensate energy and momentum

$$E_{\text{cond}}^{(0)}=-\frac{\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2\omega^2(k_0)}{2\lambda}V, \tag{4.17}$$

$$\mathbf{P}_{\text{exc}}=\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2\omega(k_0)\mathbf{k}_0V/\lambda\neq0. \tag{4.18}$$

Note that notwithstanding the presence of a nonvanishing momentum $\mathbf{P}_{\text{exc}}$ there may be no current I in the ground state since the momentum

$$\mathbf{P}_{\text{exc}}=\nabla\varphi_r\frac{\partial\text{L}}{\partial\dot{\sigma}_r},\quad r=1,2,$$

and the current

$$I=-ie\varphi\frac{\partial\text{L}}{\partial\nabla\varphi}+\text{c.c.}$$

are determined by essentially different expressions which in the case of a complicated dispersion do not reduce one to the other.

We also point out that λ rather than $\tilde{\lambda}$ occurs in the denominator of (4.17) and (4.18). This is connected with the fact that in obtaining Eqs. (4.17) and (4.18) one need not assume that the momentum conservation law is satisfied. Of course, in the nonstationary statement of the problem for the generation of a condensate in a medium at rest the total momentum, being equal to zero, is conserved. Using this, as was done when obtaining Eq. (4.6), we would be led to Eq. (4.17) but with the quantity $\tilde{\lambda}$ instead of λ since as the consequence of the conservation of the total momentum (and as a consequence of the fact that $P_{\text{exc}}\neq0$) the particles in the medium acquire a nonvanishing kinetic energy ($v\neq 0$). However, after the characteristic time t_η this energy is dissipated and for $t\gg t_\eta$ we have $v\to 0$ and the energy connected with φ reaches the value (4.17) obtained above.

Using (4.17) and (2.2) we have for the initial energy $E^{(1)}_{\rm tot}$ the expression

$$E^{(1)}_{\rm tot}=\frac{Mc^2}{\sqrt{1-v^2/c^2}}+E_{\rm med}+E^{(0)}_{\rm cond}. \qquad (4.19)$$

As the velocity of the medium increases the momentum can be transferred to the condensate of excitations with a frequency $\omega(k_0)<0$ provided we have $\delta E=E^{(2)}_{\rm tot}-E^{(1)}_{\rm tot}<0$. Assuming the magnitude of λ to be sufficiently large (in that case we have $\tilde\lambda=\lambda$) we find from (4.16) and (4.19) the corresponding critical velocity

$$v_c\simeq\frac{|\omega(k_0)|k_0\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\bigg|_{\omega(k_0)}\right]^2}{2\lambda M}V. \qquad (4.20)$$

We have thus $v_c\to 0$ as $\lambda\to\infty$. As $\lambda\to 0$ we have $\delta E>0$ and $v_c\to\infty$.

c) In the case of a nonrelativistic medium it is of interest to consider the possibility of transferring part of the momentum of the medium to a static condensate ($\varepsilon=0$). The value $\varepsilon=0$ is then no longer determined by the Dyson equation (3.14). Putting $\varepsilon=0$ in (4.7) and minimizing the energy we have for a field of the form (4.8)

$$E^{(2)}_{\rm tot}=-\frac{2\left[vk_0\left(1-\frac{\partial\Pi}{\partial\varepsilon}\bigg|_0\right)-\tilde\varepsilon(k_0)\right]^2}{\tilde\Lambda}V\theta[vk_0-\tilde\varepsilon(k_0)]$$

$$+\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}}-Mc^2+E_{\rm med}, \qquad (4.21)$$

where the value of $\tilde\varepsilon(k_0)$ is determined by the minimum of the expression

$$\tilde\varepsilon(k)=\frac{k^2}{2m}+\Pi(\varepsilon=0,k) \qquad (4.22)$$

with respect to the magnitude of k.

It is clear from Eq. (4.21) that for $(1-\partial\Pi/\partial\varepsilon|_0)>0$ and $\tilde\varepsilon(k_0)>0$ there is a critical value of the velocity

$$v_c^{\rm st}=\frac{\tilde\varepsilon(k_0)}{k_0\left(1-\frac{\partial\Pi}{\partial\varepsilon}\bigg|_0\right)} \qquad (4.23)$$

such that a static condensate field can be generated for $v>v_c^{\rm st}$. If the magnitude of (4.23) is less than the value $v_c=\varepsilon(k_0)/k_0>0$ a static condensate is generated for $v_c>v>v_c^{\rm st}$. For $v>v_c>v_c^{\rm st}$ a condensate of Bose excitations corresponding to the branch of the spectrum with $\varepsilon(k)>0$ may also appear. This possibility is realized if the parameters of the medium are such that the negative contribution to the energy (4.9) increases faster than the corresponding contribution to (4.21). In the opposite case the condensate of excitations with $\varepsilon(k)=\varepsilon(k_0)$ does not appear at all and the static condensate remains. In the $v_c<v_c^{\rm st}$ case there is a condensate of excitations with $\varepsilon=\varepsilon(k_0)$ for $v_c^{\rm st}>v>v_c$ and a static field can also appear when we have $v>v_c^{\rm st}$.

Equation (4.21) is suitable also for $\tilde\varepsilon(k_0)<0$ when there is a static condensate with $k=k_0\neq 0$ even in a medium at rest. We note that in the case considered the total momentum of the Bose subsystem $P_{\rm exc}$ is nonvanishing since the quantity $1-\partial\Pi/\partial\varepsilon|_0$ is different from zero and we have a current

$$I=\frac{\partial L}{\partial k}\bigg|_{k_0}=\left(2k-\frac{\partial\Pi}{\partial k}\right)\bigg|_{k_0}|\varphi|^2=0.$$

For $1-\partial\Pi/\partial\varepsilon|_0=0$ the excitations have a diffusive nature and we have $P_{\rm exc}=0$ in the ground state of the system, so that this effect does not occur.

