Fluctuation of number of photons in a multimode laser

A. V. Kozlovskiĭ

Lebedev Physics Institute, Russian Academy of Sciences
(Submitted 29 June 1992; resubmitted 27 April 1993)
Zh. Eksp. Teor. Fiz. **104**, 2995–3019 (September 1993)

Quantum calculations of the photon-number fluctuations in the cavity of a multimode laser are performed. Both the total number of photons and the number of photons in each individual field mode are stabilized near the lasing threshold but, unlike in a single-mode laser, the fluctuations and their dependence on the excess above the lasing threshold depend principally on the multimode-laser parameters. Calculations show that the number of photons in an individual mode as well as the total number of photons in the cavity obey quantum statistics when the decay rate of the lower laser level exceeds greatly that of the upper.

INTRODUCTION

One premise of the quantum theory of lasers is that the number of photons in a cavity is not a determined quantity. The photon-number fluctuations are governed by three factors—vacuum fluctuations, spontaneous emission, and pump noise. Quantum theory predicts a decrease of the photon-number fluctuations in one laser mode above the lasing threshold, i.e., as the generated-field intensity increases the photon-number statistics changes.[1] Near the lasing threshold the emission has threshold Bose-Einstein statistics with a stationary variance

$$\langle \Delta n^2 \rangle = n_0^2 + n_0,$$

where n_0 is the stationary expectation value of the number of photons. Above the threshold the photons tend to Poisson statistics with a variance

$$\langle \Delta n^2 \rangle \to n_0,$$

i.e., the intensity becomes stabilized.

The suppression of the photon-number fluctuations in a cavity under was considered in a number of theoretical studies[2–6] of the interaction between the active medium and one quantum-field spatial mode. An appreciable excess above threshold was assumed and the presence of the so-called regular pumping that does not contribute to the fluctuations. These studies have shown that regular pumping results in sub-Poisson statistics ($\langle \Delta n^2 \rangle < n_0$), both in the case of a gas laser[2,3] and in the case of a semiconductor laser.[2–6]

The questions of the photon-number fluctuations and statistics in a mode of a multimode laser and of the total number of photons in the cavity of a multimode laser have so far not been sufficiently investigated. Among the first to deal with this question was McCumber.[7] Assuming that the lower laser level of a four-level laser system is empty, he supplemented the system of kinetic equations with shot-noise sources and obtained in the linear approximation an expression for the spectral parameters of the intensity fluctuations:

$$\langle \Delta I_{qq}^2(\tau) \rangle = \langle a_q^+(0) a_q^+(\tau) a_q(\tau) a_q(0) \rangle - \langle a_q^+ a_q \rangle^2,$$

where a_q^+ (a_q) are creation (annihilation) operators of the qth normal mode of the field.

A fundamental difference was observed between the quantum intensity fluctuations in a mode of a multimode levels and the fluctuations of a one-mode laser. It was noted in Ref. 7 that, in view of the nonlinearity of the kinetic equations, the quantum fluctuations of the total intensity in all the lasing modes are stabilized if more than one mode is strongly pumped. In an individual mode, however, the field intensity is not stabilized.

Mandel *et al.*[8–10] studied theoretically the intensity fluctuations in a ring laser. They considered the semiclassical Van der Pol equations, without allowance for four-wave interaction, for two counterpropagating modes whose frequencies coincide with the frequency of the atomic transition, i.e., in a situation typical just for a ring laser. The simplified equations were supplemented with random sources by relating the laser field fluctuations to the spontaneous emission. Combined field-intensity probability densities were obtained analytically by changing from the stochastic Van-der-Pol equations to the Fokker–Planck equation, and the total-intensity fluctuations as well as the intensity fluctuations in the modes were calculated. The calculations have shown that the low-intensity mode resulting from mode competition has, in the limit of strong intermode coupling, fluctuations indicative of a random thermal field at a large excess above threshold, whereas the fluctuation in a high-intensity mode decreases as the pump is increased. In Refs. 9 and 10 are reported theoretical predictions, confirmed by experiment, for the case of exactly equal frequencies. No such phenomenon was observed in the case of a finite detuning.

