Action for anomaly in Fermi superfluids: quantized vortices and gap nodes

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A general expression for the topological Novikov–Wess–Zumino (NWZ) term in the hydrodynamic action of Fermi superfluids is derived. It can be applied both to systems with gap nodes in momentum space, such as superfluid ^{3}He-*A*, and to conventional superconductors with quantized vortices. The action is expressed in terms of the volume of the 6-dimensional phase space $\vec{k}$, $\vec{r}$ enclosed by the multidimensional vortex sheet swept by Green's function singularities. While in superfluid ^{3}He-*A* this action describes the chiral anomaly in 3+1 dimensions, which results from the gap nodes, in the case of vortices the NWZ term results from the spectral flow along the anomalous branches of the fermions, localized in the vortex core. At low temperatures the flow of states through $E=0$ determines the parameter D' characterizing the reactive force between the vortex and the system of normal fermions (heat bath). This reversible momentum exchange between the coherent condensate motion in three dimensions and the 1-dimensional motion of localized fermions is equivalent to the Callan–Harvey process of anomaly cancellation. The number of the anomalous branches of fermionic zero modes on vortices is related to the vortex winding number.

1. INTRODUCTION

The hydrodynamic action for the bosonic collective (Goldstone) modes of ordered Fermi systems, such as superfluids and superconductors, is well defined only if there are no singularities in the fermionic Green's function, i.e., if the fermionic spectrum has a gap everywhere in momentum ($\vec{k}$) and coordinate ($\vec{r}$) spaces. Some of the singularities in the Green's function are of topological origin, i.e., are described by the topological invariants in real space, in momentum space or in the extended $\vec{k}$, $\vec{r}$ space. These are (i) Fermi surfaces in normal metals and superconductors, which are stable topological defects in momentum space;[1] (ii) Fermi points or gap nodes as in the *A*-phase of ^{3}He (see review 2); (iii) quantized vortices which are topologically stable linear defects in real space; and (iv) combined defects in the extended $\vec{k}$, $\vec{r}$ space, appearing, e.g. in the cores of ^{3}He-*B* vortices (in a core with spontaneously broken parity, the singularity on the vortex axis is smoothed by the formation of gap nodes in the vortex core; see review 3). More exotic topological defects resulting from the topology in the extended $\vec{k}$, $\vec{r}$ space can be found in Ref. 4.

As in the phenomenon of chiral anomaly in relativistic quantum field theory,[5] in the presence of gap nodes there is a flow of the fermionic energy levels through the nodes, leading to anomalies in the hydrodynamics of superfluid ^{2}He-*A* at low temperature.[6–8] In this case the effective hydrodynamic action for the coherent (superfluid) subsystem becomes ill defined, since part of the action transfers through the gap nodes into the incoherent (normal) degrees of freedom. The anomalous part of the action, which describes the superfluid-normal exchange, is the topological Novikov–Wess–Zumino (NWZ) action.[9–13]

As was found earlier, the gap nodes in $\vec{k}$ space and the quantized vortices in $\vec{r}$ space are particular cases of more-general multidimensional vortices in the extended $\vec{k}$, $\vec{r}$ space (phase space).[14] Therefore one can expect that there is a general expression for the anomaly in the action which can be applied both to systems with gap nodes in momentum space and to systems with quantized vortices in real space. Here we derive such a generalization of the NWZ action, which comes from the Berry's phase.[15] describing the fermionic spectrum in the classical approximation in which $\vec{k}$ and $\vec{r}$ are considered as independent parameters. In the case of the $\vec{k}$-space vortices the NWZ action describes the anomaly in the hydrodynamics of ^{3}He-*A* discussed earlier,[6–11] while in the case of the $\vec{r}$-space vortices the NWZ action describes the anomaly related to the spectral flow along the anomalous branch of low-energy fermions localized on vortices; this branch was found in Ref. 16. The number of such anomalous branches is related to the vortex winding number;[17] a similar (but not equivalent) relation occurs for the strings in relativistic quantum field theory.[18,19]

The spectral flow along the anomalous branches of the fermions localized in the core of a quantized vortex in a Fermi superfluid determines at low temperature the parameter D', characterizing the reactive force between the vortex and the system of normal (thermally activated) fermions and entering the force balance for the moving vortex:

$$\rho_s(\vec{v}_s-\vec{v}_L)\times\vec{\varkappa}+D(\vec{v}_n-\vec{v}_L)-D'(\vec{v}_n-\vec{v}_L)\times\vec{\varkappa}=0. \tag{1.1}$$

Here $\vec{\varkappa}$ is the circulation vector, given $\varkappa=n\eta\pi\hbar/m$, where n is the vortex winding number (number of circulation quanta) and m is the bare mass of the fermion.

The first term is the conventional Magnus force which arises when the velocity $\vec{v}_L$ of the vortex is different from

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the velocity of the condensate $\vec{v}_s$ (superfluid velocity), and ρ_s is the superfluid density far from the vortex, which is close to the total density ρ at low temperature $T \ll T_c$. This force is due to the flux of the linear momentum from the vortex to infinity.

