Evolution of solitary waves which are approximately solitons of a nonlinear Schrödinger equation

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(Submitted 30 April 1993)
Zh. Eksp. Teor. Fiz. **104**, 3620–3629 (November 1993)

A generalized moment method is used to construct a system of equations for nonintegrable versions of a nonlinear Schrödinger equation. This system of equations allows one to analyze the evolution of the properties of a solitary wave: its amplitude, width, center of gravity, phase velocity, phase modulation depth, and phase shift. Under some auxiliary conditions, this system becomes the perturbation theory system of equations for solitons in the adiabatic approximation. Several examples of perturbations are discussed to illustrate the use of this new formalism.

1. INTRODUCTION

The topics which have recently been attracting most attention in the field of nonlinear waves are solitons and their generalizations to the periodic case. The apparent reason for this interest is the development of an analytic method for studying solitons: the method of the inverse scattering transform.[1] This method has led to striking progress in the theory of nonlinear evolution equations. The special role played by equations with soliton solutions has been demonstrated. Some new branches of mathematical physics have arisen.

On the other hand, most real problems in physics are described by equations which are not completely integrable and which therefore do not have soliton solutions. When an equation of interest is not integrable because of a small term, one can develop a perturbation theory in which a soliton solution is used as a zeroth approximation. The most systematic perturbation theory is based on the inverse scattering transform.[2] In one particular case, this approach leads to a very effective and popular adiabatic perturbation theory for solitons. That method and its development have been reviewed in detail by Kivshar and Malomed.[3]

In an effort to study the behavior of a solitary wave described by a nonlinear Schrödinger equation, Anderson[4] developed a variational method which does not make use of the complete integrability of that equation. This method was subsequently used successfully for various generalizations of the nonlinear Schrödinger equation which arise in solid state physics, plasma physics, nonlinear optics, and biophysics. It is particularly interesting to note that, if one chooses a soliton solution as a trial function, then the system of equations of adiabatic perturbation theory follows from the corresponding Euler–Lagrange equations, as was shown in Ref. 5 in the example of a nonlinear Schrödinger with a perturbation. It is quite natural to suggest that the choice of a more general trial function (a more detailed one) may lead to a more accurate description of the solution of the nonlinear Schrödinger equation with a perturbation. The variational method can be applied not only to the nonlinear Schrödinger equation but also to other evolution equations.

The Lagrangian for a variational approach in the spirit of Ref. 4 is not always easy to find. It is known[6] that a simpler way to derive the system of equations of adiabatic perturbation theory than by means of the inverse scattering transform is to work from the equations for "integrals of motion" which are not constants in the presence of perturbing terms. Along this approach, it is necessary to substitute a soliton solution of a completely integrable equation-an equation which does not contain perturbations-into the exact integrodifferential equations. This step might be thought of as analogous to choosing an appropriate trial function in the variational method.[4]

If it becomes necessary to study the behavior of solitary waves in a system which is approximately a completely integrable system (or for which nothing is known about integrability), then one can attempt to use this method of integrals of motion[6] in combination with a moment method.[7,8] As the integrals one should actually take expressions which determine the parameters of the trial function most simply and to construct for them some exact equations on the basis of the evolution equation of interest. The moments of conserved densities (the number density of particles, the energy density, the momentum density, etc.) are extremely attractive for this purpose.

The transition from an evolution equation describing a system with an infinite number of degrees of freedom to a system of ordinary differential equations for a finite number of variables is reminiscent of the transition from a microscopic description of a system to a macroscopic one. This transition might accordingly be called a "roughening procedure."