The theoretical calculations of Ref. 8 were generalized to include the case of an arbitrary number of modes by Hioe,[11] who used analogous Van-der-Pol equations and model random sources in the limit of strong mode coupling. The fluctuations in the modes, under conditions of strong and equal pumping of all the modes, are close to the thermal fluctuations typical of black-body radiation, whereas the total intensity becomes stabilized as in the single-mode case. Note that in the limit of independent modes there are no mode fluctuations at all in the model used by Hioe[11] and Mandel.[8–10]

 1063-7761/93/090393-13$10.00

In another study known to us, of the role fluctuations in multimode lasing, McMackin *et al.*[12] used the Maxwell-Bloch quasiclassical equations with Langevin noise sources. The mode intensities were determined numerically with account taken of the spatial structure of the field. Allowance for intermode interaction in a dye laser leads in their theory to irregular intensity dynamics in the field modes. The calculations have shown that the main cause of this behavior is that the equations contain not random sources that simulate quantum fluctuations, but four-level mixing. The field-intensity individual-mode irregularity observed in Ref. 12 was therefore named "deterministic fluctuations. McMackin *et al.* have shown that the mode intensities remain regular if there are no four-wave mixing terms in the stochastic equations for the field amplitudes. Similar irregular field-amplitude oscillations in the modes of a laser with inhomogeneously broadened line shape were observed earlier by Brunner *et al.*[13] who likewise used a semiclassical theory but with no allowance for noise. They have concluded that the amplitudes in the field modes of a laser with inhomogeneously broadened line are random in the case of strong coupling and competing modes (i.e., when the natural linewidth is much smaller than or of the order of the intermode spacing).

The results of Ref. 12 and their comparison with Ref. 13 lead to the conclusion that the irregular (random) field oscillations observed in laser modes near the lasing threshold are not due to quantum fluctuations but to strong intermode coupling and mode competition. On the other hand, ring-laser studies[8–10] point to a decisive role of quantum fluctuations in the behavior of the intensity of one of the counterpropagating modes above the threshold. A consistent quantum-mechanical calculation of the photon-number fluctuations in laser modes above the threshold would therefore permit an estimate of the role of quantum fluctuations in multimode lasing, would therefore be of interest. We use here therefore the quantum-mechanical Lax-Louisell-Langevin formalism to calculate the photon-number fluctuations in the cavity of a multimode laser.

1. HAMILTONIAN

According to the Lax-Louisell theory,[14–16] the active-medium-atom subsystem as well as the electromagnetic-field subsystem, which interact with one another, each undergoes disspations and fluctuations because it interacts with a corresponding statistically independent reservoir that is in thermodynamic equilibrium. We shall assume that the reservoir can be completely eliminated from the equations of motion for the atom+field system operators by introducing into the dissipation coefficients into the effective equations of motion for the quantities describing the system, to which are added fluctuation sources by using a Markov approximation and perturbation theory. The influence of the reservoirs on the behavior of the system-operator reservoirs averaged over a statistical ensemble is thus taken into account by introducing into the equations of motion the damping constants and transition probabilities, i.e., the system dissipation parameters.

We begin with the premise that the complete Hamiltonian consists of the system Hamiltonian H_S, the Hamiltonians R_A and R_F of the atomic and field subsystems, and of the interaction between them:

$$H=H_S+R_A+R_F+V_{A-R}+V_{F-R}. \tag{1}$$

It is assumed, without loss of generality, that the interaction between the subsystems and the reservoirs are described by operators of the form

$$V=\sum_i Q_iF_i, \tag{2}$$

where F_i and Q_i are respectively operators pertaining to the system and reservoir, respectively.

We expand the atom-reservoir interaction operator in terms of basis operators made up of the eigenvectors of the atoms

$$V_{A-R}=\hbar\sum_{m,n} f_{mn}|m\rangle\langle n|, \tag{3}$$

where f_{mn} is an operator pertaining to the atom reservoir. The field-reservoir interaction operator is similarly chosen, in accordance with (2), in the form

$$V_{F-R}=\sum_q \left(\frac{\hbar}{2\omega_q}\right)^{1/2} i(a_q-a_q^+)(2\hbar\omega_q)^{1/2}(g_q+g_q^+)$$
$$\approx i\hbar\sum_q (a_q^+g_q-a_qg_q^+), \tag{4}$$

where a_q^+ is the creation operator of the normal mode of the cell $A\times B\times C$, and g_q is the reservoir operator; both operators are connected with the mode q. The last term in (4) was obtained in the rotating-wave approximation.[16]

We use for the atom system the four-level Lax–Louisell model in which the atom is excited by pumping from the ground state $|0\rangle$ into the upper energy state $|3\rangle$ which has a high probability of nonradiative decay to the upper laser level $|2\rangle$, connected by a radiative transition with the lower laser level $|1\rangle$. That is to say, the effective pumping is directly to the $|2\rangle$ level.

In the expansion in terms of normal modes, the Hamiltonian of the electromagnetic field in the $A\times B\times C$ cell, interacting with an atom n of the atomic subsystem, can be expressed in the rotating-wave approximation in the form

$$H_s=\sum_q \hbar\omega_q a_q^+a_q+\sum_{j=0,1,2}\hbar\omega_j(|j\rangle\langle j|)_n$$
$$+i\hbar\mu\sum_q [a_q^+(|1\rangle\langle 2|)_n-(|2\rangle\langle 1|)_n a_q]. \tag{5}$$

The last term of this expression, which is indicative of the interaction of an atom with a multimode field, we use a dipole approximation; the coupling constant is then

$$\mu_q=\left(\frac{\omega_q}{2V\hbar\varepsilon_0}\right)^{1/2}\Big/ d_{12}\approx\mu, \tag{6}$$

where d_{12} is the matrix element of the atom's dipole-moment operator, V is the volume of the cell, and $\omega_q=\pi cq/A$.

