Mutual interference of images in gravitational optics

A. F. Zakharov and A. V. Mandzhos

Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia
(Submitted 23 March 1993)
Zh. Eksp. Teor. Fiz. **104**, 3249–3268 (October 1993)

The foundations of the mutual-coherence theory in gravitational optics are considered. The effect of the caustics on the magnitude of the mutual coherence of the images is analyzed, in particular of the caustics due to galactic microlenses lying between the quasar and the observer. The dependence of the mutual coherence on the wavelength is calculated in the case when the caustic corresponds to a fold-type singularity. Model calculations of the mutual coherence are performed for concrete values of astrophysical parameters.

1. INTRODUCTION

From the time of the discovery of the first gravitational lens in 1979 astronomers have achieved substantial successes in the study of this phenomenon. A variety of objects have been discovered involving effects due to gravitational lensing, often in an unexpected manner. There are systems with multiple images of quasars (both in the optical and in the radio frequencies) of radio rings, arcs, and arclets—extended images of distant background galaxies, viewed through rich clusters of galaxies. Lastly, there is microlensing, which is micro-splitting of the quasar images by stars in intermediates galaxies. The theoretical studies have also made significant progress. The general theory of the gravitational lens is well developed and many of ideas have been proposed for the determination of parameters of cosmic systems and the Universe, otherwise hard to obtain, on the basis of gravitational lensing effects. Some of the ideas proposed in the past have undergone further development at the contemporary level. Much has been done in modeling already discovered lens systems. On the other hand, in each concrete case the real astrophysical situation, connected with one or another lens, is so complex and has so many undetermined parameters that it is difficult to construct an unambiguous model for the mass distribution in the lens. This leads to considerable uncertainty in the determination of the searched-for astrophysical parameters. The situation is improving with increasing precision of the observations, but there is nonetheless a lack of additional manifestations of gravitational lensing effects, taken into account in the problem. For this reason we find ourselves in a situation where the search for such effects, both theoretical and experimental, is rather timely. One such possible effect is the mutual interference of images in the lenses.

This idea is not new and was, apparently, first proposed by S. Refsdal in 1964.[1] Its essence is that since multiple images of one and the same radiating surface of the cosmic object are created in the gravitational lens, the fluxes of radiation proceeding from these images should interfere with each other in pairs. The whole problem reduces to the quantitative side of the question. In the first place, the relative retardation of the signal in each pair of images could considerably exceed the coherence length of the radiation, so that the effect would altogether disappear. In the second place, if the radiating object is extended (which in fact is always the case), then from each surface element of the object the radiation arrives at the observer with its own phase and as a result the summary interference effect is strongly suppressed. The strategy of the investigation should be to determine under what circumstances these two negative effects are minimized and the mutual interference of the images is maximized. The potential significance of this effect is seen as follows. In the first place, the mutual coherence of the images is one of two proposed tests of gravitational lensing. The other test consists of the relative retardation of the signal in the images. In the case of microlensing the latter test in not accessible to present-day means of observation, owing to the small angular separation of the images. Therefore the mutual coherence of the images, should it be observed, may turn out to be the sole unambiguous test of this effect (possibly along with diffraction[2]). In the second place, the mutual interference of the images may turn out to be an effective means for the study of the fine structure of the lensing objects, in particular, quasars, since the effect depends rather strongly on the dimensions of the radiating components of the object. And finally, in the third place, in the modeling of effects of microlensing it is necessary in a number of cases to take into account the mutual interference of the images. The first calculation of the effect for a macrolens on the scale of a galaxy was performed in Ref. 3, in which it was shown that a realistically perceptible mutual coherence of images can be expected only in the case of microlensing by star-size masses and smaller. The first calculation of mutual coherence for a single lens-star was performed in Ref. 4. It was shown there that although the mutual coherence increases in that case by several orders of magnitude it is still too small to be realistically seen in observations. An analogous conclusion was reached in Ref. 5. Here the role of the caustic in the process was noted for the first time. Lastly, the case when the radiating surface of the cosmic object is intersected by the caustic arising in microlensing, was considered in Refs. 6 and 7. In that case the magnitude of the mutual coherence increases by several more orders of magnitude and reaches realistic values.

 1063-7761/93/100529-10$10.00

2. LENS EQUATIONS, CRITICAL CURVES, AND CAUSTICS

The following approximations are usually made in the description of gravitational lenses: 1) linearized Einstein theory of gravitation; 2) geometrical optics approximation; 3) the approximation of a plane lens and a plane radiating object; 4) the approximation of a small angle of refraction; 5) the description of the photon trajectory by a broken line with the break point in the plane of the lens; 6) the approximation of quasi-monochromatic radiation. Usually the dimensions of the object and the lens are many times smaller than the mutual distances in the gravitator object–observer system. We shall assume in what follows that all these approximations are valid. The question of the validity of the geometrical optics approximation as applied to the problem of mutual coherence was discussed in Ref. 7. Let us introduce into the plane of the object the local system of coordinates ξ,η, and into the plane of the lens—the coordinates x,y. Since all known gravitational lens systems are of cosmological dimensions it is necessary to describe them within the framework of a definite cosmological model. It is natural to assume the Friedmann model. Usually the local coordinate systems are not chosen arbitrarily but in an interdependent manner: a) The coordinate origin x,y is the projection of the coordinate origin ξ,η on the plane of the lens. The projection is by means of a ray of light emitted from the origin ξ,η and falling on the point of observation. b) The orientation of the axes of the coordinates ξ,η and x,y coincide. Generally speaking this condition should be formulated more rigorously in the language of parallel transport of space-like vectors, but here and below we shall adhere to a version in which relativism does not enter explicitly. Let the radiating point on the surface of the object have the coordinates $\zeta=(\xi,\eta)$, and let the ray emitted from this point and falling on the observer intersect the plane of the lens at a point with coordinates $\mathbf{r}=(x,y)$. This point can be viewed as the image of the point $(\xi,\eta$)in the lens system. Therefore the plane of the lens will play at the same time the role of the image plane. If the ray from the point (ξ,η) can fall on the observer by different paths, this means that that point has several images. In the case that both coordinate systems are chosen in an interdependent manner the coordinates of the source and its image are connected with each other by the so-called *lens equation*

$$\xi=\frac{D_s}{D_d}\mathbf{r}+D_{ds}\nabla\psi(\mathbf{r}), \tag{1}$$

where D_s, D_d, D_{ds} are the so-called angular dimension distances in the Friedmann world (see, for example, Weinberg[8]) between source and observer, lens and observer, and source and lens, respectively. The quantity ψ is the scalar potential of the lens, connected with the vector angle of refraction α as follows:

$$\alpha=\nabla\psi(x,y). \tag{2}$$

We note that in the case that the coordinate systems are not correlated the form of Eq. (1) remains unchanged, however the previous potential is replaced by a new one for which relation (2) is not valid. It is convenient to introduce dimensionless coordinates:

$$\hat{\mathbf{r}}=\frac{\mathbf{r}}{R_0};\quad \hat{\xi}=\frac{D_d}{D_s}\frac{\xi}{R_0}. \tag{3}$$

The quantity R_0 is the so-called Einstein–Khvol'son radius, see Ref. 9:

$$R_0=\sqrt{2r_g D_{ds}D_d/D_s}\ , \tag{4}$$

where r_g is the gravitational radius of the lens.