In the present paper we use the example of a nonlinear Schrödinger equation with perturbations to carry out a roughening procedure for the description of a solitary wave which is not a soliton. As a trial function we adopt a soliton-like solution, but one which has an additional phase modulation and for which the width and amplitude of the solitary wave are independent of each other. The resulting system of equations for six parameters of this wave embodies the results of adiabatic perturbation theory for solitons[3] and the results of Anderson's variational approach.[4] To illustrate the effectiveness of this method, we look at sev-

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eral examples—some examples which are quite simple but which clearly reveal the distinctions from existing results.[3]

2. DERIVATION OF THE BASIC EQUATIONS (THE ROUGHENING PROCEDURE)

We assume that the complex amplitude $q(x,t)$ of a solitary wave is determined by a nonlinear Schrödinger equation with a perturbation:

$$iq_{,t}+\sigma q_{,xx}+\mu|q|^2q=R[q]. \quad (1)$$

Here σ is a measure of the second-order dispersion, and μ characterizes a self-effect. The quantity $R[q]$ is an arbitrary function of $q(x,t)$ and of its derivatives with respect to x. In the case $R[q]=0$, Eq. (1) has soliton solutions. If $R[q]$ is in a sense a weak perturbation, then the solitary wave which is the solution of (1) is, in the same sense, regarded as being close to solitons.

In the case $R[q]=0$, the nonlinear Schrödinger equation has an infinite number of integrals of motion (under suitable boundary conditions).[1] The first two of these integrals are

$$I_1=\int_{-\infty}^{\infty}|q|^2dx, \quad I_2=\frac{1}{2}\int_{-\infty}^{\infty}(qq^*_{,x}-q^*q_{,x})dx. \quad (2)$$

For the perturbed nonlinear Schrödinger equation these integrals are not conserved, and one can verify directly with the help of (1) that the following equations hold:

$$\frac{dI_1}{dt}=i\int_{-\infty}^{\infty}(qR^*-q^*R)dx, \quad (3a)$$

$$\frac{dI_2}{dt}=-i\int_{-\infty}^{\infty}(q_{,x}R^*+q^*_{,x}R)dx. \quad (3b)$$

The integrands in (2) can be understood as certain distributions. The moments of these distributions are of interest since they are related to the properties of the solitary wave. These important integrals are

$$D_1=\int_{-\infty}^{\infty}x|q|^2dx, \quad (4a)$$

$$D_2=\int_{-\infty}^{\infty}(x-x_c)^2|q|^2dx, \quad (4b)$$

$$M_1=\int_{-\infty}^{\infty}(x-x_c)(q^*q_{,x}-qq^*_{,x})dx, \quad (4c)$$

where $x_c=D_1/I_1$—the center of gravity of the $|q(x,t)|^2$ "distribution"—is the number density of particles in the theory of the nonlinear Schrödinger equation. Using (1), we find equations which specify the changes in these integrals as a function of t:

$$\frac{dD_1}{dt}=2i\sigma I_2-i\int_{-\infty}^{\infty}x(q^*R-qR^*)dx, \quad (5a)$$

$$\frac{dD_2}{dt}=-2i\sigma M_1-\int_{-\infty}^{\infty}(x-x_c)^2(q^*R-qR^*)dx, \quad (5b)$$

$$\frac{dM_1}{dt}=2I_2\frac{dx_c}{dt}+i\int_{-\infty}^{\infty}(4\sigma|q_{,x}|^2-\mu|q|^4)dx$$
$$+2i\int_{-\infty}^{\infty}(x-x_c)(q_{,x}R^*-q^*_{,x}R)dx$$
$$+i\int_{-\infty}^{\infty}(q^*R+qR^*)dx. \quad (5c)$$

As in Ref. 9, we also need to supplement these equations with the identity

$$\int_{-\infty}^{\infty}(qq^*_{,t}-q_{,t}q^*)dx=2i\int_{-\infty}^{\infty}(\sigma|q_{,x}|^2-\mu|q|^4)dx$$
$$+i\int_{-\infty}^{\infty}(qR^*+q^*R), \quad (6)$$

which allows us to determine the law governing the phase of the envelope $q(x,t)$.