2. STOCHASTIC EQUATIONS OF MOTION

To eliminate the reservoir dimensions from the differential equations of motion for the system operator, we use in the framework of the stochastic description of the system dynamics the Markov approximation for the stochastic operators of the reservoirs, and calculate the system-operator changes due to interaction with the reservoir over a finite time interval Δt longer than the reciprocal natural frequencies of these operators in the Heisenberg representation, but shorter than the correlation times τ of the reservoir operators (values corresponding to the "collision time" of the system with the reservoir), and then setting $\Delta t \to 0$. The effective equation of motion for an arbitrary Heisenberg operator of the system, averaged over the statistical ensemble of the reservoirs, is reduced with the aid of the Langevin approach[14] to the form

$$i\hbar \frac{d}{dt}\langle M\rangle_{\mathrm{R}} = \langle [M,H_{\mathrm{S}}]\rangle_{\mathrm{R}} + i\hbar \exp\left(i\frac{H_{\mathrm{S}}+R_{\mathrm{A}}+R_{\mathrm{F}}}{\hbar}\right) \times \left\langle \frac{\Delta m}{\Delta t}\right\rangle_{\mathrm{R}} \exp\left(-i\frac{H_{\mathrm{S}}+R_{\mathrm{A}}+R_{\mathrm{F}}}{\hbar}t\right),$$

$$M(t)=\exp\left(i\frac{H_{\mathrm{S}}+R_{\mathrm{A}}+R_{\mathrm{F}}}{\hbar}t\right)m(t) \times \exp\left(-i\frac{H_{\mathrm{S}}+R_{\mathrm{A}}+R_{\mathrm{F}}}{\hbar}t\right), \tag{7}$$

$$\Delta m(t) = m(t+\Delta t) - m(t).$$

This form of the Heisenberg equation for the mean values is valid if all the correlation times are shorter than the variation time of the slow operator $m(t)$. The influence of the reservoir on the behavior of the average operators is characterized by the last term of (7). The reservoir is thus excluded, and its influence is a reflection of the displacement $\langle \Delta m/\Delta t\rangle_{\mathrm{R}}$, which is included in the equation of motion for the slow operator. The Langevin equations for the exact values are obtained from the equations for the mean values by adding Langevin noise sources with zero mean value and with zero correlation time ($\Delta t \gg \tau$).

The dissipative terms of the final expressions are expressed in terms of the transition probabilities or the damping coefficients by using an iterative solution of the equations of motion for slow operators whose variation determines completely the potential of the interaction with the reservoir, under the condition that the random processes are Markovian $[\langle Q_i(t)Q_j(t')\rangle \sim \delta(t-t')]$.[16]

The atomic subsystem of the laser is described by the operators $M_{kl}^{(n)} = (|k\rangle\langle l|)_n$, for which we obtain

$$[M_{kl}^{(n)},H_S] = \sum_q i\hbar\mu_q[\delta_{l1}a_q M_{k2}^{(n)} - \delta_{l2}M_{k1}^{(n)}a_q] + i\omega_{kl}M_{kl}^{(n)} - \sum_q i\hbar\mu_q[\delta_{k2}a_q^+ M_{1l}^{(n)} - \delta_{k1}M_{2l}^{(n)}a_q], \tag{8}$$

since[14]

$$M_{ij}^{(n)}M_{kl}^{(n)} = \delta_{jk}M_{il}^{(n)}. \tag{9}$$

This yields (without the relaxation terms)

$$\frac{d}{dt}\langle M_{22}^{(n)}\rangle = -\sum_q \mu[\langle M_{21}^{(n)}\rangle a_q + a_q^+\langle M_{12}^{(n)}\rangle] \equiv -\sum_q \mu B_q^{(n)}, \tag{10}$$

$$\frac{d}{dt}\langle M_{11}^{(n)}\rangle = \sum_q \mu B_q^{(n)}, \tag{11}$$

$$\frac{d}{dt}\langle M_{12}^{(n)}\rangle = \sum_q \mu[-\langle M_{11}^{(n)}\rangle a_q + \langle M_{22}^{(n)}\rangle a_q^+] - i\omega_{\mathrm{A}}\langle M_{12}^{(n)}\rangle, \tag{12}$$

where $\omega_{\mathrm{A}} = (E_2 - E_1)/\hbar$ is the atom transition frequency.