In dimensionless units the lens equation takes the form (the marks over the coordinates are omitted from now on)

$$\xi=\mathbf{r}+\nabla\psi. \tag{5}$$

By means of Eq. (5) the points in the plane of the lens are mapped as the points in the plane of the object. Local mapping is accomplished with the help of the Jacobi matrix:

$$\xi=A\delta\mathbf{r}, \tag{6}$$

$$A=\begin{pmatrix}\partial\xi/\partial x & \partial\xi/\partial y\\ \partial\eta/\partial x & \partial\eta/\partial y\end{pmatrix}=\begin{pmatrix}1+\psi_{11} & \psi_{12}\\ \psi_{21} & 1+\psi_{22}\end{pmatrix}. \tag{7}$$

Here

$$\psi_{ij}=\frac{\partial^2\psi}{\partial x_i\partial y_j}\ (i,j=1,2). \tag{8}$$

We shall consider here lenses corresponding to systems of point masses. Outside the points where the masses are located we have the equality (see, for example, Ref. 10)

$$\psi_{22}=-\psi_{11}. \tag{9}$$

Further we introduce the notation:

$$\psi_{11}=u,\quad \psi_{12}=v. \tag{10}$$

The amplification coefficient Λ of the lens, i.e., the ratio of the radiation flux density received by the observer to the density he would receive in the absence of the lens, equals:

$$\Lambda=\frac{1}{\det A}=\frac{1}{1-u^2-v^2}. \tag{11}$$

The curve in the plane of the lens at each point of which the amplification coefficient Λ becomes infinite is called the critical curve. It follows from (9) that the equation of the critical curve has the form:

$$1-u^2-v^2=0. \tag{12}$$

The curve that is the image of the critical curve in the plane of the object is called the caustic curve or simply the caustic. We note that sometimes a different terminology is used in which the critical curve in the lens plane is called the caustic. Two remarks should be made regarding the caustic. Since the one-to-one correspondence of the map at the points of the critical curve is violated, the smooth map is singular there. The singularity of the smooth map with smallest codimension (curve on a plane) is a fold,[11] hence in the general case the caustic is a fold. The critical curve can sometimes have a point as its map in the plane of the object, i.e., this is the case when the caustic degenerates into an isolated point. This situation takes place in the case of a point-like lens. It is not hard to see that such a singularity is structurally unstable, i.e., for a minuscule change in the parameters in the problem it disappears (or is converted into gathers and folds), since according to the familiar theorem of Whitney the only stable smooth maps of a surface on a plane are gathers and folds (see, for example, Ref. 11).

3. GENERAL PREMISES OF THE THEORY OF MUTUAL INTERFERENCE OF IMAGES IN A GRAVITATIONAL LENS

If one and the same object has several images in the gravitational lens, then in the case of coherent addition of the intensities of the radiation fields coming from all the images, the total flux density I of the radiation equals

$$I=\sum_{i=1}^{n} I_i+2\sum_{i=1}^{n} \sqrt{I_iI_j}|\gamma_{ij}|\cos\varphi_{ij}, \qquad (13)$$

where I_i is the flux density of the ith image, γ_{ij} is the degree of mutual coherence of the i–j pair of images, and φ_{ij} is the phase of that quantity as a complex number. The summation is over all pairs of images. The second term in that formula will be referred to as the interference component of the flux. The degree of mutual coherence γ_{ij} of a pair of images is the crucial theoretical concept in this problem and we discuss it first. By definition (see the monograph by Born and Wolf[12]) it is equal to

$$\gamma_{ij}=\frac{1}{\sqrt{I_iI_j}}\langle V_iV_j^*\rangle, \qquad (14)$$

where $V_{i,j}$ is the intensity of the radiation field of the ith and jth images, respectively, $\langle...\rangle$ indicates averaging over time; V_i is the instantaneous value of the intensity of the radiation field, representing the result of the addition of instantaneous radiation field intensities due to all the radiating elements of the object. The same applies of course to V_j, the difference being that V_i is determined by the radiation propagating along the trajectory of the ith image while for V_j we use the trajectory of the jth image. It will be assumed here that each element of the object radiates statistically independently of each other. Following the discussion in Born and Wolf[12] the degree of mutual coherence of two images can be written in the form

$$\gamma_{ij}=\frac{1}{\sqrt{I_iI_j}}\int\int_\sigma \frac{g(\delta t(\xi,\eta))J(\xi,\eta)\exp(i\omega\delta t(\xi,\eta))}{R_{\rm ph}^{(i)}(\xi,\eta)R_{\rm ph}^{(j)}(\xi,\eta)} \times d\xi d\eta, \qquad (15)$$

where the integration is over the entire radiating surface of the object; $J(\xi,\eta)$ is the surface brightness of the source in the reference frame of the observer; $R_{\rm ph}(\xi,\eta)$ is the photometric distance between the radiating element and the observer taking into account the action of the lens for the ith image;

$$\delta t(\xi,\eta)=t_j(\xi,\eta)-t_i(\xi,\eta)$$

is the relative time delay between the signals propagating along the trajectories of the two images, measured in the reference frame of the observer; the received radiation is quasi-monochromatic with average cyclic frequency ω; $g(\delta t(\xi,\eta))$ is the reduction coefficient of mutual coherence due to a relative phase shift in excess of the coherence time; $g(0)=1$. The radiation flux is given by the integral

$$I=\int\int_\sigma \frac{J(\xi,\eta)}{[R_{\rm ph}^{(i)}(\xi,\eta)]^2} d\xi d\eta. \qquad (16)$$

The theory of the gravitational lens gives a general expression for the photometric distance between the radiating element (ξ,η) and the observer

$$R_{\rm ph}(\xi,\eta)=D_d^{1/2}\left|\frac{D(\xi,\eta)}{D(x,y)}\right|^{1/2}, \qquad (17)$$

where D_d is the distance between the lens and the observer, $D(\xi,\eta)/D(x,y)$ is the Jacobi matrix of the lens. The quantity D_d cancels out in the formula for γ_{ij}. It is convenient to change to the dimensionless coordinates ξ,η and x,y (we do not change the notation). Then the scale factors in the expression for γ_{ij} also cancel out. We can then write in dimensionless variables

$$\gamma_{ij}=\Gamma_{ij}/\sqrt{I_iI_j}, \qquad (18)$$

$$\Gamma_{ij}=\int\int_\sigma \frac{g(\delta t(\xi,\eta))J(\xi,\eta)\exp[i\omega\delta t(\xi,\eta)]}{[|D(\xi,\eta)/D(x,y)_i|D(\xi,\eta)/D(x,y)|_j]^{1/2}} \times d\xi d\eta, \qquad (19)$$

$$\hat{I}_i=\int\int_\sigma \frac{d\xi d\eta}{|D(\xi,\eta)/D(x,y)|_i}. \qquad (20)$$