The second term is the friction force acting on the vortex when it moves with respect to the normal component (heat bath), while the last term is the reactive force which also arises when $\vec{v}_L$ deviates from the normal velocity $\vec{v}_n$. At low T the reactive force describes the reversible exchange of linear momentum between the coherent condensate motion in three dimensions and the 1-dimensional motion of localized fermions, this exchange being mediated by the spectral flow in the vortex core. This process is the realization of the Callan–Harvey phenomenon of anomaly cancellation[20] applied to the vortex.

The result for D' appears to be general; i.e., it does not depend on the vortex core structure or on the type of pairing. This corresponds to some modification of the Luttinger theorem[21] for the Fermi surface or gap nodes existing within the vortex core. The parameter D' equals the volume in momentum space confined by the singular surface. This is either the volume within the Fermi surface which appears on the vortex axis in conventional superconductors or the volume of momentum space within the surface spanned by the gap nodes which appear in the core of vortices in superfluid ^{3}He-B. This result is in agreement with the earlier calculations made for vortices in s-wave superconductors,[22] ^{3}He-B[23] and ^{3}He-A.[13,24]

In all these cases the anomalous NWZ action for vortices with winding number n is

$$S_{\mathrm{NWZ}}=\frac{1}{3}2\pi n\frac{V_{\vec{k}}}{(2\pi)^3}\int dt d\sigma\, \vec{r}\cdot\partial_t\vec{r}\times\partial_\sigma\vec{r}, \qquad (1.2)$$

where $\vec{r}(t,\sigma)$ is the vortex line position in terms of the coordinate σ along the line, and $V_{\vec{k}}$ is the corresponding volume in momentum space. In the simplest case of spherical symmetry,

$$\frac{2V_{\vec{k}}}{(2\pi)^3}=\frac{k_F^3}{3\pi^2},$$

where k_F is the magnitude of the momentum for which zeroes in the spectrum take place. The variation of action $\delta S_{\mathrm{NWZ}}/\delta\vec{r}$ gives the last reactive force in Eq. (1.1) with the parameter

$$D'=m\frac{2V_{\vec{k}}}{(2\pi)^3}. \qquad (1.3)$$

Equation (1.2) has the same form as the regular contribution to the action for the vortices [25]:

$$S_{\mathrm{reg}}=\frac{1}{3}\varkappa\rho_s\int dt d\sigma\, \vec{r}\cdot\partial_t\vec{r}\times\partial_\sigma\vec{r}, \qquad (1.4)$$

which gives the Magnus force in Eq. (1.1). As distinct from this regular term which exists both in Bose and Fermi superfluids, the NWZ action is related to the spectral flow of fermionic levels and therefore exists only in Fermi superfluids. It disappears if the spectral flow is suppressed.

In Secs. 3 and 4, using the gradient expansion of the fermionic Green's function, we derive the general NWZ term in the action from which the particular cases follow: the anomalous contribution to the hydrodynamics of ^{3}He-A with the gap nodes and the anomaly in the vortex motion in conventional superconductors and in ^{3}He-B. In Sec. 4 an analog of the Luttinger theorem is discussed, related to the volume of the 6-dimensional phase space inside the multidimensional vortex sheet spanned by the Berry's phase singularity during the vortex dynamics. In Sec. 5 the anomaly in the vortex motion is derived using the phenomenon of spectral flow along the anomalous branches of fermions in the vortex core.

2. GRADIENT EXPANSION FOR THE HYDRODYNAMIC ACTION IN TERMS OF A GREEN'S FUNCTION

The standard form of the hydrodynamic action is

$$S=\int\frac{dtd\omega}{2\pi}\mathrm{Tr}\ln \mathrm{G}=\int\frac{dtd\omega}{2\pi}d\tau \mathrm{Tr}\, \mathrm{G}\partial_\tau \mathrm{G}^{-1}, \qquad (2.1)$$

where Tr is the trace over all quantum states, and τ is an extra variable introduced to avoid the logarithm, with $\tau=1$ corresponding to the real physical world. The gradient expansion holds if the Bose fields are slow in time and/or in space. If they are slow in time, one has

$$S=\frac{1}{2}\mathrm{Tr}\int d\tau\frac{dtd\omega}{2\pi}\mathrm{G}\partial_\tau \mathrm{G}^{-1}(\mathrm{G}\partial_t \mathrm{G}^{-1}\mathrm{G}\partial_\omega \mathrm{G}^{-1} - \mathrm{G}\partial_\omega \mathrm{G}^{-1}\mathrm{G}\partial_t \mathrm{G}^{-1}, \qquad (2.2)$$

where G is a local function of time.

If we also consider the texture which is slow in space, we can neglect the spacing between the local discrete levels of fermions in the field of the order parameter of the texture. Then we have

$$\mathrm{Tr}=\mathrm{tr}\int\frac{d^3kd^3r}{(2\pi)^3}, \qquad (2.3)$$

with tr denoting the trace over the spin, Bogoliubov spin, and band indices, and the action can be expressed in terms of the spatially local Green's function $\mathrm{G}(t,\omega,\vec{k},\vec{r},\tau)$.