Part of the momentum of the medium can thus be transferred either to a nonstatic condensate or to a static condensate with $k\neq 0$. However, to avoid misunderstandings one should note that it is impossible to transfer momentum of the medium to a condensate with $k=0$ such as exists, for instance, in the ground state of condensed ^{4}He and in superconductors. However, we have already mentioned that in ^{4}He the effect considered occurs for velocities $v>v_c$ corresponding to populating the roton minimum on the dispersion law curve $\varepsilon(k)$.[1] It is also impossible to transfer momentum to a static condensate corresponding to the ground state of a system in the case of a diffusive spectrum when we have $1-\partial\Pi/\partial\varepsilon|_0=0$ and $P_{\rm exc}=0$.

5. CONDENSATE FIELD FOR ARBITRARY VELOCITIES OF THE MEDIUM

In the general case one must start directly from Eq. (2.5). Using also (3.5) and (3.6) and expressing the energy $E_{\rm exc}$ of the boson subsystem in terms of the momentum $P_{\rm exc}$ when the condensate field has the form (3.8) we get

$$E^{(2)}_{\rm tot}=Mc^2\left[1+\frac{1}{c^2}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{\mathbf{P}_{\rm exc}}{M}\right)^2\right]^{1/2}$$

$$+|\mathbf{P}_{\rm exc}|\frac{\omega(k_0)}{k_0}+\frac{\lambda P^2_{\rm exc}}{2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\bigg|_{\omega(k_0)}\right]^2k_0^2V}$$

$$+E_{\rm med}. \qquad (5.1)$$

a) Let there be a branch of the excitation spectrum with $\omega(k_0)>0$ and let us have $v\ll c$ for the velocity. Equation (5.1) can then be simplified. Minimizing the energy $E^{(2)}_{\rm tot}$ with respect to the quantity $P_{\rm exc}$ we have $v_1=v_c=\omega(k_0)/k_0$, $\mathbf{P}_{\rm exc}\|-\mathbf{v}$ and, correspondingly,

$$E_{\text{tot}}^{(2)}=Mc^2+\frac{Mv^2}{2}-\frac{(v-v_c)^2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2 k_0^2}{2\tilde{\lambda}_{\text{nrel}}}\times\theta(v-v_c)+E_{\text{med}},\qquad(5.2)$$

where we have introduced the notation

$$\tilde{\lambda}_{\text{nrel}}=\lambda\left\{1+\frac{\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2 k_0^2 V}{M\lambda}\right\}.\qquad(5.2a)$$

Hence we get

$$\delta E=-\frac{(v-v_c)^2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{2\tilde{\lambda}_{\text{nrel}}}V\theta(v-v_c),\qquad(5.3)$$

which is the same as Eq. (4.4) after one takes the nonrelativistic limit ($v\ll c$) in the latter.

In the case of arbitrary relativistic motion of the medium one can easily perform the minimization of expression (5.1) with respect to P_{exc} in the limit $\lambda\to 0$. As a result we have $v_1\simeq v_c$ and

$$E_{\text{tot}}^{(2)}=\frac{Mc^2}{\sqrt{1-v_c^2/c^2}}+Mv_c\left(\frac{v}{\sqrt{1-v^2/c^2}}-\frac{v_c}{\sqrt{1-v_c^2/c^2}}\right)+E_{\text{med}}.\qquad(5.4)$$

Hence we have $E_{\text{tot}}^{(2)}<E_{\text{tot}}^{(1)}$ for $v>v_c$. One can easily show that for $v-v_c\ll v_c$ the quantity δE following from (5.4) is the same as expression (4.5) obtained above just above criticality.

As in the case when we are just above criticality we can thus conclude that the softer the excitation spectrum (the smaller the quantity $v_c=\omega(k_0)/k_0>0$) the smaller also the resulting energy $E_{\text{tot}}^{(2)}$. As $\omega(k_0)\to 0$ the condensate field φ appears already for $v>v_c\to 0$ and the energy $E_{\text{tot}}^{(2)}$ tends to Mc^2+E_{med} instead of to the quantity $E_{\text{tot}}^{(1)}=Mc^2/\sqrt{1-v^2/c^2}+E_{\text{med}}$ as in the case when there is no excitation condensate.

b) Let there now be a branch of the excitations with a negative frequency $\omega(k_0)<0$ in the medium at rest. We have already mentioned earlier that such a branch appears, for instance, in the π^+-meson spectrum in neutron matter with a density $\rho>\rho_c^+$ ($\rho_c^+<\rho_0$). In that case there is a condensate even for $v=0$. For $\omega(k_0)<0$, as for $\omega(k_0)>0$, one can start from Eq. (5.1). In the nonrelativistic limit $v\ll c$ we have

$$P_{\text{exc}}=M\left(\frac{|\omega(k_0)|}{k_0}+v\right)>Mv,\qquad(5.5)$$

i.e., the resulting velocity of the medium for $v\neq 0$ turns out to be antiparallel to the velocity of the initial motion. The resulting energy is

$$E_{\text{tot}}^{(2)}=Mc^2+\frac{Mv^2}{2}-\frac{\left(v+\frac{|\omega(k_0)|}{k_0}\right)^2\left[\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2 k_0^2}{2\tilde{\lambda}_{\text{nrel}}}V+E_{\text{med}}.\qquad(5.6)$$

One must now compare this quantity with the expression for the initial energy

$$E_{\text{tot}}^{(1)}=Mc^2+\frac{Mv^2}{2}-E_{\text{cond}}^{(0)}+E_{\text{med}},\qquad(5.7)$$

in which the presence of a condensate already in the medium at rest has been taken into account.

One sees easily by comparing (5.6) and (5.7) that for $\omega(k_0)<0$, as for the $\omega(k_0)>0$ case, there is a critical value of the velocity of the medium starting from which it becomes energetically favorable to transfer part of the momentum from the particles in the medium to the boson subsystem. For $v>v_c$ the amplitude of the condensate field and the energy gained, δE, increase with increasing v.