Starting from the system of exact equations in (3), (5), and (6), we can construct systems of equations with various degrees of accuracy. If we assume, for example, that the solitary wave retains the shape of a soliton of the nonlinear Schrödinger equation as it evolves, i.e., assuming that the solution of (1) is

$$\tilde{q}(x,t)=A(t)\operatorname{sech}y(x,t)e^{i\Phi(x,t)},$$
$$y(x,t)=A(t)[x-x_c(t)],$$
$$\Phi(x,t)=\varphi(t)+C(t)[x-x_c(t)], \quad (7)$$

then we can find for $A(t)$, $x_c(t)$, $C(t)$, and $\varphi(t)$, the following system of equations, where $\rho=R[q]\exp(-i\Phi)$:

$$A_{,t}=\int_{-\infty}^{\infty}\operatorname{sech}y\operatorname{Im}\rho dy, \quad (8a)$$

$$C_{,t}=-\int_{-\infty}^{\infty}\operatorname{th}y\operatorname{sech}y\operatorname{Re}\rho dy, \quad (8b)$$

$$x_{c,t}=2\sigma C+\frac{1}{A^2}\int_{-\infty}^{\infty}y\operatorname{sech}y\operatorname{Im}\rho dy, \quad (8c)$$

$$\varphi_{,t}=Cx_{c,t}+\frac{1}{2}(A^2-C^2)-\frac{1}{2A}\int_{-\infty}^{\infty}\operatorname{sech}y\operatorname{Re}\rho dy. \quad (8d)$$

The only distinction from the system of equations given in Refs. 2 and 3 is in Eq. (8d), in which we need to replace sech y in the integral on the right side by $(1-y\tanh y)\operatorname{sech}y$ in order to reach complete agreement with the equations of adiabatic perturbation theory for solitons. Equation (8) is found by substituting $\tilde{q}(x,t)$ from (7) into Eqs. (3), (5a), and (6). Equations (5b) and (5c) are unnecessary in the adiabatic approximation, (7).

To find a more accurate description of the evolution of the solitary wave, we should choose a more general trial function $\tilde{q}(x,t)$. Equations (3), (5), and (6) are used fully when we choose $\tilde{q}(x,t)$ as follows:

$$\tilde{q}(x,t)=A(t)\operatorname{sech}y(x,t)\exp[i\Phi(x,t)],$$
$$y(x,t)=[x-x_c(t)]x_p^{-1}(t),$$

$$\Phi(x,t)=\varphi(t)+C(t)[x-x_c(t)]+b(t)[x-x_c(t)]^2. \tag{9}$$

This approximation could also be called an "adiabatic" approximation, but here, in contrast with the soliton in (7), we are allowing a phase modulation, and the amplitude $A(t)$ and the width $x_p(t)$ of the solitary wave appear as independent parameters.

Using (9), we can evaluate the integrals in (2) and (4):

$$I_1=2x_pA^2, \quad I_2=-2ix_pA^2C,$$

$$D_1=2x_cx_pA^2, \quad D_2=(\pi^2/6)x_p^2(x_pA^2),$$

$$M_1=i(2\pi^2/3)x_p^3A^2b.$$

From (3a) and the expression for I_1 we find the equation

$$(x_pA^2)_{,t}=x_pA\int_{-\infty}^{\infty} \operatorname{sech} y \operatorname{Im} \rho dy, \tag{10a}$$

which is a generalization of (8a). Now considering I_2, we find from (3b) and (10a) an equation for $C(t)$:

$$C_{,t}=-(x_pA)^{-1}\int_{-\infty}^{\infty} \operatorname{th} y \operatorname{sech} y \operatorname{Re} \rho dy$$
$$+2bx_pA^{-1}\int_{-\infty}^{\infty} y \operatorname{sech} y \operatorname{Im} \rho dy. \tag{10b}$$

Using (5a) and (10a), we can derive an equation for $x_c(t)$, from (5b) and (10a) we find an equation for $x_p(t)$, and from (5c), using everything derived so far, we find an equation for $b(t)$:

$$x_{c,t}=2\sigma C+x_pA^{-1}\times\int_{-\infty}^{\infty} y \operatorname{sech} y \operatorname{Im} \rho dy, \tag{10c}$$

$$x_{p,t}=4\sigma bx_p-x_p(2A)^{-1}$$
$$\times\int_{-\infty}^{\infty}(1-12y^2/\pi^2)\operatorname{sech} y \operatorname{Im} \rho dy, \tag{10d}$$

$$b_{,t}=(4\sigma/\pi^2)(x_p^{-4}-\pi^2b^2)-2\mu(A/\pi x_p)^2$$
$$-6(A\pi^2x_p^2)^{-1}\int_{-\infty}^{\infty}\left(y \operatorname{th} y-\frac{1}{2}\right)\operatorname{sech} y \operatorname{Re} \rho dy. \tag{10e}$$