A multimode field is characterized in the second-quantization representation by a set of operators $M = a_q^+ a_{q'}$. Using the equation of motion[7] without damping terms, we obtain

$$[a_{q'}H_{\mathrm{S}}] = -i\omega_{q'}a_{q'} + \mu_{q'}(|1\rangle\langle 2|)_n, \tag{13}$$

$$[a_{q'}^+H_{\mathrm{S}}] = i\omega_{q'}a_{q'}^+ + \mu_{q'}(|2\rangle\langle 1|)_n, \tag{14}$$

applying the commutation relations for Bose creation and annihilation operators

$$[a_{q'},a_q^+] = \delta_{q'q}, \quad [a_{q'},a_q] = [a_{q'}^+,a_q^+] = 0. \tag{15}$$

We find similarly that

$$[(a_{q'}^+a_{q''}),H_{\mathrm{c}}] = -i\hbar(\omega_{q'}-\omega_{q''})a_{q'}^+a_{q''} + i\hbar\mu_{q'} \times [a_{q'}^+(|1\rangle\langle 2|)_n - a_{q''}(|2\rangle\langle 1|)_n]. \tag{16}$$

Substituting expressions (10)–(14) and (16) in the corresponding equations of motion of general form (7) for atomic and field subsystems we obtain a closed set of equations for the averages $(|1\rangle\langle 1|)_n$, $(|2\rangle\langle 2|)_n$, $(|1\rangle\langle 2|)_n$ over the reservoir and a_q^+, a_q, $a_{q'}^+a_{q''}$, describing the interaction of an nth atom with a multimode field in the absence of noise. The contribution of the reservoirs to the dynamics of the system operators is manifested by the presence of relaxation terms in the equations for the mean values. Interaction with reservoirs leads to transitions between states of the atomic subsystem as well as to field damping in the cavity modes. The stochastic equations of motion needed to study the fluctuations can be obtained, in the Langevin approach, from the equations for the mean values by supplementing the latter with random noise-source operator. The result is the system of stochastic equations

$$\frac{d}{dt}a_q = -\frac{\gamma_q}{2}a_q - i\omega_q a_q + \sum_{n=1}^{N_{\mathrm{A}}}\mu(|1\rangle\langle 2|)_n + f_q(t), \tag{17}$$

$$\frac{d}{dt}(a_{q'}^+a_{q''}) = -\frac{\gamma_{q'}-\gamma_{q''}}{2}a_{q'}^+a_{q''} + i(\omega_{q'}-\omega_{q''})a_{q'}^+a_{q''} + \mu[a_{q'}^+(|1\rangle\langle 2|)_n + a_{q''}(|2\rangle\langle 1|)_n] + f_{q'q''}(t), \tag{18}$$

$$\frac{d}{dt}(|1\rangle\langle 1|)_n=-\Gamma_1(|1\rangle\langle 1|)_n+\sum_{q=1}^{Q}B_q^{(n)}+f_{11}^{(n)}(t), \tag{19}$$

$$\frac{d}{dt}(|2\rangle\langle 2|)_n=-\Gamma_2(|2\rangle\langle 2|)_n+w_{02}(|0\rangle\langle 0|)_n-\sum_{q=1}^{Q}B_q^{(n)}+f_{22}^{(n)}(t), \tag{20}$$

$$\frac{d}{dt}(|1\rangle\langle 2|)_n=-\Gamma(|1\rangle\langle 2|)_n+i\omega_A(|1\rangle\langle 2|)_n+\mu\sum_{q=1}^{Q}a_q[(|2\rangle\langle 2|)_n-(|1\rangle\langle 1|)_n]+f_{12}^{(n)}(t) \tag{21}$$

(w_{02} is the pumping rate). The dissipation coefficients γ_q, Γ_1, Γ_2, $\Gamma=\Gamma_{ph}+\Gamma_1+\Gamma_2/2$ contained in the equations (Γ_{ph} are the constants of the atom dipole-moment phase-relaxation constants as a result of the elastic collisions) include information on the reservoirs and are expressed in terms of integrals of the correlation functions of the operators pertaining to the reservoir. The Langevin noise sources are random operators pertaining to reservoirs with zero mean values and (in view of the assumption of Markovian behavior) δ-correlated second moments, which depend in general on the time:

$$\langle f_\alpha(t)f_\beta(t')\rangle=2D_{\alpha\beta}\delta(t-t'), \tag{22}$$

$$2D_{\alpha\beta}=\langle[M_\alpha(t+\Delta t)-M_\alpha(t)][M_\beta(t+\Delta t)-M_\beta(t)]\rangle/\Delta t. \tag{23}$$