The quantity Γ_{ij} will be referred to as the mutual-coherence integral. In formula (20) we can pass from integration over the surface of the object to integration over the surface of the images

$$\hat{I}_i=\int\int_\Sigma dxdy. \qquad (21)$$

4. STATIONARY CURVES OF THE MUTUAL-COHERENCE INTEGRAL

The mutual-coherence integral belongs to a class of two-dimensional integrals with a rapidly oscillating factor in the integrand. By "rapidly oscillating" is meant that within the integration region σ the quantity $\omega\delta t(\xi,\eta)$ runs over a large range of values exceeding π many times. The numerical evaluation of such integrals is very complicated while an exact integration is, as a rule, impossible. It is therefore necessary to turn to special asymptotic methods of approximate evaluation of this type of integrals. We make use here of the method of stationary phase. Although this method is a purely calculational tool, its application will permit us to draw a number of principal conclusions about the effect of mutual coherence of images. So let us consider a two-dimensional integral with a rapidly oscillating factor in the integrand

$$J=\int\int_S F(x,y)\exp[i\Omega\Phi(x,y)]dxdy, \qquad (22)$$

where $\Omega\gg 1$ is a dimensionless quantity. The stationary-phase method permits one to obtain an asymptotic expansion of the integral (22) in powers (not necessarily integer) of the quantity $1/\Omega$. The central idea of the method is that the main contribution to the integral comes from the neighborhood of the so-called stationary lines, at each point of which the following equalities are satisfied

$$\frac{\partial\Phi(x,y)}{\partial x}=0, \quad \frac{\partial\Phi(x,y)}{\partial y}=0, \qquad (23)$$

i.e., every point of a stationary line is by definition a critical point of the phase function $\Phi(x,y)$. In addition to stationary lines there can exist stationary isolated points. It is important that the integration over the remainder of the region S, i.e., outside the neighborhood of stationary curves and points, gives a negligible contribution (which can be estimated) to the complete integral. In the evaluation of integrals of the type (22) by the stationary-phase method one determines first the stationary points and curves of this integral in the integration region S. Then there exist many methods for the evaluation of the main part of this integral—the result of integration in the neighborhood of the stationary curves and stationary isolated points. Besides, an estimate of the remainder of the integral is performed. Let us analyze the situation connected with the mutual-coherence integral (19) from the point of view of the ideology of the stationary-phase method. It turns out that we can draw a number of general conclusions about the stationary curves of this integral. The phase function of this integral has the form:

$$\delta t_{ij}=\frac{2r_g}{c}(1+z_d)\delta\left\{\frac{1}{2}(\mathbf{r}-\boldsymbol{\zeta})^2+\psi(r)\right\}_{ij}, \qquad (24)$$

where r_g is the gravitational radius of the whole lens, z_d is its red shift, ζ are the coordinates of the radiating element in the plane of the object, $\mathbf{r}$ are the coordinates of the point of intersection of the lens plane by the trajectory of the light going from that element to the observer; ψ is the scalar potential of the lens; i and j are the numbers of two images. The operation $\delta\{f(x,y)\}_{ij}$ means the following:

$$\delta\{f(x,y)\}_{ij}=f(x_j,y_j)-f(x_i,y_i), \qquad (25)$$

i.e., the difference in the values of the function f at the points of the ith and jth image. Let us determine the extremum points of the phase function $\delta t(\xi,\eta,x(\xi,\eta),y(\xi,\eta))$:

$$\frac{\partial\delta t_{ij}}{\partial\xi}=\frac{2r_g}{c}(1+z_d)\delta\left\{(x-\xi)\left(\frac{\partial x}{\partial\xi}-1\right)+(y-\eta)\frac{\partial y}{\partial\xi}+\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\xi}\right\}_{ij}$$

$$=\frac{2r_g}{c}\delta\left\{\left[x-\xi+\frac{\partial\psi}{\partial x}\right]\frac{\partial x}{\partial\xi}+\left[y-\eta+\frac{\partial\psi}{\partial y}\right]\frac{\partial y}{\partial\xi}-(x-\xi)\right\}_{ij}=-\frac{2r_g}{c}(1+z_d)\delta(x-\xi)_{ij}. \qquad (26)$$

The expressions in the square brackets vanish in view of the lens equation. The quantity $\partial\delta t_{ij}/\partial\xi$ is calculated similarly. As a result we have

$$\frac{\partial\delta t_{ij}}{\partial\xi}=-\frac{2r_g}{c}(1+z_d)\delta x_{ij}, \qquad (27)$$

$$\frac{\partial\delta t_{ij}}{\partial\eta}=-\frac{2r_g}{c}(1+z_d)\delta y_{ij}. \qquad (28)$$

Consequently we have at the stationary points of the mutual-coherence integral (22)

$$\delta x_{ij}=x_i-x_j=0, \quad \delta y_{ij}=y_i-y_j=0. \qquad (29)$$

The formula (29) means that the point (ξ,η) and only the point (ξ,η) of the radiating object will be a stationary point of the integral mutual coherence of two images provided its maps are the same in these images. In brief, the images (x_i,y_i), and (x_j,y_j), of the point (ξ,η) should coincide. Then the point (ξ,η) is a stationary point of the mutual-coherence integral for the pair of images i and j. From this we can draw a more constructive conclusion: the stationary lines of the mutual-coherence integral are the caustic and only the caustic curves of the gravitational lens.