An equation for the phase $\varphi(t)$ follows from identity (6):

$$\varphi_{,t}=Cx_{c,t}-(\pi^2/12)x_p^2b_{,t}--\sigma C^2+(2/3)\mu A^2$$
$$-(\sigma/3)(x_p^{-2}+\pi^2b^2x_p^2)$$
$$-(2A)^{-1}\int_{-\infty}^{\infty} \operatorname{sech} y \operatorname{Re} \rho dy. \tag{10f}$$

Equations (10) thus make it possible to describe the evolution of a solitary wave (which need not be a soliton) for a nonlinear Schrödinger equation with a perturbation, (1), in the generalized adiabatic approximation adopted here, (9). That system of equations contains as a particular case the equations of the known adiabatic perturbation theory for solitons: System (8) follows from (10) if we discard (10d) and (10e) and if we impose the conditions $b=0$ and $x_pA=1$ in the equations which remain.

3. SOME ILLUSTRATIVE PERTURBATIONS

It is useful to consider several examples in which an analysis of the evolution of a solitary wave on the basis of Eqs. (10) leads to a result which differs from that which follows from the adiabatic soliton perturbation theory.

1. We assume that the perturbation in (1) is described by the expression

$$R[q]=-\beta|q|^4q. \tag{11}$$

Since we have $\operatorname{Im}\rho=0$ and $\operatorname{Re}\rho=-\beta A^5\operatorname{sech}^5 y$, we find a conservation law from (10a):

$$x_pA^2=W=\text{const}. \tag{12}$$

The other equations from (10) become

$$\frac{dC}{dt}=0, \quad \frac{dx_c}{dt}=2\sigma C, \tag{13a}$$

$$\begin{cases}\dfrac{dx_p}{dt}=4\sigma bx_p,\\ \dfrac{db}{dt}=(4\sigma/\pi^2)(x_p^{-4}-\pi^2b^2)-2\mu W/\pi^2x_p^3-2\tilde{\beta}W^2/\pi^2x_p^4,\end{cases} \tag{13b}$$

with $\tilde{\beta}=(16\beta/15)$. Making the substitution

$$x_p(t)/x_p(0)=r, \quad t/T=\tau,$$

where $T=[4\sigma/\pi x_p^2(0)]^{-1}$, and eliminating the variable $b(t)$ from (13b), we find an equation for $r(\tau)$:

$$\frac{d^2r}{d\tau^2}=(1-\delta)r^{-3}-\gamma r^{-2}, \tag{14}$$

where $\delta=\tilde{\beta}W/2\sigma$ and $\gamma=\mu Wx_p(0)/2\sigma$.

From (14) we find

$$\left(\frac{dr}{d\tau}\right)^2+[(1-\delta)r^{-2}-2\gamma r^{-1}]=U_0. \tag{15}$$

Here $U(r)=(1-\delta)r^{-2}-2\gamma r^{-1}$ and $U_0=U(1)+[\pi x_p^2(0)b(0)]^2$.

Equation (15) describes the motion of a material point in a $1D$ space in a potential $U(r)$ in accordance with Newton's law (14). More precisely, this is the Kepler problem.[10] With $\delta=0$, Eqs. (14) and (15) correspond to the case studied in Ref. 4; this will remain true as long as the condition $\delta<1$ holds. If the initial parameters of the solitary wave are such that we have $U_0>0$, the width of the wave will grow (this is an analog of an open orbit in the Kepler problem). If $U_0<0$, the width $r(\tau)$ periodically changes with increasing τ; here we have an analog of closed orbits in the Kepler problem. Under the condition $\delta>1$, however, there is a case of "falling on a center," i.e., a collapse of the solitary wave. The threshold value $\delta_c=1$ is reached when the energy W is equal to the critical value

$$W_c=(2\sigma/\tilde{\beta})^{1/2}. \tag{16}$$

We are assuming $\sigma>0$ and $\beta>0$ here. Such a self-compression (collapse) of an optical pulse due to a high-order nonlinearity as in (11) was recently pointed out by Azimov *et al.*[11] Note that this result does not follow in the approximation based on Eqs. (8).