The diffusion coefficients $2D_{\alpha\beta}$ for the operators $M_i=a_q$, a_q^+, $(|1\rangle\langle 1|)_n$, $(|2\rangle\langle 2|)_n$, $(|1\rangle\langle 2|)_n$ as well as a_q^+ and a_q can be calculated with the aid of the generalized Einstein equation, using the equations of motion derived above for the mean values.[14] For the field operators we obtain

$$\langle f_q^+(t)f_{q'}(t')\rangle=\delta_{qq'}\gamma_q\bar{n}_q\delta(t-t'), \tag{24}$$

$$\langle f_{q'}(t')f_q^+(t)\rangle=\delta_{qq'}\gamma_q(\bar{n}_q+1)\delta(t-t'), \tag{25}$$

here $\bar{n}_q$ is the number of photons drawn into mode q from the reservoir at a temperature T_R and is assumed hereafter to be negligibly small. The diffusion coefficients for the atomic operators are perfectly analogous to those given in Ref. 14 for the single-mode case.

We proceed next to slow system operators, eliminating partially the high-frequency time dependence of the Heisenberg operators. Assuming that the interaction with the field is the same for all atoms, and that the spatial structure of the field is immaterial for the fluctuation calculations, we consider henceforth average macroscopic operators.

We introduce the operators

$$\tilde{a}_q=a_q e^{i\omega_q t}, \tag{26}$$

$$\widetilde{a_{q'}^+a_{q''}}=a_{q'}^+e^{-i\omega_{q'}t}a_{q''}e^{i\omega_{q'}t}, \tag{27}$$

$$\sigma=\frac{1}{N_A}\sum_{n=1}^{N_A}(|1\rangle\langle 2|)_n e^{i\omega_A t}, \tag{28}$$

$$\sigma_{jj}=\frac{1}{N_A}\sum_{n=1}^{N_A}(|j\rangle\langle j|)_n, \tag{29}$$

$$F_{12}=\frac{1}{N_A}\sum_{n=1}^{N_A}f_{12}^{(n)}e^{i\omega_A t}, \tag{30}$$

$$F_{jj}=\frac{1}{N_A}\sum_{n=1}^{N_A}f_{jj}^{(n)}, \tag{31}$$

$$F_q=f_q e^{i\omega_q t}. \tag{32}$$

We shall omit henceforth the tildes over the slow field operators.

We assume here and below that the frequency ω_0 used to define the slow dipole-moment operator σ is equal to the atomic-transition frequency ω_A. Under these conditions, the equations of motion of interest to us take the form

$$\frac{d}{dt}a_q=-\frac{\gamma_q}{2}a_q+\mu N_A\sigma e^{i(\omega_q-\omega_A)t}+F_q, \tag{33}$$

$$\frac{d}{dt}(a_{q'}^+a_{q''})=-\frac{\gamma_{q'}+\gamma_{q''}}{2}a_{q'}^+a_{q''}+\mu N_A\{\sigma^+a_{q''}\times\exp[-i(\omega_{q'}-\omega_A)t]+a_{q'}^+\sigma\times\exp[i(\omega_{q''}-\omega_A)t]\}+F_{q'}^+a_{q''}+a_{q'}^+F_{q''}, \tag{34}$$

$$\frac{d}{dt}\sigma=-\Gamma\sigma+\mu\sum_{q=1}^{Q}a_q\times\exp[-i(\omega_q-\omega_A)t](\sigma_{22}-\sigma_{11})+F_{12}, \tag{35}$$

$$\frac{d}{dt}\sigma_{11}=-\Gamma_1\sigma_{11}+\sum_{q=1}^{Q}B_q+F_{11}, \tag{36}$$

$$\frac{d}{dt}\sigma_{22}=-\Gamma_2\sigma_{22}+w_{02}-\sum_{q=1}^{Q}B_q+F_{22}, \tag{37}$$

$$B_q\equiv\mu\{a_q^+\sigma\exp[i(\omega_q-\omega_A)t]+\sigma^+a_q\exp[-i(\omega_q-\omega_A)t]\}. \tag{38}$$

The diffusion coefficients (second moments) defined in (22) and (23) for random sources with zero mean values and contained in Eqs. (33–37) can be calculated directly from (7) and also, as shown in Ref. 14, by using for the correlators the equation of motion

$$\frac{d}{dt}\langle M_\alpha(t)M_\beta(t)\rangle=2\langle D_{\alpha\beta}\rangle+\langle A_\alpha M_\beta\rangle+\langle M_\alpha A_\beta\rangle, \tag{39}$$

where A_i denote the constant-displacement operators that determine the right-hand sides of the equations of motion for the operators M_i.