Let us prove this assertion. As is well known, to any point in the neighborhood of a caustic corresponds a pair of images in the neighborhood of the critical curve. These two images are located on different sides of the critical curve and coalesce into a single point on the critical curve in the limit as the radiating point approaches the caustic. In this way every point of the critical curve is the common point of two extended images. Consequently, the critical curve is the boundary of two images of an extended object, intersected by the caustic. It follows hence, in view of Eq. (29), that the caustic is always a stationary line of the mutual-coherence integral. This proves the sufficient part of the assertion. We now pass to the proof of the necessary part. Let a segment of the curve T_1T_2 intersecting the object be a stationary curve of the mutual-coherence integral of two images. The corresponding representation $T_1'T_2'$ of this curve in the plane of the lens is, according to Eq. (29), a common line of this pair of images. We shall also assume that for each of the images this line is not isolated and that the pair of the images have no two-dimensional region in common. It follows then from formula (29) that the segment $T_1'T_2'$ of the curve is the common boundary of these images. We assume, of course, that the derivatives of δt with respect to ξ and η exist in the neighborhood of the curve T_1T_2. Since outside this curve all these derivatives are different from zero, it follows from (29) that every point in the neighborhood of the caustics is represented by two points lying on different sides of the curve $T_1'T_2'$. We now turn to the lens equation. Expanding it at the point P_0' on the curve $T_1'T_2'$, and keeping only the linear terms of the series, we get

$$\xi-\xi_0=\left(\frac{\partial\xi}{\partial x}\right)_0(x-x_0)+\left(\frac{\partial\xi}{\partial y}\right)_0(y-y_0)+O_2, \quad (30)$$

$$\eta-\eta_0=\left(\frac{\partial\eta}{\partial x}\right)_0(x-x_0)+\left(\frac{\partial\eta}{\partial y}\right)_0(y-y_0)+O_2. \quad (31)$$

It follows from the existence of a pair of images for any point in the neighborhood of T_1T_2 that the determinant of this system vanishes

$$\begin{vmatrix}(\partial\xi/\partial x)_0 & (\partial\xi/\partial y)_0\\ (\partial\eta\partial x)_0 & (\partial\eta/\partial y)_0\end{vmatrix}=0. \quad (32)$$

Otherwise the linear system (30)–(31) would have only one solution for $(x-x_0)$ and $(y-y_0)$, in contradiction to the fact that to every point in the plane of the object corresponds a pair of images. It follows from Eq. (32) that P_0' is a critical point. Since similar considerations can be applied to any point of the curve $T_1'T_2'$, it follows that the curve is critical. Since this critical curve is always unambiguously represented by the caustic, it follows that the curve T_1T_2 is the caustic of the lens. This proves the second part of the assertion: only a caustic can be a stationary curve of the mutual-coherence integral of two images in a gravitational lens.

It should be noted that the assertion, that the caustic and only the caustic (naturally with certain restrictions on the class of functions under consideration) is a stationary curve of the corresponding rapidly oscillating integral, is valid in the general case of optical systems (see, for example, Ref. 13). However in the present case (gravitational lens) a simple proof of this fact, technically speaking, can be provided.

5. SOLUTION OF THE LENS EQUATION NEAR THE CRITICAL POINT

To evaluate the integral by the stationary-phase method it is necessary to know the behavior of the integrand in the neighborhood of the stationary line, which is the segment of the critical curve intersecting the object. The solution of the lens equation can be written in general form. Consider the neighborhood of a certain point O, lying on the critical curve C_R in the plane of the object, and the neighborhood of the point O', which is the map of the point O in the plane of the lens. It is assumed that the point O is an ordinary point of the critical curve (not being a return point, and the point O' lies outside the distributed mass of the gravitator). Here we introduce the system of local dimensionless coordinates $\hat{\xi}$, $\hat{\eta}$ with origin at the point O and orientation of the axis $\hat{\xi}$ along the tangent to the critical curve C_R, as well as the local coordinates $\tilde{x},\tilde{y}$ with origin at the point O' and orientation of the axis $\tilde{x}$ along the tangent to the curve C'_R. The universal solution of the lens equation near the critical curve, valid for practically any gravitator, to which correspond critical curves in the plane of the object, is written in the form (see, for example, Ref. 10)

$$\tilde{x}=\mp\frac{B}{A}\sqrt{\frac{u}{\theta_1}A\tilde{\eta}}, \quad (33)$$

$$\tilde{y}=\mp\sqrt{\frac{u}{\theta_1}A\tilde{\eta}}, \quad (34)$$

where all the quantities are expressed in terms of derivatives of the scalar $\hat{\psi}$, taken at the origin of the coordinates $\tilde{x},\tilde{y}$:

$$\tilde{u}=\frac{\partial^2\hat{\psi}}{\partial\tilde{x}^2},\quad v=\frac{\partial^2\hat{\psi}}{\partial\tilde{x}\partial\tilde{y}},\quad \theta_1=\frac{\partial^3\hat{\psi}}{\partial^2\tilde{x}\partial\tilde{y}},$$

$$A=\sqrt{2(1+u)},\quad B=\mathrm{sign}(v)\sqrt{2(1+u)}. \quad (35)$$

The two signs in the formulas for $\tilde{x},\tilde{y}$ correspond to the two branches of the local solution, which gives rise to the formation of a pair of contiguous images of the extended object. Indeed, as the radiating point S tends to the point O, both its images S'_+ and S'_- tend to each other and at the same time to the point O'. It follows hence that the tight pair of images of the object, intersected by the critical curve, will have a common boundary in the form of the segment $T_1'T_2'$ of the critical curve in the plane of the lens. We recall once more that our considerations apply near a fold-type singularity. In addition, a real solution of the lens equation exists only for radiating points from that side of the critical curve C_R for which one has the inequality:

$$\frac{u}{\theta_1}A\tilde{\eta}\geqslant 0.$$

We agree to call this the positive side. For a change in sign of the coordinate $\tilde{\eta}$, i.e., in passage of the radiating surface to the negative side of the curve C_R the solution becomes imaginary, which corresponds to disappearance of images of this point. This circumstance results in only that part of the object, which lies on the "positive" side of the critical curve, participating in the formation of the contiguous pair of images. The remaining part of the object makes no contribution to these images. Therefore the integration in (19) extends in fact only over the "positive" part of the surface of the source.

The relative delay time, the transformation determinant, and the photometric distance between the observer and the radiating element are written in the form

$$\delta t=\frac{16r_g}{3c}(1+z_G)\Phi(\tilde{\xi})\tilde{\eta}^{3/2}, \quad (36)$$

$$D(\tilde{\xi},\tilde{\eta})/D(\tilde{x},\tilde{y})=2\Phi^{-1}(\tilde{\xi})\tilde{\eta}^{1/2}, \quad (37)$$

$$R_{\mathrm{ph}}^{\pm}=\sqrt{2}D_s^2\Phi^{-1/2}(\tilde{\xi})\tilde{\eta}^{-1/4}, \quad (38)$$

where z_G is the red shift of the lens;

$$\Phi(\tilde{\xi})=\sqrt{\frac{u}{\theta_1}A\tilde{\eta}}; \quad (39)$$

in these formulas $\tilde{\xi}$ can be considered to be the geometric length along the critical curve.