In this classical approximation the quantized vortices and gap nodes are described in the same manner. This can be seen from the simple BCS model in which the Hamiltonian for fermionic quasiparticles is described by the 2×2 Bogoliubov matrix:

$$\mathrm{G}^{-1}=i\partial_t-\mathbf{H},\quad \mathbf{H}=\begin{pmatrix}\varepsilon(\vec{k},\vec{r}) & \Delta(\vec{k},\vec{r})\\ \Delta^*(\vec{k},\vec{r}) & -\varepsilon(\vec{k},\vec{r})\end{pmatrix}. \qquad (2.4)$$

Here

$$\varepsilon(\vec{k},\vec{r})=k^2/2m-\mu,\quad \mu=k_F^2/2m$$

is the chemical potential, and $\Delta(\vec{k},\vec{r})$ is the gap function. In the first case, of conventional s-wave pairing in the presence of a quantized vortex with the winding number n of the phase $\Phi(\vec{r})$ around the vortex, the gap function is

$$\Delta(\vec{k},\vec{r})=|\Delta(\vec{r})|\exp i\Phi(\vec{r})\propto(x+iy)^n, \qquad (2.5)$$

where the vortex axis is chosen along z. In the second case, of spatially homogeneous Cooper pairing into the state with n-fold gap node, the gap has the same structure as in Eq. (2.5), but in momentum space

$$\Delta(\vec{k},\vec{r}) = |\Delta(\vec{k})| \exp i\Phi(\vec{k}) \propto (k_x+ik_y)^n. \qquad (2.6)$$

These two examples represent different orientations of the same multidimensional vortex singularity in phase $\Phi(\vec{k},\vec{r})$ in the extended $\vec{k},\vec{r}$ space.[14] The real-space and momentum-space vortices thus can transform into each other; this takes place, e.g., in the core of the B-phase vortex.[3,14] The quasiparticle spectrum

$$E=\sqrt{\epsilon(\vec{k},\vec{r})^2+|\Delta(\vec{k},\vec{r})|^2}$$

vanishes at $k=k_F$, $x=0$, $y=0$ in the case of the real-space vortices and at $k=k_F$, $k_x=0$, $k_y=0$ for the momentum-space gap nodes. These zeroes are the intersections of the vortices in the gap with the former Fermi surface $k=k_F$ and represent the diabolical points of the fermionic spectrum where the quasiparticle branch touches the quasihole branch.[26]

3. CONTRIBUTION OF THE GREEN'S FUNCTION SINGULARITIES TO THE EFFECTIVE ACTION

Let us consider the contribution of the real-space vortices and the gap nodes representing vortices in momentum space to the hydrodynamic action. Equation (2.2) represents the 3-form which also describes the topological defects of π^3 homotopy group (or the π_2 group if we consider $\omega=0$). These point defects in $3D$ space, or $5D$ defects in the $8D$ space $(\tau,t,\vec{k},\vec{r})$, are the diabolical points of the fermionic spectrum where the quasiparticle branch touches the quasihole branch.[26]

It is more convenient to represent the anomalous contribution to the integral over surface (t,τ) in terms of the singularities in the Berry's phase[15] $\Phi(\vec{k},\vec{r},t,\tau)$. The singularities in the Berry's phase $\Phi(\vec{k},\vec{r},t,\tau)$ occur at the cuts which terminate on the diabolical points. Just in the same manner the string terminates on the Dirac magnetic monopole, and the doubly quantized singular vortex in the A-phase terminates at the hedgehog in the $\vec{l}$ field (see review[2]). Around the a-th cut the phase Φ changes by $2\pi n$, n being the winding number of the defect. For the cross derivatives of the Berry's phase on the cuts, one has

$$(\delta^{(1)}\delta^{(2)}-\delta^{(2)}\delta^{(1)})\Phi=\mathrm{tr}\int \frac{d\omega}{2\pi}(G\delta^{(1)}G^{-1}G\delta^{(2)}G^{-1} -G\delta^{(2)}G^{-1}G\delta^{(1)}G^{-1})G\partial_\omega G^{-1}, \qquad (3.1)$$

which means that according to Eq. (2.2) the anomalous part of the action is

$$S_{\mathrm{NWZ}}=\tfrac{1}{2}\,\mathrm{Tr}\int_{\mathrm{cuts}} dt d\tau(\partial_\tau\partial_t-\partial_t\partial_\tau)\Phi. \qquad (3.2)$$

Let us apply this first to the gap nodes. We assume that at $\tau=0$ there are no singularities, e.g., the chemical potential becomes negative, $\mu(\tau=0)<0$. Therefore no gap nodes in the fermionic spectrum exist until some critical value τ_c at which pairs of gap nodes with opposite topological charges appear. Since the cuts are absent at $\tau=0$, at $\tau>\tau_c$ they should be chosen to connect the pairs of nodes. In this case all the cuts are inside the former Fermi surface of the normal system. In the simple model of Eq. (2.4) we may choose these cuts as the segments of the multidimensional vortices discussed in Ref. 14 ($6D$ vortices in $8D$ space), which are situated within the former Fermi surface, i.e., at $k<k_F$. The different orientations of these vortices in the $\vec{k},\vec{r}$-space correspond either to real-space vortices or to the cuts between the gap nodes.