2. We assume that the perturbation in (1) is

$$R[q]=-i\kappa(|q|^2q)_{,x}-i\chi q(|q|^2)_{,x}. \tag{17}$$

We now have

$$\mathrm{Im}\ \rho=(3\kappa+2\chi)x_p^{-1}A^3\mathrm{th}\ y\ \mathrm{sech}^3\ y$$

$$\mathrm{Re}\ \rho=\kappa A^3(C+2bx_py)\mathrm{sech}^3\ y,$$

and conservation law (12) again follows from (10a). The other equations can be rewritten in light of this fact:

$$\frac{dC}{dt}=\frac{4}{3}(\kappa-\chi)Wbx_p^{-1}, \tag{18a}$$

$$\frac{dx_p}{dt}=4\sigma bx_p, \tag{18b}$$

$$\frac{db}{dt}=(4\sigma/\pi^2)\ (x_p^{-4}-\pi^2b^2)-2(\mu W-\kappa WC)/\pi^2x_p^3, \tag{18c}$$

$$\frac{dx_c}{dt}=2\sigma C+\left(\kappa W-\frac{2}{3}\chi W\right)x_p^{-1}. \tag{18d}$$

From (18a) and (18b) we find

$$\frac{dC}{dt}=-[W(\kappa-\chi)/3\sigma]\frac{d}{dt}(x_p^{-1}).$$

Hence

$$C=C_0-(\tilde{\kappa}W/3\sigma)x_p^{-1},$$

$$\tilde{\kappa}=(\kappa-\chi). \tag{19}$$

The integration constant C_0 is found from the values of C and x_p at $t=0$. Eliminating $b(t)$ from (18b) and (18c), and using (19), we find an equation like (14) for $x_p(t)$:

$$\frac{d^2x_p}{dt^2}=\left[\left(\frac{4\sigma}{\pi}\right)^2-\frac{2}{3}\left(\frac{2\tilde{\kappa}W}{\pi}\right)^2\right]x_p^{-3}-\frac{8\sigma W(\mu-\tilde{\kappa}C_0)}{\pi^2}x_p^{-2}. \tag{20}$$

From this equation and the discussion above it follows that the solitary wave can collapse. The critical energy is now given by

$$W_c=(2\sigma/|\tilde{\kappa}|)\sqrt{3/2}. \tag{21}$$

Here we are assuming $\sigma>0$.

Since $x_p(t)$ is a periodic function of t (when the initial conditions correspond to closed orbits in the Kepler problem), we find from (18d) that the center of gravity of the solitary wave "shakes"; i.e., there is a periodic change in $x_c(t)$ with increasing t.

3. A perturbation of the type

$$R[q]=i\alpha q_{,xxx} \tag{22}$$

is quite common. This perturbation corresponds to incorporation of a dispersion of higher order than in (1). If, as in the preceding cases, we restrict the discussion to R in (22), then the system of equations for the parameters of a solitary wave of the type in (9) becomes

$$\frac{d}{dt}(x_pA^2)=0,\quad \frac{dC}{dt}=0, \tag{23a}$$

$$\begin{cases}\dfrac{dx_p}{dt}=4(\sigma+3\alpha C)bx_p,\\ \dfrac{db}{dt}=\dfrac{4}{\pi^{-2}}(\sigma+3\alpha C)(x_p^{-4}-\pi^2b^2)-2\mu(A/x_p\pi)^2\end{cases} \tag{23b}$$

$$\frac{dx_c}{dt}=2\sigma C+3\alpha C^2+\alpha(x_p^{-2}+\pi^2b^2x_p^2). \tag{23c}$$

From the form of Eqs. (23b) we conclude that in this approximation the third-order dispersion leads to a renormalization of the constant which determines the second-order dispersion. When other perturbations are taken into account, Eqs. (23a) may be changed, and this renormalization may become more complicated.