6. CALCULATION OF THE DEGREE OF MUTUAL COHERENCE

The integration region in the formula for mutual coherence can be broken up mentally into two parts. One of them, $\Delta\sigma_0$, represents a narrow strip adjacent to the critical curve. Within this strip the approximate Eqs. (36)–(38) are valid. Its width is not rigorously determined. It is only necessary that the phase factor $\exp\{i\omega\delta t\}$ undergo a sufficiently large number of oscillations upon displacement along the coordinate line $\tilde{\eta}$ from the critical curve to the strip boundary. In line with the above remark, integration over this region gives the main contribution to the degree of coherence, while the result of integration over the remaining part of the surface of the source can be neglected. With these considerations in mind and as a result of passage to dimensionless variables in the integral of formula (15), the replacement $\chi=[\Phi(\tilde{\xi})]^{2/3}\tilde{\eta}$ and passage from a double to a repeated integral we obtain

$$\gamma_{ij}=\frac{1}{2\sqrt{\hat{I}_i\hat{I}_j}}\int_0^{\chi_*} f(\chi)\exp(i\Omega\chi^{3/2})\chi^{-1/2}d\chi, \quad (40)$$

$$f(\chi)=\int_{\tilde{\xi}_2(\chi)}^{\tilde{\xi}_1(\chi)} J(\tilde{\xi},\chi)\Phi^{2/3}(\tilde{\xi})d\tilde{\xi}, \quad (41)$$

where $\hat{I}=(D_d/R_0)^2$, Ω is a dimensionless constant:

$$\Omega=\frac{16r_g}{3c}(1+z_G)\omega;$$

$\tilde{\xi}_1(\chi)$ and $\tilde{\xi}_2(\chi)$ are the end points of the radiation region in the coordinate line χ; χ_* is the coordinate "width" of the region $\Delta\sigma_0$. With the replacement $\rho=\chi^{1/2}$ we are faced with the integral

$$I=\int_0^{\rho_*} F(\rho^2)\exp(i\Omega\rho^3)d\rho. \quad (42)$$

To calculate this integral we make use of an approach proposed by Gilmore.[14] We suppose that

$$F(\rho^2)=c_0+c_2\rho^2+c_4\rho^4+\ldots . \quad (43)$$

Integrating term by term we have reduced the problem to the calculation of integrals

$$\int_0^{\rho_*}\rho^{2k}\exp(i\Omega\rho^3)d\rho$$
$$=\Omega^{-(1/3+2k/3)}\int_0^{\Omega^{1/3}\rho_*} y^{2k}\exp(iy^3)dy. \quad (44)$$

We make the change of variable $y=\Omega^{1/3}\rho$. In the case that $\Omega\gg 1$ it is necessary to evaluate the integral

$$I_{2k}=\int_0^{\infty} y^{2k}\exp(iy^3)dy. \quad (45)$$

We consider the integral in the complex plane over a closed contour consisting of a sector of a circle of radius R centered at the origin of the coordinates. We take the initial angle to be $\varphi_0=0$, the final angle to be $\varphi_f=\pi/6$. Since the function $y^{2k}\exp(iy^3)$ has no poles in this region the integral over this region vanishes and, on the other hand,

$$0=I=\int_0^R x^{2k}\exp(ix^3)dx+\int_0^{\pi/6} R^{2k+1}$$
$$\times\exp[i(2k+1)\varphi]\exp[iR^3(\cos 3\varphi+i\sin 3\varphi)]id\varphi$$
$$+\int_R^0 R^{2k}\exp\left(\frac{2ik\pi}{6}\right)\exp\left[iR^3\left(\cos\frac{\pi}{2}\right.\right.$$
$$\left.\left.+i\sin\frac{\pi}{2}\right)\right]\exp\left(\frac{i\pi}{6}\right)dR. \quad (46)$$

It is not hard to see that

$$\int_0^{\pi/6} R^{2k+1}\exp[i(2k+1)\varphi]$$
$$\times\exp[iR^3(\cos 3\varphi+i\sin 3\varphi)]id\varphi\to 0, \quad (47)$$

as $R\to\infty$. Indeed, let us break up this integral into two integrals

$$\int_0^{\pi/6} R^{2k+1}\exp[i(2k+1)\varphi]$$
$$\times\exp[iR^3(\cos 3\varphi+i\sin 3\varphi)]id\varphi=\int_0^{\varphi_*}\ldots+\int_{\varphi_*}^{\pi/6}\ldots, \quad (48)$$

where the angle φ_* will be chosen below. It is clear that

$$\left|\int_{\varphi_*}^{\pi/6} R^{2k+1}\exp[i(2k+1)\varphi]\right.$$
$$\left.\times\exp[iR^3(\cos 3\varphi+i\sin 3\varphi)]id\varphi\right|\leqslant R^{2k+1}$$
$$\times\exp(-R^3\sin 3\varphi_*)\pi/6\to 0, \quad (49)$$

as $R\to\infty$

$$\left|\int_0^{\varphi_*} R^{2k+1}\exp[i(2k+1)\varphi]\right.$$
$$\left.\times\exp[iR^3(\cos 3\varphi+i\sin 3\varphi)]id\varphi\right|\leqslant R^{2k+1}\varphi_*, \quad (50)$$

and if we choose $\varphi_*=1/R^{2k+2}$ then the last integral also tends to zero as $R\to\infty$. Consequently

$$I_{2k}=\exp\left[i(2k+1)\frac{\pi}{6}\right]\int_0^{\infty} R^{2k}\exp(iR^3)dy,$$
$$=\exp\left[i(2k+1)\frac{\pi}{6}\right]\cdot\frac{1}{3}\Gamma\left(\frac{2k+1}{3}\right), \quad (51)$$

where $\Gamma(x)$ is the Euler gamma function. In this way we obtain

$$I=\sum_{k=0}^{\infty} c_{2k}\exp\left[i(2k+1)\frac{\pi}{6}\right]\cdot\frac{1}{3}\Gamma\left(\frac{2k+1}{3}\right), \quad (52)$$

where c_k are the coefficients in the expansion of the function $F(\rho)$ in a Maclaurin series. Therefore accurate up to the first term in the expansion we have the following expression for the mutual coherence

$$\gamma_{12}^{(1)}=\frac{1}{3\sqrt{\hat{I}_{+}\hat{I}_{-}}}\exp\left(\frac{i\pi}{6}\right)f(0)\Gamma\left(\frac{1}{3}\right)\times\left[\frac{16r_g}{3c}(1+z_G)\omega\right]^{-1/3}. \tag{53}$$

If we take two terms of the expansion into account, we obtain

$$\gamma_{12}^{(2)}=\gamma_{12}^{(1)}+\frac{1}{3\sqrt{\hat{I}_{+}\hat{I}_{-}}}\exp\left(\frac{i\pi}{6}\right)f'(0)\times\left[\frac{16r_g}{3c}(1+z_G)\omega\right]^{-7/3}. \tag{54}$$

With the help of the Erdélyi lemma the expression (53) was obtained in Ref. 6. The expression for $\gamma_{12}^{(1)}$ can also be obtained by making use of the asymptes of the integral:

$$F(\lambda)=\int\int_{R^2}f(x,y)\exp[i\lambda(yx^2-y^3)]dxdy, \tag{55}$$

where the function $f(x,y)$ has compact support. The asymptote of this integral is calculated in Ref. 15.

