Transport of energy, momentum, and orbital and intrinsic angular momentum of an electromagnetic wave in dispersive media

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To lowest order in λ/L and Ω/ω, equations are found for the transport of energy, momentum, and angular momentum of an electromagnetic wave propagating in a weakly absorbing uniform time-independent anisotropic and gyrotropic medium with spatial and temporal dispersion (λ and ω are the wavelength and frequency and L and $1/\Omega$ are the characteristic length and time scales on which the wave amplitude varies). It is shown that the intrinsic angular momentum (spin) of the wave is conserved only for transverse waves with circular polarization. Expressions are found for the spin density, its flux, and the power per unit volume of the losses.

INTRODUCTION

A packet of almost monochromatic electromagnetic waves propagating in a continuous medium with temporal and spatial dispersion is characterized by the density of energy, momentum, and angular momentum, and by the flux densities of these quantities. While the question regarding the transport of energy[1–5] and momentum[6–8] of the waves can be regarded as settled, the transport of angular momentum in such media has not been completely investigated theoretically. For example, there is not even an expression for the intrinsic angular momentum density and its flux density in a medium.

A. I. Sadovskii[9,10] predicted the existence of intrinsic angular momentum (spin) in an electromagnetic wave. He showed that when a wave is absorbed or its polarization changes as it passes through material a rotational angular momentum develops. This effect has been observed in the optical[11–13] and centimeter[14,15] bands. An expression for the spin density of an electromagnetic wave in a dielectric without dispersion was reported by Borgardt.[16] The status of the spin of electromagnetic waves has been discussed in review articles.[17,18] This topic has also been studied in connection with plasma physics problems.[18–20]

In the present work we consider the transport of energy, momentum, and angular momentum by an electromagnetic wave propagating in a weakly absorbent uniform time-independent anisotropic and gyrotropic medium with temporal and spatial dispersion. In order to derive expressions for the density of the intrinsic angular momentum and its flux density, as we shall see (see also Ref. 17), it is necessary to find the momentum and angular momentum transport equations up to and including terms of order $(\lambda/L)^2$, $(\Omega/\omega)^2$, and $\lambda\Omega/L\omega$, where ω and λ are the frequency and wavelength and L and $1/\Omega$ are the characteristic distance and time scales on which the wave amplitude changes. Here it is convenient in using the Maxwell equations to find an expression for the density of the free electric currents in terms of the strength of the wave electric field, retaining terms to second order in λ/L and Ω/ω (Sec. 1). The wave energy and momentum conservation laws are found to second order in Secs. 2 and 3 (the principal terms in these equations are of first order in λ/L and Ω/ω). In Sec. 4 expressions for the variation of the angular momentum density of the free currents is used to derive the transport equation for quantities which are quadratic in the complex amplitudes of the wave fields, which reduces to the transport equation for the orbital wave angular momentum [given to first (lowest) order in λ/L and Ω/ω] and to the second-order equation. The law of conservation of intrinsic angular momentum in the direction of wave propagation follows from the latter for waves with circular polarization. Expressions are found for the intrinsic angular momentum density and the density of its flux for the propagation of circularly polarized transverse waves in an isotropic gyrotropic dielectric with temporal and spatial dispersion and in a magnetized plasma parallel to the magnetic field.

1. BASIC EQUATIONS

A macroscopic electromagnetic field in matter acts on a charge q moving with velocity $\mathbf{v}$ with a force

$$\mathbf{F}=q\left(\mathbf{E}+\frac{1}{c}\,\mathbf{vB}\right). \tag{1}$$

This relation determines the strength of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in the medium. They satisfy the Maxwell equations

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \operatorname{div}\mathbf{D}=4\pi\rho_0, \tag{2a}$$

$$\operatorname{rot}\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}\,\mathbf{j}_0,\quad \operatorname{div}\mathbf{B}=0, \tag{2b}$$

where ρ_0 and $\mathbf{j}_0$ are the densities of the free charges and currents in the medium.

In the linear approximation the displacement $\mathbf{D}$ is related to $\mathbf{E}$ by

$$D_i(t,\mathbf{r})=\hat{\varepsilon}_{ij}E_j(t,\mathbf{r}). \tag{3}$$

For a plane monochromatic wave we have

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$$\hat{\varepsilon}_{ij}\mathbf{A}_0 \exp(-i\omega t + i\mathbf{kr}) = \varepsilon_{ij}(\omega,\mathbf{k})\mathbf{A}_0 \exp(-i\omega t + i\mathbf{kr}), \quad (4)$$

where $\mathbf{A}_0$ is the time-independent amplitude and $\varepsilon_{ij}(\omega,\mathbf{k})$ is the dielectric tensor of the medium (we assume that the magnetic permeability is equal to unity).

To derive transport equations for the energy, momentum, and angular momentum of an electromagnetic wave propagating in the medium we will start with the familiar variations of these quantities for free charges and currents acted on by an electromagnetic field.

The change per unit time of the kinetic energy of the free charges in an electromagnetic field is determined by

$$\frac{d}{dt}\sum_a \varepsilon_a = \sum_a q_a \mathbf{v}_a \mathbf{E}(t,\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_a(t)}, \quad (5)$$

where the summation is performed over all free charges.

If we introduce the distribution of the free charge density,

$$\rho_0(t,\mathbf{r}) = \sum_a q_a \delta[\mathbf{r}-\mathbf{r}_a(t)], \quad (6)$$

then (5) can be rewritten in the form

$$\frac{d}{dt}\sum_a \varepsilon_a - \int \mathbf{j}_0\mathbf{E}\,dV = 0, \quad (7)$$

where $\mathbf{j}_0 = \rho_0\mathbf{v}$ is the density of the free currents.

By virtue of the Maxwell equations (2) we can replace the current density $\mathbf{j}_0(t,\mathbf{r})$ of the free particles in (7) with the field properties, thereby obtaining a transport equation for the electromagnetic field energy.

Using the equations of motion for the free charges in the medium when an electromagnetic field is present, we find

$$\frac{d}{dt}\sum_a \mathbf{p}_a = \sum_a \left\{ q_a \mathbf{E}(t,\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_a(t)} + \frac{1}{c} q_a \mathbf{v}_a \times \mathbf{B}(t,\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_a(t)} \right\},$$

or

$$\frac{d}{dt}\sum_a \mathbf{p}_a - \int \left\{ \rho_0\mathbf{E} + \frac{1}{c}\mathbf{j}_0\mathbf{B} \right\} dV = 0. \quad (8)$$

After we make use of Eq. (2) to replace the quantities ρ_0 and $\mathbf{j}_0$ with field quantities, the second term in (8) determines the equation for transport of the electromagnetic field momentum.