Using (39) and taking (30) and (32) into account we obtain the diffusion coefficients

$$2D_{ijkl}=\frac{1}{N_A}\sum_{n=1}^{N_A}2D^{(n)}_{ijkl}, \tag{40}$$

$$\langle f^{(n)}_{ij}(t)f^{(n)}_{kl}(t')\rangle=\langle 2D^{(n)}_{ijkl}\rangle\delta(t-t') \tag{41}$$

$$2D_{1111}=\Gamma_1\sigma_1, \tag{42}$$

$$2D_{2222}=\Gamma_2\sigma_{22}+R/N_A, \tag{43}$$

$$2D_{1222}=\Gamma_2\sigma=2D^{+}_{2221}, \tag{44}$$

$$2D_{2112}=\sigma_{22}(\Gamma_1+2\Gamma_{ph})+R/N_A, \tag{45}$$

$$2D_{1221}=\sigma_{11}(\Gamma_2+2\Gamma_{ph}), \tag{46}$$

where $R\equiv N_A w_{02}$.

Suppose that the atomic line is wide enough, such that $\Gamma\gg\{\gamma_q\}$, Γ_1, Γ_2, and exclude adiabatically the variable σ from the set of equations (33)–(38).[1] Assuming thus that $d\sigma/dt\sim\gamma\sigma\ll\Gamma\sigma$, we neglect the time derivative in (35), whence

$$\sigma_{ad}=\frac{\mu\bar{D}}{\Gamma}\sum_q a_q\exp[-i(\omega_q-\omega_A)t]+F_{12},$$

$$\bar{D}=(\sigma_{22}-\sigma_{11}). \tag{47}$$

We substitute this adiabatic value of the polarization in the equations for the populations D, N_1, and N_2.

For

$$N_j=N_A\sigma_{jj},\quad j=1,2, \tag{48}$$

$$n_q=a^{+}_q a_q, \tag{49}$$

we obtain

$$\frac{dN_1}{dt}=-\Gamma_1N_1+\frac{2\mu^2}{\Gamma}\sum_q[N_2(n_q+1)+N_1n_q]$$
$$+\frac{\mu^2}{\Gamma}D\sum_{q'\neq q''}\{a^{+}_{q'}a_{q''}\exp[i(\omega_{q'}-\omega_{q''})t]$$
$$+a^{+}_{q''}a_{q'}\exp[-i(\omega_{q'}-\omega_{q''})t]\}+G_1, \tag{50}$$

$$\frac{dN_2}{dt}=-\Gamma_2N_2+R-\frac{2\mu^2}{\Gamma}\sum_q[N_2(n_q+1)-N_1n_q]$$
$$-\frac{\mu^2}{\Gamma}D\sum_{q'\neq q''}\{a^{+}_{q'}a_{q''}\exp[i(\omega_{q'}-\omega_{q''})t]$$
$$+a^{+}_{q''}a_{q'}\exp[-i(\omega_{q'}-\omega_{q''})t]\}+G_2. \tag{51}$$

Using (47), we obtain for the laser field operator

$$\frac{d}{dt}(a^{+}_{q'}a_{q''})=-\frac{\gamma_{q'}+\gamma_{q''}}{2}a^{+}_{q'}a_{q''}$$
$$+\frac{\mu^2}{\Gamma}D\sum_q\{a^{+}_q a_{q''}\exp[i(\omega_q-\omega_{q'})t]$$
$$+a^{+}_{q'}a_q\exp[-i(\omega_q-\omega_{q''})t]\}$$
$$+\frac{2\mu^2}{\Gamma}N_2+G_{q'q''}, \tag{52}$$

$$\frac{d}{dt}a_q=-\frac{\gamma_q}{2}a_q+\frac{\mu^2}{\Gamma}D\sum_{q'}a_{q'}\exp[-i(\omega_{q'}-\omega_q)t]+G_{a_q}. \tag{53}$$

The new random sources with zero first moments are of the form

$$G_1=N_AF_{11}+\sum_q N_AF_{B_q}, \tag{54}$$

$$G_2=N_AF_{22}-\sum_q N_AF_{B_q}, \tag{55}$$

$$F_{B_q}\equiv\frac{\mu}{\Gamma}\{a^{+}_{qc}F_{12}\exp[-i(\omega_A-\omega_q)t]$$
$$+F^{+}_{12}a_{qc}\exp[i(\omega_A-\omega_q)t)]\}, \tag{56}$$

$$a_{qc}(t)\equiv a_q(t-\varepsilon),\quad \varepsilon\to 0,$$
$$G_{a_q}=\frac{N_A\mu}{\Gamma}F_{12}\exp[i(\omega_q-\omega_A)t]+F_q, \tag{57}$$

$$G_{q'q''}=a^{+}_{q'c}F_{q'}+F^{+}_{q'}a_{q''c}$$
$$+\frac{\mu}{\Gamma}\{F^{+}_{12}a_{q''c}\exp[-i(\omega_{q'}-\omega_A)t]$$
$$+a^{+}_{q'c}F_{12}\exp[i(\omega_{q''}-\omega_A)t]\}N_A$$
$$=a^{+}_{q'c}G^{+}_{a_{q''}}G^{+}_{a_{q''}}a_{q''c}. \tag{58}$$