We call attention to the frequency dependence of the mutual coherence: $\gamma_{12}^{(1)}\propto\omega^{-1/3}$. This circumstance "ensures" the increase of the mutual coherence by several orders of magnitude in comparison with a point gravitator $\gamma_{12}\propto\omega^{-1}$.[5] In Ref. 6 an expression is given for the mutual coherence of a radiating disk of uniform brightness and small dimensions (the disk radius $R_Q\ll R_0$) from (53) one can deduce the following formula:

$$\gamma_{12}=\left(\frac{3}{16}c\right)^{1/3}\exp\left(\frac{i\pi}{6}\right)\Gamma\left(\frac{1}{3}\right)\left(\frac{D_sD_{ds}}{2D_d}\right)^{1/4}\left(\frac{u}{A\theta_1}\right)^{-1/6}\times\frac{\sqrt{1-s^2}}{(1-s)K(\sqrt{(1+s)/2})+2sE(\sqrt{(1+s)/2})}\times r_g^{-1/12}R_Q^{-1/2}\omega^{-1/3},$$

$$s\in[-1,+1], \tag{56}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, s is the distance from the center of the disk to the critical curve, expressed in units of the disc radius. We have $s>0$ when the center of the disc is on the "negative" side of the critical curve, and $s<0$ if it is on the "positive" side. The close vicinity of the point $s=+1$ is excluded, as it corresponds to a situation where only a narrow sickle of the disc participates in the formation of the image. In that case the method of stationary phase is not applicable and, consequently, the corresponding relations are not valid. For $s=1$ the degree of coherence within the framework of this approximation vanishes. Note in formula (56) the very weak dependence of the degree of coherence on the mass of the lens: $\gamma_{12}\propto r_g^{-1/12}$. The dependence on the size of the object is substantially stronger: $\gamma_{12}\propto R_Q^{-1/2}$.

7. THE CASE OF THE TWO-POINT LENS

The choice of the two-point lens for quantitative calculations of the effect of mutual coherence of contiguous images is explained by the fact that this is, on the one hand, the simplest case of a lens with critical curves in the plane of the source and, on the other hand, as was already mentioned above, it is a typical case of a broad class of lenses (see, for example, the remark on this subject in Ref. 10). It is also appropriate to note here that, in contrast to this case, the single-point lens occupies a special place, since for it the critical curve in the plane of the source degenerates into a point. It is precisely for this reason that the degree of mutual coherence of images for the single-point gravitator has extremely small values.[4,5] The surface mass density in the two-point lens is given by

$$\rho(x,y)=M_1\delta(x-p,y)+M_2\delta(x+p,y), \tag{57}$$

where M_1 and M_2 are the masses of the components of the gravitator; the radius vectors of these masses are equal to $\mathbf{p}_1=\{p,0\}$, $\mathbf{p}_2=\{-p,0\}$. The vector of the angle of refraction calculated from formula (2) is given in the present case in the form

$$\boldsymbol{\alpha}(\mathbf{r})=-r_{g_1}\frac{\mathbf{r}-\mathbf{p}_1}{(\mathbf{r}-\mathbf{p}_1)^2}-r_{g_2}\frac{\mathbf{r}-\mathbf{p}_2}{(\mathbf{r}-\mathbf{p}_2)^2}, \tag{58}$$

where $\mathbf{r}$ and $\mathbf{p}$ are two-dimensional vectors, and the quantities r_{g_1} and r_{g_2} are the gravitational radii of masses M_1 and M_2. The scalar function $\hat{\psi}$ of the lens in dimensionless variables[3] can be written as follows:

$$\hat{\psi}(\hat{\mathbf{r}})=-\frac{\mu_1}{2}\ln(\hat{\mathbf{r}}-\hat{\mathbf{p}}_1)^2-\frac{\mu_2}{2}\ln(\hat{\mathbf{r}}-\hat{\mathbf{p}}_2)^2, \tag{59}$$

where $\mu_1=M_1/(M_1+M_2)$, $\hat{\mathbf{p}}_1=\mathbf{p}_1/R_0$, $\mathbf{p}_2=\mathbf{p}_2/R_0$. The lens equation takes the form

$$\hat{\boldsymbol{\xi}}=\hat{\mathbf{r}}-\mu_1\frac{\mathbf{r}-\mathbf{p}_1}{(\mathbf{r}-\mathbf{p}_1)^2}-\mu_2\frac{\mathbf{r}-\mathbf{p}_2}{(\mathbf{r}-\mathbf{p}_2)^2}, \tag{60}$$

and the critical curves in the plane of the lens are described by the equations[4]

$$\hat{x}^2=\frac{1}{4\hat{p}^2}\left[\frac{1}{2}+(\hat{r}^2+\hat{p}^2)^2-\sqrt{\frac{1}{4}+2(\hat{r}^4+\hat{p}^4)}\right], \tag{61}$$

$$\hat{y}^2=\frac{1}{4\hat{p}^2}\left[-\frac{1}{2}-(\hat{r}^2-\hat{p}^2)^2+\sqrt{\frac{1}{4}+2(\hat{r}^4+\hat{p}^4)}\right]. \tag{62}$$

Let us suppose that we have a source in the form of a round disk, intersected by the critical curve. We then have in the plane of the lens a pair of contiguous images of the source with common boundary in the form of a segment of the critical curve $T_1'T_2'$. According to remarks made previously the radiation from that part of the disk which lies to the right side of the critical curve does not participate in the formation of the given pair of images. Although the here-presented picture in terms of dimensionless variables can correspond to a continuous set of dimensional parameters, the calculations were performed for the following set, corresponding to the following realistic situation:

A) cosmological decelerating parameter $q_0=1/2$;
the Hubble constant $H_0=55$ km/(secMpc);
the quasar (object) red shift $z_Q=1.25$;
the lens red shift $z_G=0.2$;

B) component masses of the double star (lens) $M_1=M_2=M_\odot$;
distances between the lens components $2p=R_0$ ($\hat{p}=0.5$);
radius of the disk of the quasar nucleus $R_Q=1.5\cdot 10^{16}$ cm.

Here R_0 is the radius of the effective Einstein ring of the lens. It should be noted that since the gravitational lens does not change the surface brightness of the object, the ratio of the areas of the images to the area of the source gives the lens magnification of the flux densities for the case of a uniformly radiating disk.