Let us denote by $\vec{k}^{(a)}(\vec{r},t,\tau,\sigma)$ the time- and space-dependent positions of the cut which terminates at the a-th pair of boojums in momentum space, σ being the coordinate along the cut. From the equation

$$(\partial_\tau\partial_t-\partial_t\partial_\tau)\Phi=2\pi n\int d\sigma\,\delta[\vec{k}-\vec{k}^{(a)}(\vec{r},t,\tau,\sigma)]\partial_\tau\vec{k}^{(a)}\cdot\partial_\sigma\vec{k}^{(a)}\times\partial_t\vec{k}^{(a)} \qquad (3.3)$$

one obtains the anomalous action

$$S_{\mathrm{NWZ}}=\frac{1}{4\pi^2}n\int d^3r dt d\tau d\sigma \partial_\tau\vec{k}^{(a)}\cdot\partial_\sigma\vec{k}^{(a)}\times\partial_t\vec{k}^{(a)}. \qquad (3.4)$$

It has the meaning of the volume in momentum space.

We apply this result first to the A-phase of ^{3}He, which contains two gap nodes at $\vec{k}=\pm k_F\vec{l}$, where $\vec{l}$ is the unit vector of the orbital momentum. The nodes have the topological charges ± 2 (± 1 per each spin component). Choosing the cut with winding number $n=2$ as the straight line between the nodes:

$$\vec{k}^{(a)}(\sigma,t,\tau)=\sigma\vec{l}(t,\tau),$$

where σ changes from $-k_F$ to k_F, one obtains the familiar result for the anomalous term in the dynamics of the $\vec{l}$ vector:[2,13]

$$S_{\mathrm{NWZ}}\{\vec{l}(\vec{r},t)\}=\int d^3r dt d\tau\frac{V_{\vec{k}}}{(2\pi\hbar)^3}\vec{l}\cdot\partial_\tau\vec{l}\times\partial_t\vec{l},\quad \frac{2V_{\vec{k}}}{(2\pi)^3}=\frac{k_F^3}{3\pi^2}. \qquad (3.5)$$

The same equation, (3.4), should be obtained if one considers the motion of real-space vortices with winding number n. In this case

$$(\partial_\tau\partial_t-\partial_t\partial_\tau)\Phi=2\pi n\int d\sigma\delta[\vec{r}-\vec{r}^{(a)}(\vec{k},t,\tau,\sigma)]\partial_\tau\vec{r}^{(a)}\cdot\partial_\sigma\vec{r}^{(a)}\times\partial_t\vec{r}^{(a)}, \qquad (3.6)$$

where $\vec{r}^{(a)}(\vec{k},t,\tau,\sigma)$ is the position of the vortex line, and one has

$$S_{\mathrm{NWZ}}\{\vec{r}_a\}=\frac{1}{2\pi^2}n\int d^3k dt d\tau d\sigma\partial_\tau\vec{r}^{(a)}\cdot\partial_\sigma\vec{r}^{(a)}\times\partial_t\vec{r}^{(a)}. \qquad (3.7)$$

We again assume that at $\tau=0$ there are no singularities even inside the vortex core; e.g., the chemical potential is negative at $\tau=0$, and no Fermi surface exists until some critical value τ_c, at which the Fermi surface appears on the vortex axis. Therefore the cuts $\vec{r}^{(a)}(\vec{k},t,\tau,\sigma)$ are concentrated only within the Fermi surface. In this region the variables $\vec{r}^{(a)}$ do not depend on $\vec{k}$, and therefore the integral over $\vec{k}$ gives the volume $V_{\vec{k}}$ inside the Fermi surface, which is $4\pi k_F^3/3$ for the spherically symmetric case. The variation of Eq. (3.7) over $\vec{r}^{(a)}(t,\tau,\sigma)$,

$$\delta S_{\rm NWZ}\{\vec{r}_a\}=\pi n\frac{2V_{\vec{k}}}{(2\pi)^3}\int dt d\sigma \delta\vec{r}^{(a)}\cdot\partial_\sigma\vec{r}^{(a)}\times\partial_t\vec{r}^{(a)}$$

$$=n\frac{\pi\hbar}{m}D'\int dt d\sigma\delta\vec{r}^{(a)}\cdot\partial_\sigma\vec{r}^{(a)}\times\partial_t\vec{r}^{(a)},$$

$$D'=m\frac{2V_{\vec{k}}}{(2\pi)^3}, \quad (3.8)$$

gives the reactive force on the vortex from the normal component

$$\frac{\delta S_{\rm NWZ}}{\delta\vec{r}^{(a)}}=\kappa D'\partial_\sigma\vec{r}^{(a)}\times\partial_t\vec{r}^{(a)}, \quad \kappa=n\frac{h}{2m}. \quad (3.9)$$