4. Up to this point we have been discussing nondissipative perturbations. We now assume

$$R[q]=-i\Gamma q, \tag{24}$$

so we have $\mathrm{Im}\ \rho=-\Gamma A\ \mathrm{sech}\ y$. For this perturbation, Eqs. (10) become

$$\frac{d}{dt}(x_pA^2)=-2\Gamma x_pA^2, \tag{25a}$$

$$\frac{dC}{dt}=0,\quad \frac{dx_c}{dt}=2\sigma C, \tag{25b}$$

$$\begin{cases}\dfrac{dx_p}{dt}=4\sigma bx_p,\\ \dfrac{db}{dt}=\dfrac{4\sigma}{\pi^2}(x_p^{-4}-\pi^2b^2)-2\mu(A/\pi x_p)^2.\end{cases} \tag{25c}$$

It follows from (25a) that we have $x_pA^2=W\exp(-2\Gamma t)$. Eliminating the variable $b(t)$ from (25c), we find the equation

$$\frac{d^2x_p}{dt^2}=\left(\frac{4\sigma}{\pi}\right)^2x_p^{-3}-\frac{8\sigma\mu W}{\pi^2}\exp(-2\Gamma t)x_p^{-2}, \tag{26}$$

which was derived in Ref. 12 by a variational approach.

4. CONCLUSION

System of Eqs. (10), derived here, gives an approximate (roughened) description of the evolution of a solitary wave determined by a nonlinear Schrödinger equation with a perturbation. If instead of trial function (9) we had used a more detailed function, the approximation would have been more accurate. A disadvantage of this method is that we cannot make an *a priori* estimate of the accuracy. This situation is typical for methods of this sort (e.g., the variational method). At the same time, it is possible to derive results which are more rigorous than can be done in adiabatic perturbation theory for solitons. For example, a periodic change in the width of a solitary wave as it approaches a soliton value is frequently observed in numerical simulations (e.g., Ref. 13). The fact that the oscillations of $x_p(t)$ are not damped is a consequence of our ignoring radiative losses in (9) [i.e., our ignoring the formation of a nonsoliton part of the solution of Eq. (1)].

Multisoliton effects and emission from a soliton subjected to perturbations have attracted much interest.[3] These questions have been taken up by a perturbation theory based on the inverse scattering transform. It would be useful to extend the roughening procedure developed here to those problems.

A roughening procedure can also be carried out for other integrable equations, and possible corrections to the adiabatic perturbation theory can be found. A more attractive possibility, however, is to analyze nonintegrable, physically meaningful systems and multidimensional generalizations of integrable equations with perturbing increments. Classical field theory, the theory of gravitation, and wave processes in plasma and hydrodynamics contain numerous interesting examples for such studies.

We should also point out that if we set $\chi=0$ in example 2 then the resulting equation can be converted into a modified nonlinear Schrödinger equation (a nonlinear Schrödinger equation with a derivative) through a change of variables:

$$iQ_{,t}+\sigma Q_{,ss}+i\kappa(|Q|^2Q)_{,s}=0,$$

where

$$s=x-2\sigma(\mu/\kappa)t,$$

$$Q(t,s)=q(t,x)\exp[-i(\mu/\kappa)x+i\sigma t(\mu/\kappa)^2].$$

This equation can be integrated by the inverse scattering transform.[14] One can show that, as in Ref. 15, a solution corresponding to an initial value $q(t=0,x)$, taken in the form in (3) with $b(0)=0$, with the increase of t does not undergo collapse. This example shows that it is very important to choose the trial function correctly. In this case we see the need to consider in the calculation the nonsoliton part of the solution, viz., the radiation, which stops the collapse as an ordinary dissipation.

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Translated by D. Parsons