The degree of mutual coherence for a pair of images was calculated from formulas (53)–(56). Here the group B of parameters was varied, while the parameters from group A were kept constant. The core of the quasar was represented by a circle of uniform brightness. Its center was translated along the straight line $\hat{\eta}=0.4$. The location of the disk was determined by the quantity s, equal to the distance of the center from the critical curve, expressed in units of the radius of the nucleus. In the extreme left location the disk was tangent to the critical curve from the left side ($s=-1$), in the extreme right location—from the right side ($s=1$). In Fig. 1 we show graphs of $|\gamma_{12}(s)|$ for the case $M_1=M_2=M_\odot$ for a disk radius 10^{16} cm, respectively, in the optical ($\lambda=4000$ Å) and radio ($\lambda=18$ cm) regions. We note first of all the sharp increase in the degree of coherence as the object disk is shifted from left to right. Near the point $s=1$ the mutual coherence reaches its largest values. Calculations were not carried out in the immediate neighborhood of this point, since in that case the number of oscillations of the phase factor $\exp(i\omega\delta t)$ between the limits of the part of the disc forming the image turns out to be too small and the application of the stationary phase method is not correct.

As can be seen in Fig. 1 the modulus of the degree of coherence reaches tens of percent in the radio region. This result exceeds by several orders of magnitude the values obtained in Refs. 3–5 and permits a realistic prospect for the detection of this effect. However in the optical region under the same circumstances the coherence is measured in fractions of a percent. In Fig. 2 we demonstrate the case $M_1=M_2=0.01\ M_\odot$, $R_Q=10^{13}$ cm, when the mutual coherence reaches a few percent.

The dependence of the degree of coherence on the wave length of the radiation λ is of principal significance. In Fig. 3 we show the graph of the dependence of $|\gamma_{12}(\lambda)|$ for the case $M_1=M_2=M$, $R_Q=10^{15}$ cm for coordinates of the center of the disk $\hat{\xi}_0=0.15$, $\hat{\eta}_0=-0.4$. This graph demonstrates the fact that the degree of coherence varies widely with varying wavelength. This should have as a direct consequence variations in the overall spectrum of the quasar core over the period of intersection of the latter by the critical curve of the microlens. This is connected

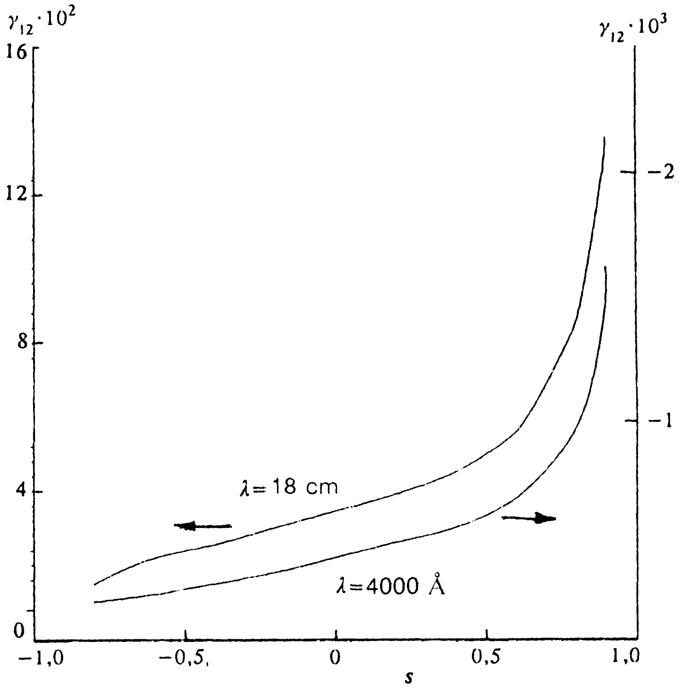

FIG. 1. Modulus of the degree of mutual coherence of contiguous images of the quasar core in a two-point lens ($M_1=M_2=M_\odot$) as a function of the location of its center (radius of the quasar core $R_Q=10^{14}$ cm).

with the fact that the degree of coherence enters into the formula for the overall radiation flux density. Such variations in the continuous spectrum of the quasar core are one of two observable manifestations of the effect of mutual

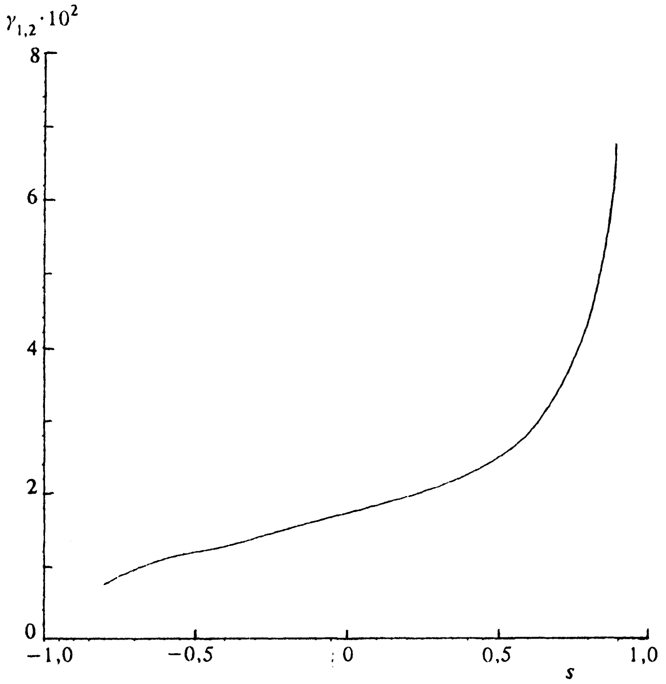

FIG. 2. Modulus of the degree of mutual coherence of contiguous images of the quasar core in a two-point lens ($M_1=M_2=0.01M_\odot$) as a function of the location of its center (radius of the quasar core $R_Q=10^{13}$ cm). Optical region ($\lambda=4000$ Å).

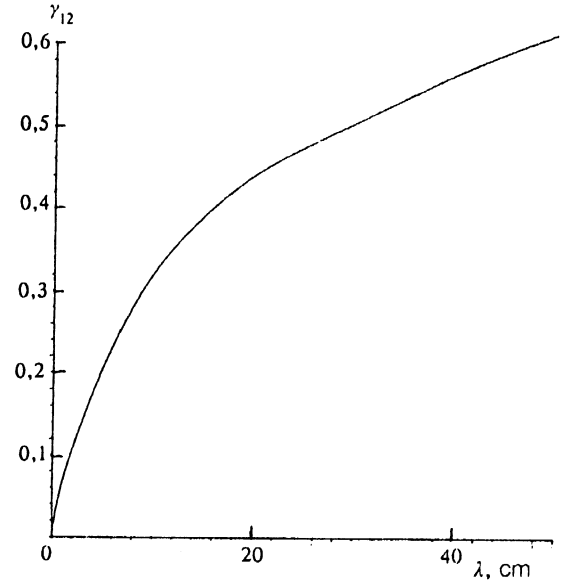

FIG. 3. Modulus of the degree of mutual coherence of contiguous images of the quasar core in a two-point lens ($M_1=M_2=M_\odot$) as a function of the wave length of the radiation (radius of the quasar core $R_Q=10^{15}$ cm, coordinates of the center of the disk $\xi_0=0.159$, $\eta_0=-0.4$).

coherence of images of the microlensing quasar. It should be noted here that there exists another reason for the variation of the spectrum over the microlensing period: the dependence of the lens amplification coefficient on the wave length due to different dimensions of the core in different frequency regions. This question requires a special study.