One should again note that these effects occur for characteristic times $t\ll t_\eta$. For $t\gtrsim t_\eta$ the energy

$$\delta\tilde{E}=\frac{Mc^2}{\sqrt{1-v_1^2/c^2}}-Mc^2\qquad(5.8)$$

changes to thermal energy and it is removed from the system together with the corresponding momentum

$$\delta\tilde{P}=Mv_1/\sqrt{1-v_1^2/c^2}.\qquad(5.9)$$

The resulting energy is then lowered by an amount $\delta\tilde{E}$ and the condensate field corresponding to a frequency $\omega(k_0)<0$ reaches the value occurring in the system at rest.

6. CONDENSATION OF BOSE EXCITATIONS IN A ROTATING MEDIUM

Above we considered the simplest case of uniform and rectilinear motion of the medium. The interaction between the medium and the walls then served as the mechanism for the generation of excitations. One also required the velocity of the medium to exceed a certain critical magnitude.

It is natural to consider nonuniform as well as rectilinear uniform motion. The presence of walls is then no longer necessary. Indeed, the existence of acceleration or deceleration guarantees another mechanism for the generation of excitations. Thus, an excitation condensate can occur in, let us say, a variable field or in a rotating medium.

We now turn directly to the study of the case of a rotating medium which is of physical interest. Instead of the momentum conservation law used earlier we must then take into account the angular momentum conservation law. Otherwise the discussion is similar to the one given above for the case of rectilinear uniform motion. To simplify the calculations we restrict ourselves to the case of nonrelativistic rotation with $\Omega a \ll 1$, where Ω is the rotation frequency and a a characteristic dimension of the system. It would be necessary to take general relativistic effects connected with changes in the metric into account for $\Omega a \sim 1$. To begin with we shall also neglect deformations of the system occurring during rotation, i.e., for simplicity we shall assume that the density of the medium and the shape of the body are fixed.

In what follows we consider the two cases of a rotating cylinder and a rotating spheroid which are the simplest cases of most physical interest.

a) Rotation of a cylinder.

The angular momentum of a cylinder of radius a rotating as a whole around the z-axis with frequency Ω is

$$\mathbf{L}=\int m^*\rho[\mathbf{r},[\Omega,\mathbf{r}]]dV=\tfrac{1}{2}m^*V\rho a^2\Omega, \tag{6.1}$$

where m^* is the mass of the particles forming the medium and we have $r=\sqrt{x^2+y^2+z^2}$.

The Bose excitations which are generated subject to a $\omega(k)$ dispersion law with a finite momentum $k\neq 0$ may also have a finite angular momentum. The angular momentum conservation law has the form

$$\mathbf{L}=\mathbf{L}_1+\mathbf{L}_{\text{exc}}, \tag{6.2}$$

where $\mathbf{L}$ is the initial angular momentum, $\mathbf{L}_1$ is the finite angular momentum connected with the particles of the medium, and $\mathbf{L}_{\text{exc}}$ is the angular momentum transferred to the Bose excitations. In the case of a cylinder we have

$$\mathbf{L}_1=\tfrac{1}{2}m^*V\rho a^2\Omega_1, \tag{6.3}$$

where Ω_1 is the resulting rotation frequency,

$$\mathbf{L}_{\text{exc}}=\int\left[\mathbf{r},\frac{d\mathbf{P}_{\text{exc}}}{dV}\right]dV, \tag{6.4}$$

and $d\mathbf{P}_{\text{exc}}/dV$ is the density of the total momentum of the Bose subsystem.

To find the quantities $\mathbf{P}_{\text{exc}}$ and $\mathbf{L}_{\text{exc}}$ we must know the structure of the Bose excitation field. A field of the form (3.7) is now unsuitable since for it in the case in question of a rotating medium we have $\bar{\mathbf{P}}_{\text{exc}}=0$ and hence $\bar{\mathbf{L}}_{\text{exc}}=0$, where the bar indicates averaging over the volume. Also unsuitable are cylindrically symmetric solutions of the equation of motion and spherically symmetric solutions of the form

$$\varphi=\varphi_0\exp(ik_0\tilde{r}-i\omega t),\quad \tilde{r}=\sqrt{x^2+y^2};$$

$$\varphi=\varphi_0\exp(ik_0 r-i\omega t),\quad r=\sqrt{x^2+y^2+z^2}. \tag{6.5}$$

In what follows we shall use the variational principle; as a test function we choose a field of the form

$$\varphi=\varphi_0\exp\left[ik_0\tilde{r}\sin\left(\psi-\frac{\alpha\omega t}{k_0\tilde{r}}\right)-i\beta\omega t\right]. \tag{6.6}$$

The variables $\tilde{r}=\sqrt{x^2+y^2}$, ψ, z determine a cylindrical coordinate system; the parameters φ_0, k_0, and also α, β are chosen by minimizing the energy.

Expression (6.6) was chosen so that if we neglect small terms $\sim\mathscr{O}(1/k_0 a)$ we have

$$\Delta\varphi=-k_0^2\varphi \tag{6.7}$$

the same as for the one-dimensional solution (3.7), while the momentum density $d\bar{P}_{\text{exc}}/dV$ and the angular momentum $\bar{L}_{\text{exc}}$ are nonvanishing.

Substituting (6.6) into (3.6) we get for the ψ component of the momentum density of the Bose subsystem

$$\frac{d\mathbf{P}_{\text{exc}}}{dV}=\mathbf{n}_\psi k_0\varphi_0^2\cos\left(\psi-\frac{\alpha\omega t}{k_0\tilde{r}}\right)\times\left\{2\alpha\omega\cos\left(\psi-\frac{\alpha\omega t}{k_0\tilde{r}}\right)+2\beta-\frac{\partial\Pi(\tilde{\omega},k_0)}{\partial\tilde{\omega}}\right\}, \tag{6.8}$$

where $\tilde{\omega}=\alpha\omega\cos(\psi-\alpha\omega t/k_0\tilde{r})+\beta\omega$, and $\mathbf{n}_\psi$ is a unit vector defining the ψ direction, which is conveniently written in the form

$$\mathbf{n}_\psi=\left[\frac{\Omega}{\Omega},\frac{\mathbf{r}}{r}\right]. \tag{6.9}$$

The direction of the vector $\mathbf{L}_0$ is then

$$\left[\frac{r}{r},n_\psi\right]=\frac{\Omega}{\Omega}-\frac{r(\Omega r)}{r^2\Omega}. \tag{6.10}$$