Similarly, we find an expression which describes the transport of the angular momentum of the free charges and the field,

$$\frac{d}{dt}\sum_a \mathbf{M}_a = \sum_a \mathbf{r}_a \frac{d\mathbf{p}_a}{dt} = \sum_a \mathbf{r}_a \left\{ q_a \mathbf{E}(t,\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_a(t)} + \frac{1}{c} q_a \mathbf{v}_a \times \mathbf{B}(t,\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_a(t)} \right\}, \quad (9)$$

from which

$$\frac{d}{dt}\sum_a \mathbf{M}_a - \int \mathbf{r}\left\{ \rho_0\mathbf{E} + \frac{1}{c}\mathbf{j}_0\mathbf{B} \right\} dV = 0. \quad (10)$$

For an arbitrary electromagnetic field in vacuum, substituting for ρ_0 and $\mathbf{j}_0$ from (2) in (10), we find by virtue of the symmetry of the Maxwell stress tensor the law for conservation of angular momentum of a closed system consisting of fields and free particles.

From (7), (8), and (10) we see that to derive equations for the transport of energy, momentum, and angular momentum of an electromagnetic wave we need to express $\mathbf{j}_0(t,\mathbf{r})$ and $\rho_0(t,\mathbf{r})$ in terms of the wave electric field using Eq. (2).

When the wave is weakly damped, $|\varepsilon_{ij}^A| \ll |\varepsilon_{ij}^H|$ [here ε_{ij}^A and ε_{ij}^H are the Hermitian and anti-Hermitian parts of $\varepsilon_{ij}(\omega,\mathbf{k})$], the damping rate is proportional to ε_{ij}^A and the amplitude of the electromagnetic wave is a slowly varying function of position and time in comparison with the phase $\omega t - \mathbf{kr}$. We write the electric field of the wave in the form

$$\mathbf{E}(t,\mathbf{r}) = \mathbf{E}(t,\mathbf{r};\omega,\mathbf{k})\exp(-i\omega t + i\mathbf{kr}). \quad (11)$$

The field (11) is represented as a superposition of monochromatic fields with frequencies and wave vectors close to ω and $\mathbf{k}$ respectively (a wave packet). Using the familiar technique[21] we find

$$D_i = \hat{\varepsilon}_{ij}E_j(t,\mathbf{r};\omega,\mathbf{k}) = \exp(-i\omega t + i\mathbf{kr}) \times \left\{ \varepsilon_{ij}(\omega,\mathbf{k})E_j(t,\mathbf{r};\omega,\mathbf{k}) + i\frac{\partial\varepsilon_{ij}}{\partial\omega}\frac{\partial E_j}{\partial t} - i\frac{\partial\varepsilon_{ij}}{\partial k_m}\frac{\partial E_j}{\partial x_m} - \frac{1}{2}\frac{\partial^2\varepsilon_{ij}}{\partial\omega^2}\frac{\partial^2 E_j}{\partial t^2} + \frac{\partial^2\varepsilon_{ij}}{\partial\omega\partial k_m}\frac{\partial^2 E_j}{\partial t\partial x_m} - \frac{1}{2}\frac{\partial^2\varepsilon_{ij}}{\partial k_m\partial k_l}\frac{\partial^2 E_j}{\partial x_m\partial x_l} + \dots \right\}. \quad (12)$$

To the same accuracy, retaining terms with derivatives of the slowly varying electric field amplitude through second order [i.e., through terms proportional to $(\lambda/L)^2$, $(\Omega/\omega)^2$, and $\lambda\Omega/L\omega$], we find from Eq. (2a)

$$\mathbf{B}(t,\mathbf{r}) = \exp[i(\mathbf{kr}-\omega t)]\,\frac{c}{\omega}\left\{ \left(1 - \frac{i}{\omega}\frac{\partial}{\partial t} - \frac{1}{\omega^2}\frac{\partial^2}{\partial t^2}\right)(\mathbf{kE}) - i\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t}\right)\nabla\mathbf{E} \right\}. \quad (13)$$

Substituting (12) and (13) in (2b) we have after some simple transformations

$$-4\pi j_{0i} = \exp(-i\omega t + i\mathbf{kr})\left\{ -i\omega M_{ij}E_j + \frac{\partial\omega M_{ij}}{\partial\omega}\frac{\partial E_j}{\partial t} - \frac{\partial\omega M_{ij}}{\partial k_m}\frac{\partial E_j}{\partial x_m} + \frac{i}{2}\frac{\partial^2\omega M_{ij}}{\partial\omega^2}\frac{\partial^2 E_j}{\partial t^2} - i\frac{\partial^2\omega M_{ij}}{\partial\omega\partial k_m}\frac{\partial^2 E_j}{\partial t\partial x_m} + \frac{i}{2}\frac{\partial^2\omega M_{ij}}{\partial k_m\partial k_l}\frac{\partial^2 E_j}{\partial x_m\partial x_l} \right\}, \quad (14)$$

where

$$M_{ij}(\omega,\mathbf{k})=\varepsilon_{ij}(\omega,\mathbf{k})-N^2(\delta_{ij}-\varkappa_i\varkappa_j), \tag{15}$$

$$N=\frac{ck}{\omega}, \quad \varkappa=\frac{\mathbf{k}}{k} \tag{16}$$

(here M_{ij} is the Maxwell tensor and N is the index of refraction).

Equation (14), which follows from the Maxwell equations, relates the density of the free currents to the electric field strength of the waves to second order in λ/L and Ω/ω. This equation is the basis for deriving the transport equations.

Note that the expansion (15) can be derived formally to any order, in analogy with (12), if we write (2b) in the form

$$-4\pi j_{0i}=\hat{f}_{ij}E_j(t,\mathbf{r}), \quad \hat{f}_{ij}=\frac{\partial}{\partial t}\hat{M}_{ij}, \tag{17}$$

where for an amplitude $\mathbf{A}_0$ which is independent of space and time we have

$$\hat{f}_{ij}\mathbf{A}_0\exp(-i\omega t+i\mathbf{kr})=f_{ij}(\omega,\mathbf{k})\mathbf{A}_0\exp(-i\omega t+i\mathbf{kr}), \tag{18}$$
$$f_{ij}(\omega,\mathbf{k})=-i\omega M_{ij}(\omega,\mathbf{k}).$$