As can be seen from formula (56) and the graphs, the effect of mutual coherence depends substantially on the dimensions of the source. Here it should be kept in mind that what is relevant is not necessarily the entire core but a separate bright detail on its surface. In order for such a detail to indeed give rise to an appreciable effect of mutual coherence, it is necessary that its glitter at the instant of intersection with the critical curve should be comparable with the glitter of the core against the background of the entire quasar.

8. DISCUSSION

First of all it is necessary to call attention to the so-called coherence time of the radiation under consideration. Connected with it is the question of the validity of the application of the stationary-phase method and the validity of this work in general. The coherence time $\delta\tau$ is not an independent characteristic of the source but is determined by the acceptance width of the receiver according to the formula:[12]

$$\delta\nu\delta\tau \simeq \frac{1}{4\pi}. \tag{63}$$

If for the given element of the source the relative retardation time δt does not exceed the coherence time $\delta\tau$

$$(\delta t \leqslant \delta\tau), \tag{64}$$

then at the point of the observer the two radiation fluxes from the pair of images will add up coherently; if instead $\delta t \gg \delta\tau$ then the addition will be incoherent. That part of the surface of the source for which the inequality (64) is satisfied can be called the "coherence zone." This zone represents a strip near the critical curve, since δt vanishes on the critical curve. The width of this strip is expressed in terms of the width of the spectral strip with the use of formulas (36), (63) and (64). The application of the stationary phase method is correct if the number of oscillations N_ω of the phase factor is sufficiently large. It follows from formulas (63) and (64) that the relative width of the transmission strip is expressed in terms of N_ω as follows:

$$\delta\lambda/\lambda \simeq (4\pi N_\omega)^{-1}. \tag{65}$$

For example, with $N_\omega=50$ for $\lambda=18$ cm the width of the strip needed from the point of view of the coherence time equals $\Delta\nu \approx 3$ MHz, and for $\lambda=4000$ Å we need $\Delta\lambda \approx 8$ Å. The width of the transmission strip can turn out to be inadequate from the point of view of sensitivity of the apparatus. The entire transmission range can be broken up into substrips of satisfactory width from the point of view of the coherence time. Coherent reception takes place within the boundaries of each substrip. Then the fluxes of all substrips are added up incoherently. If the images of the quasar are mutually coherent then the results of the observations in such a multi-channel setting will differ from the results of the observations of the same quasar over the whole range. This effect is the second expected manifestation of mutual coherence of the images of the microlensing quasar.[1)] Let us also indicate a possible observable manifestation of a sizeable coefficient of mutual interference. In that case, as is not hard to see, the intensity of the radiation near the caustic will be significantly different from the sum of the intensities of different images [in accordance with relation (13)], which could be one of the reasons for the observed variability of quasars. The study of the variability of quasars and its connection with microlensing has been performed previously using the method of imitation modeling (see, for example, the dissertation by Wamsganss[16]), thus the discussed in this work asymptotic approach substantially complements the numerical analysis of Ref. 16.

As a consequence of the proper relative motion of the observer, the intermediate galaxy and the quasar, a trend of the latter relative to the multiple curves of microlensing occurs and is caused by the stars of the galaxy. For the problem studied in this paper it is important to know how often intersections of the critical curves by the quasar occur. An answer to this question is given by model calculations.[17,18] According to these calculations, in the passage of the quasar radiation through the galaxy zones located sufficiently close to its center (as is the case for the quasar Q 2237+030 or the image of the quasar Q 0957 +561), such events take place once every few years, i.e., often enough from the point of view of possible observation. As regards the duration of the "transit" of the quasar core through the critical curve, that depends on the dimensions of the latter and could amount to between a few weeks to a few years.

In conclusion we thank V. S. Imshennik and D. K. Nadezhin for interest in the work, and to G. Beskin and V. L. Oknyanskiĭ for discussions. We also thank R. Kayser, T. Schramm, and V. Faraoni for sending us preprints used in the work on this article, and also P. Schneider for sending us materials from the monograph, Ref. 9, prior to publication, in which were discussed applications of the method of catastrophe theory in problems of gravitational lensing. We also thank the reviewer for valuable critical remarks, which made possible an improved presentation of the results.

One of the authors (A.F.Z.) expresses deep gratitude to the American Physical Society and the International Science Fund for partial support of this work.

[1)]The possibility of such a multi-channel reception in this problem was suggested by V. L. Oknyanskiĭ and V. K. Dubrovich.

[1]S. Refsdal, Month. Not. Roy. Ast. Soc. **128**, 295 (1964).
[2]A. F. Zakharov, Preprint ITEP 6, Moscow (1993).
[3]A. V. Mandzhos, Pis'ma v Astron. zh. **7**, 387 (1981).
[4]P. Schneider and J. Schmid-Burgk, Astron. and Astroph. **2**, 369 (1985).
[5]O. P. Verkhoglyadova and A. V. Mandzhos, Preprint Inst. Theoret. Phys. Acad. Sci. Ukr. SSR. N89-4E, Kiev (1989).
[6]A. V. Mandzhos, Astron. zh. **68**, 22 (1991).
[7]A. V. Mandzhos, Astron. zh. **68**, 236 (1991).
[8]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[9]P. Schneider, J. Ehlers, and E. Falco, *Gravitational Lensing* (Springer, New York, 1992).
[10]P. Schneider and A. Weiss, Astron. and Astroph. **164**, 237 (1986).
[11]V. I. Arnol'd, *Catastrophe Theory* (Springer-Verlag, Berlin, 1984).
[12]M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1969).
[13]V. I. Arnol'd, A. N. Varchenko, and S. M. Gusein-Zade, *Singularities of Differentiable Maps, II* (Birkhauser, Boston, 1984).
[14]R. Gilmore, *Catastrophe Theory for Scientists and Engineers* (Wiley, New York, 1981).
[15]M. V. Fedoryuk, *The Saddle-point Method* [Nauka, Moscow, 1977] [in Russian].
[16]J. Wamsganss, *Gravitational Microlensing, Dissertation der Fakultät für Physik der Ludwig-Maximilians-Universität*, München (1990).
[17]B. Paczynski, Astrophys. J. **301**, 503 (1986).
[18]R. Kayser, S. Refsdal, and R. Stabel, Astron. and Astroph. **166**, 36 (1986).

Translated by Adam M. Bincer