Substituting (6.10) into (6.8) we have from (6.4) in the cylindrical case considered

$$\bar{\mathbf{L}}_{\text{exc}}=\mathbf{n}_z\left(2\omega\alpha-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\bigg|_{\omega(k_0)}\frac{k_0 a}{3}\varphi_0^2 V. \tag{6.11}$$

The bar indicates here as before averaging over the volume or, which is the same, over the angle ψ. To determine the value of the quantity

$$\overline{2\cos\left(\psi-\frac{\alpha\omega t}{k_0\tilde{r}}\right)\frac{\partial\Pi}{\partial\omega}}=\frac{\partial\bar{\Pi}}{\partial\omega}, \tag{6.12}$$

which occurs in Eq. (6.11) we must expand the polarization operator $\Pi(\tilde{\omega},k_0)$ in a series in the parameter $\tilde{\omega}$, to determine the quantity $\partial\Pi/\partial\tilde{\omega}$, carry out the averaging, and after that again form the series.

The excitation spectrum is determined from the equation of motion. After we neglect fast oscillating terms it takes the form

$$\bar{\omega}^2=m^2+k^2+\bar{\Pi}(\tilde{\omega},k^2), \tag{6.13}$$

where we have introduced the notation $\bar{\omega}^2=\frac{1}{2}\alpha^2\omega^2+\beta^2\omega^2$. Equation (6.13) is completely analogous to Eq. (3.8) used earlier. It determines the excitation spectrum $\omega(k)$. In the case of a nonretarded interaction we have $\Pi(\tilde{\omega},k^2)=\Pi(0,k^2)$. In that case the averaging in (6.13) does not change the shape of the spectrum if we put $\alpha=\sqrt{2}$, $\beta=0$. We can proceed also in the case of a quadratic dependence

$$\Pi(\tilde{\omega},k^2)\simeq\Pi(0,k^2)+\frac{1}{2}\frac{d\Pi}{d\tilde{\omega}^2}\tilde{\omega}^2.$$

In the case of a linear dependence,

$$\Pi(\tilde{\omega},k^2)\simeq\Pi(0,k^2)+\frac{d\Pi}{d\tilde{\omega}}\tilde{\omega}+...$$

we must take into account that we have $\beta\neq 0$. In the following calculations for simplicity we put $\beta=0$, $\alpha=\sqrt{2}$ and assume that the function $\tilde{\omega}(k)$ has the form shown in one of the figures 1a or 1b.

If there are no excitations the energy of a rotating liquid is

$$E_{\rm tot}^{(1)}=\frac{L^2}{2I}+E_{\rm med}, \tag{6.14}$$

where $E_{\rm med}$ is the internal energy of the medium in the system rotating together with its center of inertia and I is the moment of inertia. For a cylinder we have

$$I^{\rm cyl}=\frac{m^*\rho a^2}{2}V=\frac{Ma^2}{2}. \tag{6.15}$$

If there are Bose excitations present the energy is

$$E_{\rm tot}^{(2)}=\frac{L_1^2}{2I}+E_{\rm med}+E_{\rm exc}. \tag{6.16}$$

For a field of the form (6.6) we have from (3.5) for $\alpha=\sqrt{2}$, $\beta=0$

$$E_{\rm exc}=\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\omega(k_0)\varphi_0^2V+\frac{\lambda\varphi_0^4}{2}V. \tag{6.17}$$

Assuming the change in the frequency Ω due to the condensation of Bose excitations to be small, using (6.2), (6.11), and (6.17), we get from (6.14) and (6.16)

$$E_{\rm tot}^{(2)}=\frac{L^2}{2I}+E_{\rm med}+\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\times\left[\omega(k_0)-\frac{\sqrt{2}k_0\Omega a}{3}\right]\varphi_0^2V+\frac{\lambda_1\varphi_0^4}{2}V, \tag{6.18}$$

where

$$\lambda_1=\lambda\left[1+\frac{2\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}k_0^2a^2}{9I^{\rm cyl}\lambda}\right]. \tag{6.18a}$$

Minimizing (6.18) with respect to φ_0^2 we have

$$E_{\rm tot}^{(2)}=\frac{L^2}{2I}+E_{\rm med}-\frac{\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2\left[\frac{\sqrt{2}}{3}k_0\Omega a-\omega(k_0)\right]^2}{2\lambda_1}\times V\theta\left[k_0\Omega a-\frac{3}{\sqrt{2}}\omega(k_0)\right]. \tag{6.19}$$

From this we can determine the critical value of the rotational frequency for $\omega(k_0)>0$:

$$\Omega_c^{\rm cyl}=\frac{3\omega(k_0)}{\sqrt{2}k_0a}. \tag{6.20}$$

In the $\omega(k_0)<0$ case we have

$$E_{\rm tot}^{(1)}=\frac{L^2}{2I}+E_{\rm med}+E_{\rm cond}^{(0)}, \tag{6.21}$$

where the quantity $E_{\rm cond}^{(0)}$ is given by (4.17). Here we use the fact that the field φ which minimizes the energy has the form (3.7) in a medium at rest. Assuming that the magnitude of λ is large, i.e., assuming that we can consider the second term in the square brackets in (6.18a) to be a correction, we get from the condition $\delta E<0$

$$\Omega_c^{\rm cyl}=\frac{\sqrt{2}}{3}\frac{|\omega(k_0)|k_0\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{\lambda Ma}V. \tag{6.22}$$

Equations (6.18) to (6.22) are completely analogous to the corresponding equations obtained in Sec. 4 for the case of uniform rectilinear motion. The changes are connected with the more complicated coordinate and momentum dependence of the test function φ. As for the uniform motion one can also show easily that for nonrelativistic rotation ($\Omega a\ll 1$) Eq. (6.19) is valid not only under conditions where we are just above criticality but also for any distance from criticality.

b) Rotation of a spheroid.