2. THE ENERGY TRANSPORT EQUATION

Substituting (14) in (7), after averaging over the wave period $T=2\pi/\omega$ we find the energy conservation law in differential form to second order in λ/L and Ω/ω:

$$\frac{\partial w}{\partial t}+\operatorname{div}\mathbf{S}+q=-\overline{\mathbf{j}_0\mathbf{E}}, \tag{19}$$

where

$$w=\frac{1}{16\pi}\left\{\frac{\partial\omega M_{ij}^H}{\partial\omega}E_i^*E_j+\frac{i}{2}\frac{\partial^2\omega M_{ij}^H}{\partial\omega^2}\left(E_i^*\frac{\partial E_j}{\partial t}-E_j\frac{\partial E_i^*}{\partial t}\right)-\frac{i}{2}\frac{\partial^2\omega M_{ij}^H}{\partial\omega\partial k_m}\left(E_i^*\frac{\partial E_j}{\partial x_m}-E_j\frac{\partial E_i^*}{\partial x_m}\right)\right\},$$

$$S_m=-\frac{1}{16\pi}\left\{\frac{\partial\omega M_{ij}^H}{\partial k_m}E_i^*E_j+\frac{i}{2}\frac{\partial^2\omega M_{ij}^H}{\partial\omega\partial k_m}\left(E_i^*\frac{\partial E_j}{\partial t}-E_j\frac{\partial E_i^*}{\partial t}\right)-\frac{i}{2}\frac{\partial^2\omega M_{ij}^H}{\partial k_m\partial k_l}\left(E_i^*\frac{\partial E_j}{\partial x_l}-E_j\frac{\partial E_i^*}{\partial x_l}\right)\right\},$$

$$q=-\frac{1}{16\pi}\left\{2i\omega M_{ij}^AE_i^*E_j-\frac{\partial\omega M_{ij}^A}{\partial\omega}\left(E_i^*\frac{\partial E_j}{\partial t}-E_j\frac{\partial E_i^*}{\partial t}\right)+\frac{\partial\omega M_{ij}^A}{\partial k_m}\left(E_i^*\frac{\partial E_j}{\partial x_m}-E_j\frac{\partial E_i^*}{\partial x_m}\right)\right\}. \tag{20}$$

Here w, $\mathbf{S}$, and q are the energy density, energy flux density, and the rate of energy losses per unit volume respectively, averaged over the wave period.

We can write down the energy conservation law (19) for any eigenwave ($\mathbf{j}_0=0$). In this case the wave amplitude $\mathbf{E}(t,\mathbf{r};\omega,\mathbf{k})$ can be written as a power series in λ/L and Ω/ω (Ref. 22)

$$\mathbf{E}=\mathbf{E}^0+i\mathbf{E}^1+\mathbf{E}^2+..., \tag{21}$$

where the terms are given by a hierarchy of equations which follow from (14) in the limit $|M_{ij}^A|\ll|M_{ij}^H|$ and $\mathbf{j}_0=0$ (we restrict ourselves to writing down the first three equations of this hierarchy):

$$-i\omega M_{ij}^HE_j^0=0, \tag{22a}$$

$$\omega M_{ij}^HE_j^1-i\omega M_{ij}^AE_j^0+\frac{\partial\omega M_{ij}^H}{\partial\omega}\frac{\partial E_j^0}{\partial t}-\frac{\partial\omega M_{ij}^H}{\partial k_m}\frac{\partial E_j^0}{\partial x_m}=0, \tag{22b}$$

$$-i\omega M_{ij}^HE_j^2+\omega M_{ij}^AE_j^1+i\frac{\partial\omega M_{ij}^H}{\partial\omega}\frac{\partial E_j^1}{\partial t}-i\frac{\partial\omega M_{ij}^H}{\partial k_m}\frac{\partial E_j^1}{\partial x_m}+\frac{\partial\omega M_{ij}^A}{\partial\omega}\frac{\partial E_j^0}{\partial t}-\frac{\partial\omega M_{ij}^A}{\partial k_m}\frac{\partial E_j^0}{\partial x_m}+\frac{i}{2}\left[\frac{\partial^2\omega M_{ij}^H}{\partial\omega^2}\frac{\partial^2E_j^0}{\partial t^2}-2\frac{\partial^2\omega M_{ij}^H}{\partial\omega\partial k_m}\frac{\partial^2E_j^0}{\partial t\partial x_m}+\frac{\partial^2\omega M_{ij}^H}{\partial k_m\partial k_l}\frac{\partial^2E_j^0}{\partial x_m\partial x_l}\right]=0. \tag{22c}$$

From Eq. (22a) we find the dispersion relation for the eigenmodes

$$M^H(\omega,\mathbf{k})=0 \quad (M^H\equiv\det M_{ij}^H) \tag{23}$$

and the polarization vector $\mathbf{e}$ for an arbitrary eigenmode,

$$\mathbf{E}^0=\mathbf{e}(\omega,\mathbf{k})E_0(\omega,\mathbf{k};t,\mathbf{r}), \quad e_i=\frac{e_{ix}}{\sqrt{e_{xx}\operatorname{Sp}e}}, \tag{24}$$

where

$$|\mathbf{e}|=1, \quad e_{ij}\equiv M^H(M_{ij}^H)^{-1}, \quad M_{ij}^{-1}M_{jn}=\delta_{in}.$$

Here we have made use of the relation (cf. Refs. 23 and 24 for $M^H=0$)

$$e_{ij}e_{mn}=e_{in}e_{mj}. \tag{25}$$

In evaluating the components of the polarization vector $\mathbf{e}$ it is convenient to use the formulas [cf. Eqs. (24) and (25)]

$$e_i=\frac{e_{ij}}{\sqrt{e_{jj}\operatorname{Sp}e}}=\frac{e_{ij}}{|e_{ij}|}\sqrt{\frac{e_{ii}}{\operatorname{Sp}e}}. \tag{26}$$

In (26) there is no summation over the subscripts i and j, each of which can assume one of the three values x, y, or z; for example, for $j=x$ we have

$$e_x=\frac{e_{xx}}{|e_{xx}|}\sqrt{\frac{e_{xx}}{\operatorname{Sp}e}}, \quad e_y=\frac{e_{yx}}{|e_{yx}|}\sqrt{\frac{e_{yy}}{\operatorname{Sp}e}}, \quad e_z=\frac{e_{zx}}{|e_{zx}|}\sqrt{\frac{e_{zz}}{\operatorname{Sp}e}}. \tag{27}$$

From (24) and (25) it follows that

$$e_i^*e_j=\frac{e_{ix}^*e_{jx}}{e_{xx}\operatorname{Sp}e}=\frac{e_{xi}e_{jx}}{e_{xx}\operatorname{Sp}e}=\frac{e_{ji}e_{xx}}{e_{xx}\operatorname{Sp}e}=\frac{e_{ji}}{\operatorname{Sp}e}. \tag{28}$$

Retaining only the lowest-order term of the expansion of $\mathbf{E}^0$ in Eq. (20) we find from (21) in the form (24), using (28),

$$w_0=\frac{1}{16\pi}\frac{\partial\omega M_{ij}^H}{\partial\omega}E_i^{0*}E_j^0$$

$$=\frac{\omega}{16\pi}|\mathbf{E}_0|^2\frac{\partial M_{ij}^H}{\partial\omega}\frac{e_{ji}}{\mathrm{Sp}\,e}$$

$$=\frac{\omega}{\mathrm{Sp}\,e}\frac{|\mathbf{E}^0|^2}{16\pi}M^H(M_{ji}^H)^{-1}\frac{\partial M_{ij}^H}{\partial\omega}$$

$$=\frac{\omega}{\mathrm{Sp}\,e}\frac{\partial M^H(\omega,\mathbf{k})}{\partial\omega}\frac{|\mathbf{E}^0|^2}{16\pi}. \qquad (29)$$