This is, in a sense, the Luttinger theorem applied to a vortex. At $T\ll T_c$ the parameter D'equals the volume inside the Fermi surface, which appears in the core of the vortex. To elucidate this theorem for the general case let us consider the change of the anomalous action in the adiabatic process during which the system finally returns to the initial state. Examples are as follows: (i) a closed vortex loop is virtually created from the vacuum and then vanishes again; (ii) a vortex line moves along some path and returns to its initial position; (iii) gap nodes are created and annihilated, etc. We consider first the process (i). As follows from Eqs. (3.8) or (1.2) the change in action is expressed in terms of the volume $V_{\vec{r}}$ inside the surface spanned by the vortex loop in real space:

$$\Delta S_{\rm NWZ}=2\pi n\hbar\frac{V_{\vec{k}}V_{\vec{r}}}{(2\pi\hbar)^3}. \quad (3.10)$$

This expression can be easily modified for the general case of the dynamics of a multidimensional vortex, including the gap node motion. The change in action is proportional to the number N of quantum states within the volume of the phase space $V_{\vec{k},\vec{r}}$ inside the singular surfaces spanned by vortex sheet in $6D$ space:

$$\Delta S_{\rm NWZ}=\pi\hbar nN, \quad N=\frac{2V_{\vec{k},\vec{r}}}{(2\pi\hbar)^3}. \quad (3.11)$$

This shows that in such a process a certain part of the superfluid action is transferred into the incoherent degrees of the fermionic system due to the flow of states through the vortex sheet in the phase space. Since N is even due to double degeneracy of states (two spin components), the quantity $\exp\{iS_{\rm NWZ}/\hbar\}$ does not change in the process of virtual creation of a vortex loop in real space or of a multidimensional vortex sheet in the phase space. The situation is different for the system with one spin projection, such as the A_1=phase of ^{3}He. In this case the number N can be odd and $\exp\{iS_{\rm NWZ}/\hbar\}$ can change sign.

This result is somewhat different from that obtained by Haldane and Wu,[27] who stated that the phase change contains the *mean number of superfluid particles* enclosed by the surface swept by the vortex loop. Their consideration is related to the regular term in action, Eq. (1.4), which gives the conventional Magnus force and is also valid for Bose superfluids (e.g., superfluid ^{4}He). We consider an additional phase change coming from the chiral anomaly term related to the spectral flow in the vortex core. Such an anomaly exists only in Fermi superfluids, That is why it is the number of fermionic quantum states, which gives the anomalous (topological) contribution to the phase change. This number can be very different from the mean number of superfluid particles.

Above we considered the classical spectrum of fermions and neglected the quantum (discrete) character of the levels localized in the texture of the $\vec{l}$ vector in the case of the gap nodes or within the vortex core in the case of quantized vortices. Below we take into account the discreteness of levels and show explicitly how the process of spectral flow of quantum states in the vortex core results in the linear momentum anomaly described by the NWZ action. The linear momentum anomaly appears to exist only if the relevant energy scales are larger than the interlevel spacing of localized fermions, which is of order T_c^2/E_F in conventional vortices, while at the lower energy scale the process of level flow is suppressed and $D'\to 0$.

4. CALLAN-HARVEY EFFECT FOR VORTICES

In axially symmetric quantized vortices in Fermi superfluids the energy levels $E(n_r,n_l,k_z)$ of the fermionic quasiparticles localized in the vicinity of the vortex core are characterized by the quantum numbers appropriate for the axial symmetry,[27] namely, linear momentum along the vortex axis, k_z, orbital angular momentum n_l and radial quantum number n_r. The distance between levels with different n_r and that between levels with different n_l at given n_r are characterized by two energy scales, ω_r and ω_l correspondingly. In the singular vortex ω_r is of order of the gap parameter $\Delta\sim T_c$, while $\omega_t\sim\Delta^2/E_F\ll\Delta$.[16] The vortex dynamics and thermodynamics are essentially different in the two regions of low temperatures $\Delta\gg T\gg\omega_l$ and $T<\omega_l$.

We show here that the results of the previous sections are valid only in the region $\Delta\gg T\gg\psi_l$. On this scale there exist one or several anomalous (chiral) branches of localized fermions, corresponding to $n_r=0$. The spectrum $E(n_r=0,n_l,k_z)$ forms the a with nearly equidistant levels $E(n_r=0,n_l,k_z)\approx\omega_l(k_z)n_l$, which as a function of (discrete) n_l "crosses" zero energy and thus produces a finite density of states $N(\omega)$ if $\omega\gg\omega_l$.[16] Due to the discrete nature of levels there is no real crossing of zero energy, but this is unimportant if one considers an energy scale larger than ω_l. [Whether the spectrum $E(n_r,n_l,k_z)$ really crosses zero as a function of k_z depends on the vortex core structure.[28]] So we assume that either ω or the energy level width τ^{-1} is larger than the interlevel distance ω_l between the n_l levels, and in this case the energy spectrum can be considered as a continuous function of n_l.