We consider a spheroid defined by the equation $r^2/a^2+z^2/b^2=1$. In that case we have instead of (6.3)

$$\mathbf{L}_1=\frac{2}{5}Ma^2\Omega_1=I\Omega_1,\quad M=m^*\rho V. \tag{6.23}$$

For a field of the form (6.6) we get instead of (6.11) ($\alpha=\sqrt{2}$, $\beta=0$):

$$\bar{\mathbf{L}}_{\rm exc}=\frac{3\pi}{32}\mathbf{n}_z\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}k_0a\varphi_0^2V. \tag{6.24}$$

Taking these obvious changes into account we have instead of (6.18) and (6.18a)

$$E_{\rm tot}^{(2)}=\frac{L^2}{2I}+E_{\rm med}+\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\left[\omega(k_0)-\frac{3\pi}{16\sqrt{2}}k_0\Omega a\right]\varphi_0^2V+\frac{\lambda_2\varphi_0^4}{2}V, \tag{6.25}$$

where

$$\lambda_2=\lambda\left\{1+\frac{1}{2}\left(\frac{3\pi}{16}\right)^2\frac{\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2k_0^2a^2V}{I^{\rm sp}\lambda}\right\}. \tag{6.25a}$$

Minimizing (6.25) with respect to φ_0^2 we get

$$E_{\rm tot}^{(2)}=\frac{L^2}{2I}+E_{\rm med}$$

$$-\frac{\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2\left[\frac{3\pi}{16\sqrt{2}}k_0\Omega a-\omega(k_0)\right]^2}{2\lambda_2}$$

$$\times V\theta\left[\frac{3\pi}{16\sqrt{2}}k_0\Omega a-\omega(k_0)\right], \qquad (6.26)$$

and for the critical rotation frequency we have for $\omega(k_0)>0$

$$\Omega_c^{\rm sp}=\frac{16\sqrt{2}}{3\pi}\frac{\omega(k_0)}{k_0 a}. \qquad (6.27)$$

For $\omega(k_0)<0$ and large values of λ when the second term in the square brackets in (6.25a) can be regarded as a correction, we get

$$\Omega_c^{\rm sp}=\frac{5}{4\sqrt{2}}\frac{3\pi}{16}\frac{|\omega(k_0)|k_0\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{\lambda Ma}V. \qquad (6.27a)$$

It is convenient to rewrite Eqs. (6.19) and (6.26) in new variables, using instead of the quantity Ω the conserved angular momentum L. As a result we have for $L>L_c$

$$E_{\rm tot}^{(2)}=\frac{L^2}{2I}-\frac{L^2}{2I^*}+E_{\rm med}+L\Omega^*+E_{\rm cond}, \qquad (6.28)$$

where

$$E_{\rm cond}=-\frac{\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2\omega^2(k_0)}{2\lambda_i}V=-\frac{\Omega^{*2}I^*}{2}, \qquad (6.28a)$$

$$I^*=\frac{\lambda_i I_i^2}{\alpha_i^2 k_0^2 a^2\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2 V}$$

$$\leqslant I,\quad \Omega^*$$

$$=\frac{\alpha_i k_0\omega(k_0)\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2}{\lambda_i I_i}V, \qquad (6.28b)$$

$$\lambda_i=(\lambda_1,\lambda_2),\quad \alpha_i=\left(\frac{\sqrt{2}}{3},\frac{3\pi}{16\sqrt{2}}\right),$$

$$I_i=\beta_i Ma^2,\quad \beta_i=\left(\frac{1}{2},\frac{2}{5}\right)$$

and we have $i=1,2$, respectively, for a cylinder and a spheroid. The initial energy is then

$$E_{\rm tot}^{(1)}=\frac{L^2}{2I}+E_{\rm med}+E_{\rm cond}^{(0)} \qquad (6.29)$$

and the quantity $E_{\rm cond}^{(0)}$ is given by Eq. (4.17) and has the meaning of the energy of the condensate corresponding to a field of the form (3.7) which exists in a medium at rest even before it rotates.

For $\omega(k_0)>0$ the quantity

$$L_c=I^*\Omega^*=\frac{\omega(k_0)\beta_i Ma^2}{\alpha_i k_0}, \qquad (6.30)$$

determined by the equation $\delta E=0$ for a fixed shape of the system has the meaning of the critical angular momentum.

For small values of λ when we can neglect unity in comparison with the second terms in the square brackets of (6.18a) and (6.25a) we can simplify Eqs. (6.28a) and (6.29a). As a result we have

$$I^*\simeq I,\quad \Omega^*\simeq\Omega_c\simeq\frac{\omega(k_0)}{\alpha_i k_0 a},\quad E_{\rm cond}\simeq-\frac{\Omega^{*2}I}{2}. \qquad (6.31)$$

For $\omega(k_0)<0$ we can simplify the expression for the quantity L_c considerably in the limiting case of large λ values used above. Equating (6.28) and (6.29) we have in that case

$$L_c=\frac{\alpha_i^2 I^*|\Omega^*|\left[\left(2\omega-\frac{\partial\bar{\Pi}}{\partial\omega}\right)\Big|_{\omega(k_0)}\right]^2 k_0^2 V}{2\lambda M\beta_i}\simeq\frac{I|\Omega^*|}{2}. \qquad (6.32)$$

7. CHANGE IN SHAPE AND VOLUME OF ROTATING BODIES

Above we assumed that the shape and the volume of the system were fixed and remained unchanged when the condensate appeared. In the general case this is not true. Even before the appearance of a condensate the shape and volume of a drop can change when it rotates. To determine these quantities we must minimize the energy with respect to the characteristic dimensions for a fixed value of L. It is then advantageous for an initially spherical drop to become a rotating spheroid. Moreover, the value of the equilibrium density ρ_m also changes. For a fixed number of particles, A, this means a change in the volume. When the condensate appears the dependence of the energy on the characteristic dimensions and the density changes. This leads to a change in the shape and the volume.