The expression for $\mathbf{S}_0$ is found from (20) in a similar fashion:

$$\mathbf{S}_0=-\frac{1}{16\pi}\frac{\partial\omega M_{ij}^H}{\partial\mathbf{k}}E_i^{0*}E_j^0=\mathbf{v}_g w_0, \qquad (30)$$

where

$$\mathbf{v}_g=-\frac{\partial M^H/\partial\mathbf{k}}{\partial M^H/\partial\omega}. \qquad (31)$$

Finally, for the quantity q_0 we have

$$q_0=-\frac{i\omega}{8\pi}\varepsilon_{ij}^A e_i^* e_j|E_0|^2=2\gamma_0 w_0, \qquad (32)$$

where

$$\gamma_0=-i\frac{\varepsilon_{ij}^A e_{ji}}{\partial M^H/\partial\omega}. \qquad (33)$$

We recall that it is necessary to substitute $\omega=\omega(\mathbf{k})$ in the right-hand sides of Eqs. (29)–(33). Here $\omega(\mathbf{k})$ is the eigenfrequency which satisfies the dispersion relation (23). Thus, for an eigenmode the energy conservation law (19) in lowest order assumes the form

$$\frac{\partial w_0}{\partial t}+\mathrm{div}\,\mathbf{S}_0+q_0=0. \qquad (34)$$

In contrast to the usual[1-5] expressions used for w_0, the expression (29) derived here has a simpler form, since it does not explicitly contain the wave polarization vector ($|\mathbf{E}_0|^2=|E_0|^2$). Equation (30) is the most direct technique for deriving a relation between the energy flux density and the energy density for an arbitrary eigenmode. Note that this relation is valid only in lowest order and becomes meaningless when we take into account subsequent terms in the expansion of the field amplitude in powers of λ/L and Ω/ω [compare Eqs. (19) and (20)]. From (29) it follows that when the condition

$$\frac{\omega}{\mathrm{Sp}\,e}\frac{\partial M^H(\omega,\mathbf{k})}{\partial\omega}>0 \qquad (35)$$

holds the wave energy density is positive. This condition generalizes the familiar conditions for waves in an isotropic plasma[25] and for irrotational plasma oscillations in a magnetic field[26] to the case of an arbitrary dispersive medium.

Equation (34) describes the variation of the absolute value $|E_0(\omega,\mathbf{k};t,\mathbf{r})|$ of the wave amplitude as a function of time and position. We define the phase $\alpha_0(\omega,\mathbf{k};t,\mathbf{r})$ of the wave amplitude by the relation

$$E_0(\omega,\mathbf{k};t,\mathbf{r})=|E_0(\omega,\mathbf{k};t,\mathbf{r})|\exp\{i\alpha_0(\omega,\mathbf{k};t,\mathbf{r})\}. \qquad (36)$$

The change of this phase as a function of time and space is determined by the imaginary part of the equation obtained by multiplying (22b) by E_i^0,

$$\frac{\partial\alpha_0}{\partial t}+\mathrm{div}(\mathbf{v}_g\alpha_0)=0, \qquad (37)$$

i.e., the phase α_0 varies only in the direction of the group velocity.[22]

3. MOMENTUM TRANSPORT EQUATION

From the continuity equation for the free charge,

$$\frac{\partial\rho_0}{\partial t}+\mathrm{div}\,\mathbf{j}_0=0, \qquad (38)$$

we can express ρ_0 in terms of $\mathbf{j}_0$ to second order in λ/L and Ω/ω:

$$\rho_0(t,\mathbf{r})=\exp-i\omega t+i\mathbf{kr})\left\{\frac{\mathbf{k}}{\omega}\left(1-\frac{i}{\omega}\frac{\partial}{\partial t}-\frac{1}{\omega^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{j}_0\right.$$
$$\left.-\frac{i}{\omega}\left(1-\frac{i}{\omega}\frac{\partial}{\partial t}\right)\nabla\mathbf{j}_0\right\}. \qquad (39)$$

Using (39) and (13) we have

$$\left\{\overline{\rho_0\mathbf{E}}+\frac{1}{c}\overline{\mathbf{j}_0\mathbf{B}}\right\}_n=\frac{1}{2}\mathrm{Re}\left\{\frac{k_n}{\omega}j_{0i}\left(1+\frac{i}{\omega}\frac{\partial}{\partial t}-\frac{1}{\omega^2}\frac{\partial^2}{\partial t^2}\right)E_i^*\right.$$
$$-\frac{i}{\omega^2}\left[\frac{\partial}{\partial t}(k_i j_{0i}E_n^*)+\frac{i}{\omega}k_i j_{0i}\frac{\partial^2 E_n^*}{\partial t^2}\right.$$
$$\left.-\frac{i}{\omega}E_n^*\frac{\partial^2}{\partial t^2}(k_i j_{0i})\right]$$
$$+\frac{i}{\omega}j_{0i}\left(1+\frac{i}{\omega}\frac{\partial}{\partial t}\right)\frac{\partial E_i^*}{\partial x_n}-\frac{i}{\omega}\frac{\partial}{\partial x_i}(j_{0i}E_n^*)$$
$$\left.+\frac{1}{\omega^2}\left(j_{0i}\frac{\partial^2 E_n^*}{\partial t\partial x_i}-E_n^*\frac{\partial^2 j_{0i}}{\partial t\partial x_i}\right)\right\}. \qquad (40)$$

If we substitute expression (14) in (40) for the free current j_{0i}, then we obtain the wave momentum transport equation to second order in the variation of the amplitude of the electromagnetic field. This transport equation has the form of a conservation law,

$$\frac{\partial p_n}{\partial t}+\frac{\partial}{\partial x_m}S_{nm}+q_n=-\left(\overline{\rho_0\mathbf{E}}+\frac{1}{c}\overline{\mathbf{j}_0\mathbf{B}}\right)_n. \qquad (41)$$

Here p_n is the electromagnetic wave momentum density averaged over a wave period $2\pi/\omega$,