Note that expressions (50) and (51), as well as (54–56), were obtained in the adiabatic limit $\Gamma\to\infty$, and it is assumed that $2\mu^2/\Gamma$ remains finite nonetheless.[15,16] Since the random processes are Markovian, it is assumed also that $\langle F_{11}(t)a_{qc}(t)\rangle=0$ as $\varepsilon\to 0$. We obtain for the corresponding diffusion coefficients

$$\langle G_i(t)G_j(t')\rangle=\langle 2D_{ij}\rangle\delta(t-t'), \tag{59}$$

$$2D_{11}=\Gamma_1N_1+S\Pi\sum_{q=1}^{Q}n_q+\Pi N_2Q+\Sigma_1, \tag{60}$$

$$2D_{22}=R+\Gamma_2N_2+S\Pi\sum_{q=1}^{Q}n_q+\Pi N_2Q+\Sigma_1, \tag{61}$$

$$2D_{12}=-\Pi S\sum_{q=1}^{Q}n_q-\Pi N_2Q-\Sigma_1, \tag{62}$$

$$2D_{qq}=S\Pi n_q+\gamma n_q+\Pi N_2, \tag{63}$$

$$2D_{1n_q}=S\Pi n_q+\Pi N_2+\Sigma_2, \tag{64}$$

$$2D_{2n_q}=-\Pi Sn_q-\Pi N_2-\Sigma_2, \tag{65}$$

$$S=N_1+N_2,\quad \Pi=\frac{2\mu^2}{\Gamma}, \tag{66}$$

$$\Sigma_1=\Pi\sum_{q'\neq q''}\{a^{+}_{q'}a_{q''}\exp[i(\omega_{q'}-\omega_{q''})t]N_1$$
$$+a_{q'}a^{+}_{q''}\exp[-i(\omega_{q'}-\omega_{q''})t]N_2\}, \tag{67}$$

$$\Sigma_2=\Pi\sum_{q'\neq q}\{a_{q'}^{+}a_q\exp[i(\omega_{q'}-\omega_q)t]N_1 +a_{q'}^{+}a_{q'}^{+}\exp[-i(\omega_{q'}-\omega_q)t]N_2\}. \tag{68}$$

Equations (50)–(53) contain bilinear field operators proportional to $a_{q'}^{+}a_{q''}$, where $q'\neq q''$, and oscillating at the intermode-beat frequency. These variables, which are responsible for the dynamic interaction of the modes, will be omitted in the calculations that follow, meaning that we use the balance-equations approximation widely used in multimode-laser theory. A similar approximation is used by us also for the diffusion coefficients (60)–(65), i.e., we neglect the influence of Σ_1 and Σ_2 on the photon-number fluctuations.

The resultant stochastic equations contain only the operators n_q, N_1, and N_2 (or D) which commute with one another. However, the second moments and the diffusion coefficients of the corresponding random processes depend, as shown by the calculations, on the sequence in which the random sources G enter the correlators. Using the results of Ref. 15, we shall consider normally ordered diffusion coefficients, i.e., we choose the sequence of the operators to be a_q^{+}, N_1, N_2, and a_q. We shall assume hereafter that expressions (48)–(65) written in the normal order are c-numbers, and that all the variables commute with one another.

3. QUANTUM MULTIMODE LASING STATIONARY OPERATING POINT

Following Ref. 16, we discard random sources from Eqs. (33)–(38) reduced to normally ordered form, and regard then as c-number equations relating the quantities $a_q=\langle a_q\rangle$, $a_q^{*}=\langle a_q^{+}\rangle$, $\sigma=\langle\sigma\rangle$, $\sigma^{*}=\langle\sigma^{+}\rangle$ etc. We shall leave out in the present section the averaging-operation symbols. Using the resultant equations, we obtain the quantum stationary values of the laser quantities n_{q0}, N_{10}, N_{20}, and also D_0.

To determine the field and atomic variables indicative of the dynamics of a multimode laser in an established stationary regime n_{q0}, N_{10}, N_{20}, and D_0, we equate to zero the time derivatives of the slow mean values (33)–(38). We assume by the same token that the basic high-frequency time dependences of the Heisenberg variables a_q and $|1\rangle\langle 2|$ have the form $\exp(-i\omega_q t)$ and $\exp(-i\omega_A t)$, and are completely cancelled out by introducing slow operators in accordance with (26) and (28), while the populations of the laser levels in the stationary state are constant in time.