The way the energy of a drop at rest depends on its shape and volume is determined by the equation

$$E_{\rm med}=E_S S+E_V V, \qquad (7.1)$$

where $E_S(\rho)$ is the surface energy density, $E_V(\rho)$ the volume energy density, and S the surface area. In what follows we assume that the density satisfies $\rho\simeq$const. In fact, in the general case the density of the medium may depend on the distance from the axis of rotation. Moreover, we shall assume that the drop has the shape of a rotating spheroid. In the limiting $a\gg b$ case we have $S\simeq 2\pi a^2$ and for $a \ll b$ we have $S \simeq \pi^2 ab$. We restrict our discussion to those two possibilities. Putting $a \gg b$ we have for the energy of the system before condensation

$$E_{\rm tot}^{(1)} = \frac{L^2}{2M\beta_2 a^2} + E_S 2\pi a^2 + \frac{E_V A}{\rho}, \quad \beta_2 = 2/5. \tag{7.2}$$

Minimizing (7.2) with respect to a for fixed volume V (density ρ) we find

$$a^2 = L(4M\beta_2 \pi E_S)^{-1/2}, \tag{7.3}$$

$$E_{\rm tot}^{(1)} = 2L\sqrt{\pi E_S}/\sqrt{M\beta_2} + E_V A/\rho. \tag{7.3a}$$

The condition $a \gg b$ used in deriving (7.3) is satisfied for

$$L \gg (M\beta_2 E_S)^{1/2}(3A/\rho)^{2/3} 2^{-1/3} \pi^{-1/6}. \tag{7.4}$$

For sufficiently large values of L the rotating system thus acquires the shape of a strongly flattened rotational ellipsoid.

When there is an excitation condensate field present we have, using (6.28) and (7.1),

$$E_{\rm tot}^{(2)} = \frac{\gamma_0 L^2}{a^2} + \frac{\gamma_1 L}{a} + E_S S + \tilde{E}_V, \tag{7.5}$$

where the quantities

$$\gamma_0 = \frac{1}{2\beta_2 M} - \frac{a^2}{2I^*} > 0, \quad \gamma_1 = \Omega^* a,$$

$$\tilde{E}_V = E_V A/\rho - \Omega^{*2} I^*/2 \tag{7.5a}$$

are independent of the transverse dimension a; we have $\gamma_1 > 0$ for $\omega(k_0) > 0$ and $\gamma_1 < 0$ for $\omega(k_0) < 0$.

To begin with, let us have $\omega(k_0) > 0$ and $L > L_{\rm cr}$. One can easily minimize expression (7.5) in the limiting case of a small magnitude of λ, i.e., when $\gamma_0 \simeq 0$ [see (6.31)]. Putting $a \gg b$ we have in that case

$$a = \left(\frac{\gamma_1 L}{4\pi E_S}\right)^{1/3}, \tag{7.6}$$

$$E_{\rm tot}^{(2)} \simeq (\gamma_1 L)^{2/3} E_S^{1/3} \frac{3}{2} (4\pi)^{1/3} + \tilde{E}_V. \tag{7.6a}$$

The condition $a \gg b$ which we have used is satisfied for

$$L \gg \frac{3A}{\rho \gamma_1} E_S. \tag{7.7}$$

We now consider the case $\omega(k_0) < 0$. In that case, even when we neglect the contribution E_S, there is a finite value of a which is determined by the minimization of (7.5). Hence we get for $E_S = 0$

$$a = 2\gamma_0 L/\gamma_1, \tag{7.8}$$

$$E_{\rm tot}^{(2)} = -\frac{1}{4}\frac{\gamma_1^2}{\gamma_0} + \tilde{E}_V. \tag{7.8a}$$

The value (7.8) can be either less than or greater than b. For

$$L \gg \frac{\gamma_1}{2\gamma_0}\left(\frac{3A}{4\pi\rho}\right)^{1/3} \tag{7.9}$$

we have $a \gg b$, which corresponds to a strongly flattened spheroid. For large values of λ we have $\gamma_1 \sim 1/\lambda$ and condition (7.9) is satisfied even for not too large values of L.

The expression for the energy was minimized above with respect to the characteristic dimensions for a fixed volume. However, the volume can also change. The expressions obtained must therefore still be minimized with respect to ρ. To carry out this procedure one must know how the various parameters depend on the density ρ. As a result the gain in energy due to a decrease in kinetic energy and a change in the shape and volume of the drop may be sufficient for it to be unstable for a given value of L, or the system may go over into a metastable state even for $L=0$. The observation of such metastable states would undoubtedly be of interest.

8. PHYSICAL CONSEQUENCES

1. In the case of condensed ^{4}He one must look for

a) A modulated density component and b) a streaming effect such that the superfluid and the normal components have different momenta.

One can observe a modulated density component in neutron scattering experiments. The effects considered occur not only for uniform rectilinear flow of ^{4}He in a capillary[1] but also for rotation. In the latter case the ^{4}He pressure is diminished on the walls of the vessel. The effect can also be observed from the change in the moment of inertia. All these effects occur for velocities for which $v > v_c$.

2. We list here some astrophysical consequences.

a) It is well known that when a compact object rotates with a frequency Ω for which $\Omega > \Omega_c^G \sim \sqrt{GM/R^3}$, where G is the gravitational constant, M the mass, and R the radius, matter is stripped from the surface. It is thus commonly assumed that Ω_c^G is the maximum rotational frequency of a star. For a neutron star we have $M \sim M_\odot$, $R \sim 10$ km, and we get $\Omega_c \sim 10^4\ {\rm s}^{-1}$. Rotation with such a frequency corresponds to a critical value of the angular momentum, $L_c^G \sim MR^2\Omega_c$. Since the angular momentum is conserved one usually assumes that a neutron star is formed in the collapse of a star with angular momentum $L < L_c^G$. We have already mentioned above that for $\Omega > \Omega_c$ an excitation condensate can occur which removes part of the angular momentum. Moreover, for $\rho_c > \rho > \rho_c^+$ there is a branch in the π^+ meson spectrum with $\omega(k_0) < 0$ and the excess angular momentum can be transferred to the π^+ condensate. As a result a neutron star can become stable for values of the angular momentum L considerably larger than L_c^G.