Since the result does not depend on details and is completely determined by topology, we consider here the simplest (and well-known) case of an axisymmetric singular vortex with a single circulation quantum $n=1$ in a superfluid or superconductor with s-wave pairing. The orbital quantum number n_l is considered here as a continuous variable, so one can use the quasiclassical approximation for the fermions localized in the vortex core. The fermions with the given spin projection are described by the Bogoliubov–Nambu Hamiltonian obtained from Eq. (2.4) by the substitution $\vec{k}=\vec{q}-i\vec{\nabla}$ with $\vec{q}^2=k_F^2$:

$$H=\hat{\tau}_3\vec{q}\cdot(-i\vec{\nabla})/m+\hat{\tau}_1\mathrm{Re}\Delta(\vec{r}_\perp)-\hat{\tau}_2\mathrm{Im}\Delta(\vec{r}_\perp), \quad (4.1)$$

where the gap function in the axisymmetric vortex with winding number n is given by Eq. (2.5).

In this description we shall use instead of k_z the magnitude of the transverse momentum of a quasiparticle

$$q_\perp=|\vec{q}_\perp|=\sqrt{k_F^2-k_z^2},$$

and instead of the orbital quantum number n_l the continuous impact parameter

$$\tilde{y}=\hat{z}\cdot(\vec{r}\times\vec{q}_\perp)/q_\perp ,$$

which is related to the orbital angular momentum by $\hbar n_l=q_\perp\tilde{y}$. Introducing the coordinate $\tilde{x}=\vec{r}\cdot\vec{q}_\perp/q_\perp$ along $\vec{q}_\perp$ such that $r_\perp^2=\tilde{y}^2+\tilde{x}^2$ and assuming that in the important regions one has $|\tilde{y}|\ll|\tilde{x}|$, one obtains the dependence of the gap function in the singly quantized vortex ($n=l$) on $\tilde{x}$ and $\tilde{y}$,

$$\Delta(\vec{r}_\perp)\approx|\Delta(|\tilde{x}|)|\left(\mathrm{sign}(\tilde{x})-i\frac{\tilde{y}}{|\tilde{x}|}\right), \quad (4.2)$$

and the Hamiltonian:

$$\mathbf{H}=\mathbf{H}^{(0)}+\mathbf{H}^{(1)},$$

$$\mathbf{H}^{(0)}=-i\hat{\tau}_3\frac{q_\perp}{m}\partial_{\tilde{x}}+\hat{\tau}_1|\Delta(|\tilde{x}|)|\mathrm{sign}(\tilde{x}),$$

$$\mathbf{H}^{(1)}=\hat{\tau}_2\tilde{y}\frac{|\Delta(|\tilde{x}|)|}{|\tilde{x}|}. \quad (4.3)$$

The Hamiltonian $\mathbf{H}^{(0)}$ is supersymmetric and has zero eigenvalue corresponding to $n_r=0$ with the eigenfunction

$$\Psi^{(0)}\propto(1-\hat{\tau}_2)\exp-\frac{m}{q_\perp}\int_0^{|\tilde{x}|}dr_\perp|\Delta(r_\perp)|. \quad (4.4)$$

Using the first order in perturbation, $\mathbf{H}^{(1)}$, one obtains the lowest energy levels:

$$E(n_r=0,n_l,k_z)$$
$$\approx\langle 0|\mathbf{H}^{(1)}|0\rangle$$
$$=-\tilde{y}\left\langle\frac{|\Delta(|\tilde{x}|)|}{|\tilde{x}|}\right\rangle$$
$$=-n_l\omega_l(q_\perp),$$

$$\omega_l(q_\perp)=\frac{1}{q_\perp}\frac{\int_0^\infty dr_\perp|\Psi^{(0)}(r_\perp)|^2\frac{|\delta(r_\perp)|}{r_\perp}}{\int_0^\infty dr_\perp|\Psi^{(0)}(r_\perp)|^2}. \quad (4.5)$$

This is the anomalous branch of the low-energy localized fermions obtained in Ref. 16. If the energy spectrum is considered as a continuous function of n_l, this anomalous branch crosses zero at $n_l=0$ (for more complicated vortices the crossing occurs at finite n_l; this takes place for the ^{3}He-B vortices with broken parity and broken axial symmetry[1]) and in the ^{3}He-A vortices with broken parity[24]). If one takes into account the spin degrees of freedom, there are two anomalous branches or fermionic zero modes corresponding to two spin projections. It is shown below that the number of such branches, N_{zm}, is completely determined by the number n of circulation quanta, i.e., $N_{\mathrm{zm}}=2n$. A similar relation, $N_{\mathrm{zm}}=n$, between the number of fermionic zero modes in the core of the string and the string winding number n holds in the relativistic quantum field theories.[18,19] The difference is that in the core of the string the spectrum of the chiral fermionic modes resulted from the nontrivial topological number n of the string exactly crosses zero as a function of continuous parameter k_z. In condensed matter vortices the corresponding anomalous branches of topological origin are functions of discrete orbital quantum number n_l. They look continuous only on the scale $\omega\gg\omega_l$. (Branches of the localized fermions which cross zero as a function of k_z can also exist in condensed matter vortices, but the number of such branches is not connected with the topology of vortex line; it depends on the detailed structure of the vortex core, such as the core radius and the symmetry of the order parameter field within the core.[28] These branches are not important here, since they are not of topological origin and do not contribute to the NWZ action.)