$$
\begin{aligned}
p_n=\frac{k_n}{\omega}\Big\{ w+\frac{1}{16\pi}\Big[& M_{ij}^H E_i^* E_j-\frac{i}{\omega} M_{ij}^H\Big(E_i^* \frac{\partial E_j}{\partial t} \\
& -E_j \frac{\partial E_i^*}{\partial t}\Big)-\frac{i}{2\omega}\frac{\partial \omega M_{ij}^H}{\partial k_m}\Big(E_i^* \frac{\partial E_j}{\partial x_m}-E_j \frac{\partial E_i^*}{\partial x_m}\Big)\Big]\Big\} \\
& -\frac{1}{16\pi}\Big\{\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial \omega}\Big(\frac{k_i}{\omega}\frac{\partial E_j}{\partial t} E_n^*-\frac{k_j}{\omega}\frac{\partial E_i^*}{\partial t} E_n\Big) \\
& -\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial k_m}\Big(\frac{k_i}{\omega}\frac{\partial E_j}{\partial x_m} E_n^*-\frac{k_j}{\omega}\frac{\partial E_i^*}{\partial x_m} E_n\Big) \\
& +M_{ij}^H\Big(\frac{k_i}{\omega} E_j E_n^*+\frac{k_j}{\omega} E_i^* E_n\Big)+M_{ij}^A\Big(\frac{k_i}{\omega} E_j E_n^* \\
& -\frac{k_j}{\omega} E_i^* E_n\Big)+\frac{i}{\omega} M_{ij}^H\Big[\frac{k_i}{\omega}\Big(E_j \frac{\partial E_n^*}{\partial t}-E_n^* \frac{\partial E_j}{\partial t}\Big) \\
& -\frac{k_j}{\omega}\Big(E_i^* \frac{\partial E_n}{\partial t}-E_n \frac{\partial E_i^*}{\partial t}\Big)\Big]\Big\} \\
& -\frac{1}{32\pi}\frac{i}{\omega}\Big\{\frac{\partial \omega M_{ij}^H}{\partial \omega}\Big(E_i^* \frac{\partial E_j}{\partial x_n}-E_j \frac{\partial E_i^*}{\partial x_n}\Big) \\
& +M_{ij}^H\Big(E_i^* \frac{\partial E_j}{\partial x_n}-E_j \frac{\partial E_i^*}{\partial x_n}\Big)+M_{ij}^H\Big[E_j \frac{\partial E_n^*}{\partial x_i} \\
& -E_n^* \frac{\partial E_j}{\partial x_i}-\Big(E_i^* \frac{\partial E_n}{\partial x_j}-E_n \frac{\partial E_i^*}{\partial x_j}\Big)\Big]\Big\}. \qquad (42)
\end{aligned}
$$

The flux density S_{nm} of the nth component of the momentum in the x_m direction is equal to

$$
\begin{aligned}
S_{nm}=\frac{k_n}{\omega}\Big\{ & S_m+\frac{1}{32\pi}\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial k_m}\Big(E_i^* \frac{\partial E_j}{\partial t}-E_j \frac{\partial E_i^*}{\partial t}\Big)\Big\} \\
& -\frac{1}{16\pi}\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial \omega}\Big(\delta_{im} E_n^* \frac{\partial E_j}{\partial t}-\delta_{jm} E_n \frac{\partial E_i^*}{\partial t}\Big) \\
& +\frac{1}{16\pi}\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial k_l}\Big[\delta_{im} E_n^* \frac{\partial E_j}{\partial x_l}-\delta_{jm} E_n \frac{\partial E_i^*}{\partial x_l}\Big] \\
& -\frac{1}{16\pi} M_{ij}^H(\delta_{im} E_j E_n^*+\delta_{jm} E_i^* E_n) \\
& -\frac{1}{16\pi} M_{ij}^A(\delta_{im} E_j E_n^*-\delta_{jm} E_i^* E_n) \\
& -\frac{1}{32\pi}\frac{i}{\omega} M_{ij}^H\Big[\delta_{im}\Big(E_j \frac{\partial E_n^*}{\partial t}-E_n^* \frac{\partial E_j}{\partial t}\Big) \\
& -\delta_{jm}\Big(E_i^* \frac{\partial E_n}{\partial t}-E_n \frac{\partial E_i^*}{\partial t}\Big)\Big] \\
& +\frac{1}{16\pi}\delta_{mn}\Big\{M_{ij}^H E_i^* E_j+\frac{i}{2\omega}\frac{\partial \omega M_{ij}^H}{\partial \omega}\Big(E_i^* \frac{\partial E_j}{\partial t} \\
& -E_j \frac{\partial E_i^*}{\partial t}\Big)-\frac{i}{2\omega}\frac{\partial \omega M_{ij}^H}{\partial k_l}\Big(E_i^* \frac{\partial E_j}{\partial x_l}-E_j \frac{\partial E_i^*}{\partial x_l}\Big) \\
& -\frac{i}{2\omega} M_{ij}^H\Big(E_i^* \frac{\partial E_j}{\partial t}-E_j \frac{\partial E_i^*}{\partial t}\Big)\Big\} \\
& +\frac{1}{32\pi}\frac{i}{\omega}\frac{\partial \omega M_{ij}^H}{\partial k_m}\Big(E_i^* \frac{\partial E_j}{\partial x_n}-E_j \frac{\partial E_i^*}{\partial x^n}\Big). \qquad (43)
\end{aligned}
$$

Finally, the quantity q_n which describes momentum dissipation is equal to

$$
\begin{aligned}
q_n=\frac{k_n}{\omega}\Big[& q-\frac{1}{16\pi} M_{ij}^A\Big(E_i^* \frac{\partial E_j}{\partial t}-E_j \frac{\partial E_i^*}{\partial t}\Big)\Big] \\
& -\frac{1}{16\pi} M_{ij}^A\Big(E_i^* \frac{\partial E_j}{\partial x_n}-E_j \frac{\partial E_i^*}{\partial x_n}\Big). \qquad (44)
\end{aligned}
$$

Here w, S_m, and q are defined in Eqs. (20).

Note that in the expressions for the momentum density $\mathbf{p}$, Eq. (42), and for the momentum flux S_{nm}, Eq. (43), when the right-hand side is nonzero, i.e., for forced oscillations, we have the terms

$$
p_n'=\frac{k_n}{\omega} M_{ij}^H \frac{E_i^* E_j}{16\pi}-\frac{k_i}{\omega} M_{ij}^H \frac{E_j E_n^*}{16\pi}-\frac{k_j}{\omega} M_{ij}^H \frac{E_i^* E_n}{16\pi}, \qquad (42a)
$$

$$
\begin{aligned}
S_{nm}'=\frac{1}{16\pi}\delta_{mn} M_{ij}^H E_i^* E_j-\frac{1}{16\pi} M_{ij}^H(\delta_{im} E_j E_n^* \\
+\delta_{jm} E_i^* E_n), \qquad (43a)
\end{aligned}
$$

which are comparable in magnitude with the lowest-order terms retained in the expressions $(k_n/\omega)w$ and $(k_n/\omega)S_m$ respectively, and also the terms proportional to the anti-Hermitian part of the tensor M_{ij}.

To lowest order, neglecting terms of order λ/L and Ω/ω in Eqs. (42)–(44) and taking into account (22a), we find for the eigenmodes

$$
\frac{\partial p_{0n}}{\partial t}+\frac{\partial}{\partial x_m} S_{0nm}+q_{0n}=0, \qquad (45)
$$

where

$$
p_{0n}=\frac{k_n}{\omega} w_0,
$$

$$
S_{0nm}=\frac{k_n}{\omega} S_{0m},
$$

$$
q_{0n}=\frac{k_n}{\omega} q_0,
$$

$$
\rho_0\mathbf{E}+\frac{1}{c}\mathbf{j}_0\mathbf{B}=\frac{\mathbf{k}}{\omega}(\mathbf{j}_0\mathbf{E}). \qquad (46)
$$

Thus, the momentum conservation law (45) to lowest order including (46) can be derived from the energy conservation law (34) in the same approximation if each term in (34) is multiplied by $\mathbf{k}/\omega$ (see, e.g., Ref. 18). This relation between the wave momentum and energy densities, $\mathbf{p}_0=(\mathbf{k}/\omega)w_0$, is the same as that between the momentum and energy of a photon in quantum theory, but it fails when the subsequent terms in the expansion powers of λ/L and Ω/ω are taken into account in the expressions for $\mathbf{p}$ and w.