b) The energy of a rotating star with a sufficiently small radius R consisting of an ultrarelativistic degenerate perfect Fermi gas is given by the equation

$$E_{\rm tot}^{(1)} = -A_0 \frac{GM^2}{R} + A_1 \frac{M^{4/3}}{R} + A_2 \frac{L^2}{MR^2}, \tag{8.1}$$

where A_0, A_1, and A_2 are numerical coefficients. It is clear from this expression that the rotational energy, which is proportional to L^2/R^2, increases with decreasing radius faster than the gravitational attraction energy. It is thus commonly assumed that the rotation of a compact object prevents its collapse. If we take condensation of excitations into account the last term in (8.1) decreases. In the limiting case when $\lambda\to 0$ we get for $\omega(k_0)>0$ instead of (8.1)

$$E^{(2)}_{\rm tot}=-A_0\frac{GM^2}{R}+A_1\frac{M^{4/3}}{R}+A_3\frac{Lv^*}{R},$$

$$v^*=\omega(k_0)/k_0. \qquad (8.2)$$

Rotation with an angular momentum $L\lesssim GM^2/v^*$ thus does not prevent collapse.

c) A rotating neutron star loses energy and angular momentum through radiation. As a result it slows down. The equation for the time-dependence of the rotational frequency has a different form depending on whether there is a condensate removing part of the angular momentum. The energy is lost mainly through magnetic dipole emission. The rate of energy emission is in this case given by the equation[6]

$$\dot{E}=-A\Omega^4, \qquad (8.3)$$

where we have $A=B_p^2R^6\sin^2\alpha/6c$, B_p is the magnetic field at the magnetic pole of the star, and α is the angle giving the orientation of the magnetic moment relative to the axis of rotation. If there is no condensate the equation determining $\Omega(t)$ has the form

$$I\Omega\dot{\Omega}=-A\Omega^4. \qquad (8.4)$$

For the pulsar in the Crab Nebula the exact age t_0 is known and there are also very exact data allowing us to establish the values of $\Omega(t_0)$, $\dot{\Omega}(t_0)$, and $\ddot{\Omega}(t_0)$. The solution of Eq. (8.4) does not satisfy these data. One can satisfy the experimental values of $\Omega(t_0)$ and $\dot{\Omega}(t_0)$ in a combined model taking into account the possibility of gravitational radiation by a rotating spheroid with a well defined nonvanishing eccentricity ε. However, the value of $n=\Omega\ddot{\Omega}/\dot{\Omega}^2$ is then found to be 3.43 (according to 1972 data) instead of the experimental value $n=2.515\pm0.005$.

Taking condensation into account the equation determining the rate of energy loss according to (6.28) and (6.23) has the form:

$$I_{\rm eff}\Omega\dot{\Omega}+I\Omega^*\dot{\Omega}=-A\Omega^4,\quad I_{\rm eff}=I(1-I/I^*)>0, \qquad (8.5)$$

we have $\Omega^*>0$ when there is a condensate of excitations with $\omega(k_0)>0$ and $\Omega^*<0$ in the case of a π^+ condensation for $\rho_c^+<\rho<\rho_c$ when we have $\omega(k_0)<0$. Introducing the notation

$$\alpha=I_{\rm eff}/(2A),\quad \beta=I\Omega^*/(3A) \qquad (8.6)$$

we easily get from (8.5)

$$\alpha\Omega+\beta=\Omega^3(t+t_1). \qquad (8.7)$$

Hence we can find the function $\Omega(t)$. We can determine the constants α, β, and t_1 from the experimental values of $\Omega(t_0)$, $\dot{\Omega}(t_0)$, and $\ddot{\Omega}(t_0)$. According to (8.7) we have

$$n=\Omega\ddot{\Omega}/\dot{\Omega}^2=4-\frac{2\alpha\Omega}{2\alpha\Omega+3\beta}. \qquad (8.8)$$

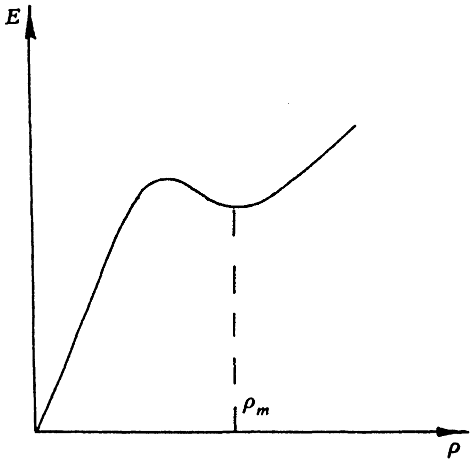

FIG. 2. Sketch of the density dependence of the energy of a rotating nuclear drop with a condensate for large values of the angular momentum L. The magnitude of the density ρ_m corresponds to a possible metastable state.

Using 1972 data [$t_0=921$ yrs, $\Omega(t_0)=1.898\cdot10^2$ s^{-1}, $\Omega(t_0)\dot{\Omega}(t_0)=-4.59\cdot10^{-7}$ s^{-3}, $n(t_0)=2.515$], we find $\alpha\simeq2.04\cdot10^{15}$ s^{-1}, $\beta\simeq-0.84\cdot10^{17}$ s^{-2}, and $t_1\simeq1.53\cdot10^{10}$ s. Since $\beta<0$, we have $\Omega^*<0$, i.e., the experimental data can be explained by saying that part of the angular momentum was transferred to the π^+ condensate.

d) The π^+ condensate subsystem is superfluid. It therefore interacts very weakly with the normal subsystem. When the period of the pulsar is disrupted the frequency of rotation of the normal subsystem changes. It takes a long time to equalize the frequencies between the normal and the condensate subsystems. The disruptions of periods in the sufficiently fast rotating pulsars (for $\Omega>\Omega_c$) may be connected with the accumulation of vortices in the condensate region and the subsequent release of the stored energy. This mechanism for disruptions of pulsar periods and their subsequent relaxation is completely analogous to the standard mechanism which is commonly assumed to be connected with *nn* and *pp* pairing and superfluidity.