Now let us consider a vortex moving with the velocity $\vec{v}_L$ relative to the heat bath. In this case the coordinate $\vec{r}_\perp$ is replaced by $(\vec{r}_\perp-\vec{v}_Lt)$; as a result the parameter $\tilde{y}$ entering the quasiparticle energy in Eq. (4.5) is also shifted with time:

$$E(n_r=0,n_l,k_z,t)$$
$$=-\left(\tilde{y}-\frac{\epsilon(\vec{q})}{q_\perp}t\right)q_\perp\,\omega_l(q_\perp)$$
$$=-(n_l-\epsilon(\vec{q})t)\omega_l(q_\perp), \quad (4.6)$$

where $\epsilon(\vec{q})=\hat{z}\cdot(\vec{v}_L\times\vec{q})$ acts on fermions localized in the core in the way that electric field acts on fermions localized on a string in relativistic quantum field theory. Under this field there is a steady flow of levels across $E=0$, and the number of the fermionic levels crossing zero per unit time is

$$\partial_t n_l=\epsilon(\vec{q})=\hat{z}\cdot(\vec{v}_L\times\vec{q}). \quad (4.7)$$

The net linear momentum $\vec{P}$ transferred per unit time from the vacuum (from the levels below zero) along the anomalous branch to the heat bath is thus

$$\partial_t\vec{P}=\sum\vec{q}\partial_t n_l$$

$$=\frac{1}{2}N_{\rm zm}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\vec{q}\epsilon(\vec{q})$$

$$=\pi n\frac{k_F^3}{3\pi^2}\hat{z}\times\vec{v}_L. \tag{4.8}$$

The factor 1/2 compensates for the double counting of particles and holes. This gives the D' parameter in the force acting from the normal component if the vortex moves with respect to the normal heat bath:

$$D'=m\frac{k_F^3}{3\pi^2}=m\frac{2V_{\vec{k}}}{(2\pi)^3}. \tag{4.9}$$

Here it is implied that all the quasiparticles, created from the negative levels of the vacuum state, immediately become part of the normal component, i.e., there is a nearly reversible transfer of linear momentum from fermions to the heat bath. This should be valid in the limit of large scattering rate: $1/\tau\gg\omega_l$. The small retardation in this process leads to the effective friction force $D\propto\omega_l\tau D'$ (see Refs. 22–24). In the opposite limit, when the characteristic energy scale is less than the interlevel spacing, the transition between the discrete levels vanishes, and $D'\to 0$.[22–24]

Note also that the parameter D', though being very close to the Fermi liquid density ρ, nevertheless is not exactly equal to ρ. The mass density equals $m(k_F^3/3\pi^2)$ only in the normal Fermi liquid where the Luttinger theorem is valid, while in superfluids ρ is not equal to $m(k_F^3/3\pi^2)$ but is close to this value in the limit of small gap, $\Delta\ll E_F$. For a singular vortex in s-paired superfluids and superconductors the parameter D' corresponds to the density at the vortex axis, where the gap is zero and thus the Luttinger theorem does hold:

$$D'=\rho(r=0)=m\frac{k_F^3}{3\pi^2}.$$

It is is important however that, as follows from the general arguments of previous sections, k_F here is the Fermi momentum of the Fermi surface that exists on the vortex axis; therefore this k_F coincides with the Fermi momentum of bulk Fermi liquid above T_c only within our BCS model.

The general relation between the number of fermionic zero modes, $N_{\rm zm}$, and the winding number n can be reproduced using our simplified model. The number of the anomalous branches of the spectrum $E(\tilde{y})$ which cross zero as a function of $\tilde{y}$, coincides with the number of topological zeroes of the classical energy $E(\tilde{y},\tilde{x},p_{\tilde{x}})$. The classical limit of Hamiltonian (4.1) is expressed in terms of the vector function $\vec{m}(\vec{s})$ in the $3D$ space of parameters $\vec{s}=(\tilde{y},\tilde{x},p_{\tilde{x}})$:

$$\hat{H}_{\rm class}(\vec{s})=\vec{\tau}\cdot\vec{m}(\vec{s}). \tag{4.10}$$

The components of $\vec{m}(\vec{s})$ are

$$m_3(\vec{s})=q_\perp p_{\tilde{x}}/m,$$

$$m_1(\vec{s})={\rm Re}\Delta(\tilde{x},\tilde{y}),$$

$$xm_2(\vec{s})=-{\rm Im}\Delta(\tilde{x},\tilde{y}). \tag{4.11}$$

The number of zeroes of this vector function [points $\vec{s}_0$ where $\vec{m}(\vec{s}_0)=0$ and therefore $E(\vec{s}_0)=0$] and thus the number of anomalous branches is given by the topological invariant [2]:

$$N_{\rm zm}=\frac{1}{8\pi}\int dS^i e_{ikl}|\vec{m}(\vec{s})|^{-3}\left(\vec{m}\cdot\frac{\partial\vec{m}}{\partial_{s_{\hat{k}}}}\times\frac{\partial!\vec{m}}{\partial s_1}\right), \tag{4.12}$$

where the integral is taken over the closed surface σ about zeroes. For the gap function

$$\Delta(\vec{r})=e^{in\phi}|\Delta(r_\perp)|$$

in the vortex with winding number n one obtains $N_{\rm zm}=n$, which should be multiplied by two if one takes into account two spin projections. Thus the general relation, which does not depend on the detailed structure of the vortex, is

$$N_{\rm zm}=2n. \tag{4.13}$$

Note that the quantity $N_{\rm zm}$ changes sign for vortices with negative winding number. This is because the quantity $N_{\rm zm}$ is an algebraic quantity, since it also shows whether $E(n_l)$ increases or decreases when crossing zero energy is crossed.

The general NWZ action in Eq. (3.2) describes the process of the transfer of linear momentum from the vacuum of Fermi superfluids to the normal motion of fermions. At low temperatures this process corresponds to the Callan–Harvey phenomenon of anomaly cancellation.[20] The anomaly in 3+1 dimensions—nonconservation of the linear momentum of 3-dimensional superfluid Bose-condensate—is cancelled by the opposite anomaly in the 1+1 dimensional world of the normal chiral fermions which exist in the vicinity of the topological singularity of the fermionic Green's function. In the case of the hydrodynamics of ^{3}He-A these low-energy chiral fermions are concentrated in the vicinity of the gap nodes. The 1-dimensional character of these fermions was employed in Ref. 7 to introduce this analogy with the Callan–Harvey effect.

In the case of the vortex motion it is a 1+1 dimensional anomaly in the dynamics of the $1D$ fermions in the vortex core which compensates for the 3+1 dimensional anomaly. This superfluid-normal exchange gives rise to reactive force between the vortex and the heat bath. This force looks very similar to the Magnus force but is essentially different. The Magnus force exerted on a vortex by the superfluid motion results from the reversible flux of linear momentum from the vortex to infinity. On the contrary the reactive force related to the Callan–Harvey effect is a consequence of the reversible flux of momentum from the vortex into the region near the vortex axis, i.e., into the core region. Within the core the linear momentum of the vortex transforms to the linear momentum of $1D$ fermions when the fermionic levels on anomalous branches cross the chemical potential. Due to the quasiparticle scattering, this momentum is further transferred to the heat bath of thermal fermions or to the walls of the container, thus producing the reactive force between the vortex and the normal component of the liquid.

This version of the Callan–Harvey effect does not depend on the detailed structure of the vortex core or even on the type of pairing, provided the important energy scales are larger than the interlevel spacing of the fermions localized in the vortex core. The effect is the same for the singular and continuous vortices and depends only on the number of fermionic zero modes, as expressed through the topological winding number n of the vortex according to Eq. (4.13). The topological result (4.13) for the number of anomalous branches which as functions of the impact parameter $\tilde{y}$ cross zero energy remains valid for any superfluids, in particular, for vortices in the p-wave superfluids of ^{3}He–A and ^{3}He-B. In the case of singly quantized vortices in ^{3}He-B two anomalous branches have been obtained by Schopohl (see footnote[1)]). While for the most symmetric vortex they cross zero at $\tilde{y}=0$ as in Eq. (4.5), for the vortex with the broken symmetry in the core the crossing occurs at finite $\tilde{y}$. This however does not change the (4.7) for the spectral flow or (4.9) for D'.

The similarity of the anomaly in the vortex dynamics and the anomaly in the dynamics of the gap nodes in superfluid ^{3}He-A is a consequence of the topology in the extended space (phase space). The same kind of intercoupling of the real-space and momentum-space topologies has been found in 2-dimensional films of superfluid ^{3}He-A, where the effective Chern-Simons action determining the quantum statistics of solitons was obtained. This term contains topological invariants in both real and momentum spaces.[29] This is just the conventional reduction of the Wess–Zumino term to lower dimensions. Since a homogeneous film of ^{3}He-A and a vortex in a 2-dimensional film of a conventional superconductor are just different particular cases of multidimensional vortices in the phase space, one can expect that the reduced NWZ action should also determine the quantum statistics of $2D$ vortices. The possibility of different quantum statistics for $2D$ vortices was discussed in Refs. 30 and 27. However, Ref. 30 and 27 took the phenomenological approach using the conventional dynamics of vortices based on the Magnus force, which is insensitive to the fermionic degrees of freedom in the vortex core. It seems that only the microscopic phenomena of the chiral anomaly occurring in the core vortices in Fermi superfluids should be responsible for the quantum statistics of vortices.

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