4. TRANSPORT EQUATIONS FOR ANGULAR MOMENTUM, ORBITAL ANGULAR MOMENTUM, AND INTRINSIC ANGULAR MOMENTUM

Now from (10) and (40) we derive an equation for the transport of wave angular momentum in differential form:

$$-\left\{\mathbf{r}\times\left(\overline{\rho_0\mathbf{E}}+\frac{1}{c}\overline{\mathbf{j}_0\mathbf{B}}\right)\right\}_s$$

$$=-\frac{1}{2}\operatorname{Re} e_{skn}x_k\left\{\frac{k_n}{\omega}j_{0i}\left(1+\frac{i}{\omega}\frac{\partial}{\partial t}\right)E_i^*\right.$$

$$\left.-\frac{i}{\omega^2}\frac{\partial}{\partial t}(E_n^*k_i j_{0i})+\frac{i}{\omega}j_{0i}\frac{\partial E_i^*}{\partial x_n}-\frac{i}{\omega}\frac{\partial}{\partial x_i}(j_{0i}E_n^*)\right\},$$

(47)

where e_{skn} is the Levi–Civita fully antisymmetric third-order unit tensor [terms of higher order in λ/L and Ω/ω have been omitted from the right-hand side of (47)].

The largest term on the right-hand side of (47) is the first:

$$-\frac{1}{2}\operatorname{Re}\left\{\left(\mathbf{r}\frac{\mathbf{k}}{\omega}\right)_s(\mathbf{j}_0\mathbf{E}^*)\right\}. \tag{48}$$

Substituting expression (19) in (48), we find the transport equation for the "orbital" angular momentum of the wave:

$$\mathbf{r}\frac{\mathbf{k}}{\omega}\left(\frac{\partial w}{\partial t}+\operatorname{div}\mathbf{S}+q\right)=-\left(\mathbf{r}\frac{\mathbf{k}}{\omega}\right)(\overline{\mathbf{j}_0\mathbf{E}}), \tag{49}$$

or

$$\frac{\partial M_i}{\partial t}+\frac{\partial}{\partial x_j}\left\{\left(\mathbf{r}\frac{\mathbf{k}}{\omega}\right)_i S_j\right\}+q\left(\mathbf{r}\frac{\mathbf{k}}{\omega}\right)_i+\left(\frac{\mathbf{k}}{\omega}\mathbf{S}\right)_i$$

$$=-\left(\mathbf{r}\frac{\mathbf{k}}{\omega}\right)_i(\overline{\mathbf{j}_0\mathbf{E}}). \tag{49a}$$

Taking $\mathbf{j}_0=0$ in (49a) and retaining only the lowest-order terms in the left-hand side [cf. Eqs. (29)–(33)], we have

$$\frac{\partial M_{0i}}{\partial t}+\frac{\partial}{\partial x_j}(v_{gj}M_{0i})+2\gamma_0 M_{0i}=(\mathbf{v}_g\mathbf{p}_0)_i, \tag{50}$$

where $\mathbf{M}_0$ is the orbital angular momentum density of the wave,

$$\mathbf{M}_0=\left[\mathbf{r}\frac{\mathbf{k}}{\omega}\right]w_0=[\mathbf{r}\mathbf{p}_0]. \tag{51}$$

If the phase velocity $\mathbf{v}_{ph}=(\omega/k)\varkappa$ and the group velocity $\mathbf{v}_g$ are not collinear (e.g., for propagation of waves in a magnetized plasma or in crystals with a symmetry axis above second order), then for the components of the orbital angular momentum density vector lying in the same plane as the vectors $\mathbf{v}_{ph}$ and $\mathbf{v}_g$, Eq. (50) assumes the form of a conservation law. The component of the vector $\mathbf{M}_0$ perpendicular to this plane, as follows from (50), are not conserved.

We transform the last term in (47):

$$\operatorname{Re}\frac{i}{2\omega}e_{skn}x_k\frac{\partial}{\partial x_i}(j_{0i}E_n^*)=\frac{\partial}{\partial x_i}\operatorname{Re}\frac{i}{2\omega}e_{skn}x_k j_{0i}E_n^*$$

$$-\operatorname{Re}\frac{i}{2\omega}e_{sin}j_{0i}E_n^*. \tag{52}$$

We see that in (47), when we take (52) into account, there is only one term,

$$-\operatorname{Re}\frac{i}{2\omega}(\mathbf{j}_0\mathbf{E}^*)_s, \tag{53}$$

which does not explicitly contain the radius vector $\mathbf{r}$. This term is precisely the one which determines the transport equation after substitution of (14):

$$-\operatorname{Re}\frac{i}{2\omega}(\mathbf{j}_0\mathbf{E}^*)_s=\frac{1}{8\pi}\operatorname{Re}\frac{i}{\omega}e_{sin}E_n^*\left\{-i\omega M_{ij}E_j\right.$$

$$\left.+\frac{\partial\omega M_{ij}}{\partial\omega}\frac{\partial E_j}{\partial t}-\frac{\partial\omega M_{ij}}{\partial k_m}\frac{\partial E_j}{\partial x_m}\right\}, \tag{54}$$

which under conditions to be specified below goes over to the law of conservation of wave intrinsic angular momentum.

Let us investigate (54) for the case of the propagation of an eigenmode. Setting $\mathbf{j}_0=0$ and substituting the first two terms of the expansion from (21) in the right-hand side of (54), using (22a) we have

$$\frac{1}{8\pi}\operatorname{Re} ie_{sin}e_n^*\left\{M_{ij}^H E_j^1 E_0^*+\frac{\partial M_{ij}^H}{\partial\omega}e_jE_0^*\frac{\partial E_0}{\partial t}\right.$$

$$\left.-\frac{\partial M_{ij}^H}{\partial k_m}e_jE_0^*\frac{\partial E_0}{\partial x_m}-iM_{ij}^A e_j\left|E_0\right|^2\right\}=0. \tag{55}$$