3. When determining the conditions for the stability of a nuclear system with a large angular momentum L one must also take into account the possibility of condensation of excitations with $\omega(k_0)>0$ or a π^+ condensation with $\omega(k_0)<0$. In those cases for some density ρ_m a new metastable state may appear which is stable against small changes in density. We sketch in Fig. 2 the behavior of the total energy $E(\rho)$ corresponding to the possibility of such a metastable state. To find the values of ρ_m and $E(\rho_m)$ we must minimize the energy with respect to the density, the isotopic composition, and the characteristic dimensions. For $\omega(k_0)>0$ one can use Eq. (7.6a), if one assumes a strongly flattened spheroid, and Eq. (7.9a) for $\omega(k_0)<0$. One must then take into account the explicit density dependence of all the parameters k_0, $\omega(k_0)$, $\partial\bar{\Pi}/\partial\omega$, E_S, E_V, and λ.

a) A metastable state may occur for a nuclear drop with the number of neutrons approximately equal to the number of protons. In that case it corresponds to the presence of low-lying excitations of the zero-sound type, for instance, spin-isospin sound with a frequency $\omega \simeq k_0 v_F$ (see Fig. 1b).

b) Metastable states are possible of neutron matter with a large angular momentum and a density $\rho \sim \rho_c^+ < \rho_0$ in which the π^+ condensate subsystem ($\omega(k_0) < 0$) accounts for part of the angular momentum. Such neutron drops have not too large a size, but larger than the coherence length, since in the opposite case the energy of the electric field is essentially lost. To balance it in a large system ($Z > 1/e^3$) electrons are generated which carry a large kinetic energy.[7,5]

Note that the lifetime of such neutron drops may be very long. This is connected with the fact that a π^+ condensate with $\omega(k_0) < 0$ is superfluid and the interaction between the π^+ condensate and the normal subsystems is difficult. Moreover, the π^+ condensate in a finite system, even in the ground state, possesses a finite angular momentum.

Drops of π^+ condensate may be formed during the collapse of a supernova when it changes to a neutron star. In that case they take away part of the total angular momentum of the star which, as we mentioned already earlier, might in the standard scenario prevent the collapse.

The presence of metastable π^+ condensate neutron drops may be observed also during collisions of high-energy nuclei.

Some time ago there was a broad discussion in the literature of the problem of the so-called anomalons.[8] In nucleus-nucleus collisions with energies $\gtrsim$GeV/nucleon specific fragments of nucleus projectiles were observed flying off at small angles and possessing anomalously large cross-sections for interaction with matter. It turned out that these fragments made up several per cent. They possess a very long lifetime. A memory effect was also observed—after interacting an anormalon decayed into anormalons.

From those experiments it was concluded that one is dealing with stable or metastable nuclear systems which for a given atomic weight have an anomalously large size and are generated in the peripheries of the colliding nuclei with energies larger than a well defined critical value ($\sim$GeV/nucleon).

Notwithstanding a multitude of models which were proposed to explain the anomalon effect, up to now there does not exist any convincing theoretical explanation of this effect. The statistics of the experiments are not very good and they also do not make it possible to reach unambiguous conclusions.[8]

We note here that a π^+ condensate neutron drop may serve as a good candidate to explain the anomalon effect.

Firstly, it is just in a peripheral collision of nuclei that a fragment with large angular momentum may appear. A rough estimate of the maximum possible angular momentum gives

$$L \sim \frac{m_N A v}{\sqrt{1-v^2/c^2}} \sim E_{\rm kin} \frac{Av}{c^2} R, \tag{8.9}$$

where $E_{\rm kin}$ is the collision energy of the nuclei in the *laboratory* system, A is the atomic number of the colliding nuclei, and R is a characteristic dimension. This value of L is larger than the magnitude L_c starting from some critical value of the collision energy of the nuclei (for nuclei of a given atomic weight).

Secondly, since there is a metastable state for a neutron drop with a π^+ condensate one can explain the long lifetime of an anomalon and the memory effect.

Thirdly, since for the equilibrium density value we have $\rho_m \sim \rho_c^+ < \rho_0$ while the shape of the drop probably corresponds to a strongly flattened spheroid, its cross-section for interactions with matter will exceed considerably the cross-section of a normal fragment with the same atomic weight.

Finally, since the metastable π^+ condensate state occurs through a phase transition and on average the value of the angular momentum of fragments is small, the probability for the appearance of such anomalous fragments is rather small.

4). An excitation condensate with $\omega = k_0 v_F[\rho(t)]$ may appear also in the central region of collisions of nuclei. One usually assumes that a nuclear fireball is formed in a nucleus-nucleus collision which afterwards expands into vacuum. During the expansion of the fireball the characteristic velocity of the nucleons increases and for $v > v_F[\rho(t)] \simeq \frac{1}{3}(\rho/\rho_0)^{1/3}$ an excitation condensate may occur with $\omega \simeq k_0 v_F$. According to the model of Refs. 4 and 9 the minimum density of a nuclear fireball in the stage where it breaks up is $\rho_b \simeq 0.6\rho_0$ and a velocity v larger than v_c can be reached for nucleon energies $\geqslant$0.4 GeV/nucleon. In the breakup of a nuclear fireball the excess momentum of the excitation condensate is transferred back to the nucleons thereby effectively heating the nucleon subsystem. This fact may, even if only partially, explain that the effective temperature of the nucleons (experimentally determined from the slope of the invariant differential cross-section) is slightly higher than the corresponding temperature of the pions.

9. CONCLUDING REMARKS

As we have already noted earlier the essence of the effect considered here is that if excitations can be generated with a sufficiently large momentum and a sufficiently small energy (which is determined from the dispersion law $\omega(k)$ and the velocity of the medium) the particles of the moving medium transfer part of their momentum to the excitations, thereby lowering the energy of the system on the whole. Since this effect is quite general many of the results obtained are model-independent, although they were obtained above for the specific model of a two-component scalar field. We have formulated some physical consequences of the effect considered in actual systems which might make it possible to observe it experimentally. A minute description of the details of the effect in actual systems was not the object of the present paper; the effect is of interest in itself.

1) At the same time the characteristic size of the system must be larger than the coherence length, i.e., the characteristic length over which the amplitude of the condensate field changes, since otherwise the concept of a "quasicondensate" loses its meaning.

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