Equation (55) is to be compatible with Eq. (22a) and (22b). We obtain the compatibility condition by taking the dot product of (55) with $\mathbf{a}$, determined by

$$a_s e_{sin}e_n^*=Ae_i^*, \tag{56}$$

where A is an arbitrary proportionality coefficient. Then, taking into account that by virtue of (22a) we have $e_i^*M_{ij}^H=(M_{ji}^H e_i)^*=0$, after we take the dot product of (55) with the vector $\mathbf{a}$ the first term, which is proportional to E_j^1, vanishes and we find the desired compatibility condition. Before writing this condition down explicitly, let us find the vector $\mathbf{a}$. Solving (56) for a_s, we obtain a relation which must be satisfied by the components of the polarization vector for the eigenmode in question (along with the normalization condition $|\mathbf{e}|^2=1$):

$$e_x^{*2}+e_y^{*2}+e_z^{*2}=0. \tag{57}$$

From (27) it follows that e_x and e_z are real, while e_y is purely imaginary. Condition (57) can therefore in general be satisfied only as a result of an additional relation between ω and $\mathbf{k}$, different from that which follows from the dispersion relation, which has been eliminated. It is easy to see that condition (57) can be satisfied only for a wave with circular polarization:

$$e_x=\frac{1}{\sqrt{2}},\quad e_y=\pm\frac{i}{\sqrt{2}},\quad e_z=0, \tag{58}$$

and the vector $\mathbf{a}$ that satisfies (56) together with (57) is equal to

$$\mathbf{a}=A(\mathbf{e}^*\mathbf{e}). \tag{59}$$

If we set $A=-i$, then the vector $\mathbf{a}$ by virtue of (58) is equal to

$$\mathbf{a}=-i(\mathbf{e}^*\mathbf{e})=\pm\mathbf{e}_z. \tag{60}$$

Using (60), we take the dot product of (55) with $\mathbf{e}_z$,

$$\frac{1}{8\pi}\operatorname{Re} i(-\mathbf{e}_z\mathbf{e}^*)_i\left\{M_{ij}^H E_j^1 E_0^* + \frac{\partial M_{ij}^H}{\partial\omega} e_j E_0^* \frac{\partial E_0}{\partial t}\right.$$
$$\left.-\frac{\partial M_{ij}^H}{\partial k_m} e_j E_0^* \frac{\partial E_0}{\partial x_m} - iM_{ij}^A e_j \left|E_0\right|^2\right\}=0. \tag{61}$$

Then

$$\begin{aligned}\mathbf{e}_z\mathbf{e}^* &= \pm(\pm\mathbf{e}_z\mathbf{e}^*)\\ &= \mp i(\mathbf{e}^*\mathbf{e})\mathbf{e}^*\\ &= \mp i\{\mathbf{e}(\mathbf{e}^*\mathbf{e}^*)-\mathbf{e}^*(\mathbf{e}\mathbf{e}^*)\}\\ &= \pm i\mathbf{e}^*.\end{aligned} \tag{62}$$

Substituting (62) in (61) we find from (61) the compatibility condition

$$\pm\frac{1}{8\pi}\operatorname{Re}\left\{\frac{\partial M_{ij}^H}{\partial\omega} e_i^* e_j E_0^* \frac{\partial E_0}{\partial t} - \frac{\partial M_{ij}^H}{\partial k_m} e_i^* e_j E_0^* \frac{\partial E_0}{\partial x_m}\right.$$
$$\left.-i\varepsilon_{ij}^A e_i^* e_j \left|E_0\right|^2\right\}=0. \tag{63}$$

Using (29), (30), and (32), we can rewrite (63) in the form

$$\frac{\partial}{\partial t}\left(\pm\frac{w_0}{\omega}\right)+\operatorname{div}\left(\pm\frac{\mathbf{S}_0}{\omega}\right)+2\gamma_0\left(\pm\frac{w_0}{\omega}\right)=0. \tag{64}$$

Derived from the angular momentum transport equation (47), this equation has the form of a conservation law for the intrinsic angular momentum (spin) of a wave with circular polarization directed perpendicular to the plane of polarization:

$$\frac{\partial\mu_z}{\partial t}+\operatorname{div}\mathbf{S}_{\mu z}+2\gamma_0\mu_z=0, \tag{65}$$

where

$$\mu_z=\pm\frac{w_0}{\omega},$$

$$\mathbf{S}_{\mu z}=\pm\frac{\mathbf{S}_0}{\omega}=\pm\mathbf{v}_g\frac{w_0}{\omega}=\mathbf{v}_g\mu_z. \tag{66}$$

The quantity μ_z is the spin density averaged over the wave period, $\mathbf{S}_{\mu z}$ is the spin flux density, and $2\gamma_0\mu_z$ is the rate at which spin is lost per unit volume [the $\pm$ signs in (64) and (66) correspond to the signs in the polarization vector $\mathbf{e}(2^{-1/2},\pm i\cdot 2^{-1/2},0)$. Let us consider a few examples of transverse waves with the circular polarization (58), which have the spin (66) propagating in various media.

a) A wave propagates in an isotropic gyrotropic medium whose dielectric tensor for arbitrary spatial dispersion is determined by the expression[25]

$$\varepsilon_{ij}(\omega,\mathbf{k})=\varepsilon_l(\omega,k)\varkappa_i\varkappa_j+\varepsilon_t(\omega,k)(\delta_{ij}-\varkappa_i\varkappa_j)$$
$$+if(\omega,k)e_{ijs}k_s. \tag{67}$$

In this case we have

$$M=\varepsilon_1(\varepsilon_t-N^2-kf)(\varepsilon_t-N^2+kf),$$

$$\operatorname{Sp} e=2\varepsilon_1(\varepsilon_t-N^2)+(\varepsilon_t-N^2)^2-k^2f^2,$$

and the dispersion relation for transverse waves with the polarization (58) takes the form

$$\varepsilon_t-N^2=\pm kf. \tag{68}$$

The energy density w_0, the group velocity $\mathbf{v}_g$, and γ_0 for transverse waves with the dispersion (68) are equal to [cf. Eqs. (29) and (31)]

$$w_0=\omega\frac{\partial}{\partial\omega}(\varepsilon_t-N^2\mp kf)\frac{|\mathbf{E}_0|^2}{16\pi},$$

$$\mathbf{v}_g=-\frac{\partial(\varepsilon_t-N^2\mp kf)/\partial\mathbf{k}}{\partial(\varepsilon_t-N^2\mp kf)/\partial\omega}, \tag{69}$$

$$\gamma_0=\frac{\varepsilon_t''\mp kf''}{\partial(\varepsilon_t-N^2\mp kf)/\partial\omega}.$$

[In the right-hand sides of Eqs. (69) we need to substitute (68), $\varepsilon_t''=\operatorname{Im}\varepsilon_t$, and $f''=\operatorname{Im} f$ in the expressions operated on by the derivatives.] Substituting (69) in (66) and (65), we find explicit expressions for the spin of a wave and its conservation law.

Note that if we put $\varepsilon_1=\varepsilon_t=1$, $f=0$, and $\gamma_0=0$ in Eqs. (65)–(69), then we find the spin and its conservation law for a transverse wave propagating in vacuum.

b) In a "cold" plasma in an external magnetic field $\mathbf{B}_0$ the dielectric tensor takes the form

$$\varepsilon_{ij}(\omega)=\varepsilon_1(\omega)\delta_{ij}+(\varepsilon_3-\varepsilon_1)h_ih_j+i\varepsilon_2e_{ijs}h_s, \tag{70}$$

where

$$\varepsilon_1=1-\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega^2-\omega_{c\alpha}^2},\quad \varepsilon_3=1-\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega^2},$$

$$\varepsilon_2=-\sum_\alpha\frac{\omega_{c\alpha}\omega_{p\alpha}^2}{\omega(\omega^2-\omega_{c\alpha}^2)},$$

$$\omega_{c\alpha}=\frac{e_\alpha B_0}{m_\alpha c},\quad \omega_{p\alpha}^2=\frac{4\pi n_\alpha e_\alpha^2}{m_\alpha},\quad \mathbf{h}=\frac{\mathbf{B}_0}{B_0}, \tag{71}$$

and e_α, n_α, and m_α are the charge, density, and mass respectively of particles of species α. For simplicity we assume that the waves are not collisionally damped ($\operatorname{Im}\varepsilon_1=\operatorname{Im}\varepsilon_2=\operatorname{Im}\varepsilon_3=0$).

For waves propagating parallel to the magnetic field $\mathbf{B}_0$ ($\varkappa=\mathbf{h}=\mathbf{e}_z$), we have

$$M=\varepsilon_3(\varepsilon_1-N^2-\varepsilon_2)(\varepsilon_1-N^2+\varepsilon_2), \tag{72}$$

$$\mathrm{Sp}\, e=2\varepsilon_3(\varepsilon_1-N^2)+(\varepsilon_1-N^2)^2-\varepsilon_2^2. \quad (73)$$

The dispersion relation for transverse waves with circular polarization given by Eq. (58) takes the form

$$\varepsilon_1-N^2=\pm\varepsilon_2. \quad (74)$$

The energy density w_0, the group velocity $\mathbf{v}_g$, and the energy flux density $\mathbf{S}_0$ for such waves are equal to ($\gamma_0=0$)

$$w_0=\omega\frac{\partial}{\partial\omega}(\varepsilon_1-N^2\mp\varepsilon_2)\frac{|\mathbf{E}^0|^2}{16\pi},$$

$$\mathbf{v}_g=-\frac{\partial(\varepsilon_1<-N^2\mp\varepsilon_2)/\partial\mathbf{k}}{\partial(\varepsilon_1-N^2\mp\varepsilon_2)/\partial\omega}=\frac{2\boldsymbol{\kappa}(c/\omega)N}{\partial(\varepsilon_1-N^2\mp\varepsilon_2)\partial\omega},$$

$$\mathbf{S}_0=\boldsymbol{\kappa}cN\frac{|\mathbf{E}^0|^2}{8\pi}. \quad (75)$$

[In the expressions on the right-hand sides of Eqs. (75) acted on by the derivatives we must substitute (74).] Substituting (75) in (66) and (65) we obtain explicit expressions for the spin of the wave and its conservation law.

If the conditions $\omega_p^2\gg\omega|\omega_{ce}|$, $\omega\ll|\omega_{ce}|$, ($|\varepsilon_1|\ll|\varepsilon_2|\ll|\varepsilon_3|$), hold in the cold plasma, then a "whistler" can propagate. For propagation parallel to the magnetic field $\mathbf{B}_0$ the dispersion relation for a whistler and the polarization factor take the forms ($\varepsilon_2=-|\varepsilon_2|$)

$$N^2=|\varepsilon_2|,\quad \mathbf{e}=\frac{1}{\sqrt{2}}(\mathbf{e}_x+i\mathbf{e}_y). \quad (76)$$

Explicit expressions for the spin in its conservation law in the case of a whistler can be derived from (65) and (66) (taking the + sign) by substituting (75) with $\varepsilon_1-N^2\mp\varepsilon_2$ replaced by $-N^2+|\varepsilon_2|$.

CONCLUSION

This treatment of the transport of energy, momentum, and angular momentum of a nearly monochromatic electromagnetic wave packet propagating in a weakly absorbing uniform time-independent medium with temporal and spatial dispersion shows that for a broad packet there is a conservation law for the intrinsic angular momentum in the differential form (65) (but only for waves with circular polarization), from which expressions for the intrinsic angular momentum density, its flux density, and the power lost per unit volume per unit time are obtained [see Eqs. (66)]. This conservation law can be derived only by treating higher terms in the expansion of the field in powers of λ/L and Ω/ω. [The role of the finite size of the wave packet was remarked even in the early works of Abragam, Einstein, and Ehrenfest on the subject of the intrinsic angular momentum of electromagnetic waves (see Ref. 17).]

We have shown that there is a conservation law for the components of the wave orbital angular momentum in the same plane as the phase vectors and the group velocity, although the component perpendicular to this plane is not conserved [cf. Eq. (50)].

We have derived laws for the conservation of energy and momentum in differential form to within terms of higher order in the quantities λ/L and Ω/ω [Eqs. (19), (20), and (41)–(44)]. Note that the expressions for the momentum density and its flux contain terms proportional to the anti-Hermitian part of the tensor ε_{ij}, and also (in the case of driven oscillations) terms of the same order as the leading terms for the eigenmodes (when $\mathbf{j}_0=0$).

We have shown that the momentum conservation law is a consequence of the energy conservation law only to leading order; in this approximation the relation between the wave momentum and energy is the same as that for a photon in quantum theory.

To lowest order we have found an expression simpler than the usual one (see, e.g., Refs. 4 and 5) for the energy density of an electromagnetic wave, which does not explicitly contain the polarization vector. This expression allows the condition for the positivity (or negativity) of the wave energy density to be generalized to the case of an arbitrary dispersive medium.

Note that the question of the transfer of angular momentum to the medium can also be treated in the case of a very broad wave packet ($L\to\infty$) resulting from the formation of a scattered wave by a system of distributed dipole moments, and also due to diffraction effects at the boundary of a scattering (or absorbing) dielectric plate.[18]

In the above investigation we have not specified the mechanism by which waves are absorbed, described by the anti-Hermitian part of the tensor ε_{ij}. Thus, we have not considered the fate of the absorbed wave energy, momentum, and angular momentum. To answer this question, each case must be analyzed separately. For example, in a collisionless plasma the absorbed energy (and momentum and angular momentum) of a wave packet are equal to the energy (momentum, angular momentum) acquired by the resonant particles. When collisions are taken into account this energy goes into increasing the particle thermal energy, etc. In order to study these processes we must use a more detailed description of the system, e.g., by means of the kinetic equations. In this case it may be necessary to take into account nonlinear effects, such as the entrainment currents[18] with the accompanying processes of charge separation in the field of a damped wave packet[27] and momentum transfer from the resonant particles to the nonresonant particles,[7,28] the inverse Faraday effect (production of a wave packet by a quasisteady magnetic field[18]), etc. But these topics lie outside the scope of the present work.

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