From the equation for a_q

$$\frac{d}{dt}a_q=0=-\frac{\gamma_q}{2}a_{q0}+\mu N_A\sigma_0\exp[i(\omega_q-\omega_A)t], \tag{69}$$

we obtain

$$a_{q0}=\frac{\mu N_A\sigma_0}{(\gamma_q/2)}\exp[i(\omega_q-\omega_A)t]. \tag{70}$$

Substituting this expression in (35)

$$\frac{d}{dt}\sigma=0=-\Gamma\sigma_0+\mu\sum_{q=1}^{Q}a_{q0} \times\exp[i(\omega_A-\omega_q)t](\sigma_{22,0}-\sigma_{11,0}), \tag{71}$$

we obtain from the stationary condition as the working value of the population difference

$$D_0=N_A(\sigma_{22,0}-\sigma_{11,0})=\frac{\gamma\Gamma}{2\mu^2 Q}=\frac{\gamma}{\Pi Q},\quad \gamma\approx\gamma_q,\forall q. \tag{72}$$

Changing to Eqs. (36) and (37) for σ_{11} and σ_{22} and using the c-number relations obtained from (70)

$$\begin{aligned} a_{q'0}&=a_{q0}\exp[-i(\omega_q-\omega_{q'})t],\\ a_{q'0}^{*}&=a_{q0}^{*}\exp[i(\omega_q-\omega_{q'})t], \end{aligned} \tag{73}$$

we obtain for the quantities B_q indicative of the stimulated emission and absorption

$$\begin{aligned} B_{q0}=&\frac{\mu^2}{\Gamma N_A}\Big\{a_{q0}^{*}\sum_{q'=1}^{Q}a_{q'0}\exp[-i(\omega_{q'}-\omega_A)t]D_0\\ &\times\exp[i(\omega_q-\omega_A)t]+\sum_{q'=1}^{Q}a_{q'0}^{*}\\ &\times\exp[i(\omega_{q'}-\omega_A)t]D_0\exp[-i(\omega_q-\omega_A)t]a_{q0}\Big\}\\ =&\frac{\mu^2}{\Gamma N_A}D_0\sum_{q'=1}^{Q}\{a_{q0}^{*}a_{q0}\exp[-i(\omega_q-\omega_{q'})t]\\ &\times\exp[-i(\omega_{q'}-\omega_A)t]\exp[i(\omega_q-\omega_A)t]\\ &+a_{q0}^{*}a_{q0}\exp[i(\omega_q-\omega_{q'})t]\exp[i(\omega_{q'}-\omega_A)t]\\ &\times\exp[-i(\omega_q-\omega_A)t]\}=\Pi\frac{D_0}{N_A}Q|a_{q0}|^2. \end{aligned}$$

We get then from (36)

$$-\Gamma_1\sigma_{11,0}+\sum_{q=1}^{Q}B_{q0}=0=-\Gamma_1\sigma_{11,0}+\gamma|a_{q0}|^2Q/N_A,$$

and, in view of (72), the populations are

$$N_{10}\equiv N_A\sigma_{11,0}=\frac{\gamma}{\Gamma_1}Q|a_{q0}|^2, \tag{74}$$

$$N_{20}\equiv N_A\sigma_{22,0}=\frac{\gamma}{\Gamma_1}Q|a_{q0}|^2+\frac{\gamma}{\Pi Q}. \tag{75}$$

We determine the stationary mean value of the number of photons in one mode from (37), by substituting (75),

$$-\Gamma_2\sigma_{22,0}+w_{02}-\sum_{q=1}^{Q} B_{q0}=0,$$

$$n_{q0}=|a_{q0}|^2=\frac{w_{02}-\Gamma_2\gamma/\Pi Q N_A}{\gamma Q(1+\Gamma_2/\Gamma_1)}$$

$$=\frac{\bar{\Gamma}}{\Pi Q^2}\left(\frac{\Pi Q}{\Gamma_2\gamma}w_{02}N_A-1\right)=\frac{\bar{\Gamma}}{\Pi Q^2}(\xi-1), \tag{76}$$

$$\xi\equiv\frac{R}{R_{\text{th}}}=\frac{\Pi Q}{\Gamma_2\gamma}R,\quad R\equiv w_{02}N_A,\quad \bar{\Gamma}\equiv\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2},$$

where $\xi-1$ is the excess of the pump above the threshold value

$$R_{\text{th}}=\frac{\Gamma_2\gamma}{\Pi Q}.$$

We use the equivalent-mode approximation, wherein $\gamma\approx\gamma_q$ and $\Pi\approx\Pi_q$ for all q, meaning also that the stationary mean value is

$$N_0=\sum_{q=1}^{Q} n_{q0}=Qn_0. \tag{77}$$

Note that the obtained working inversion $\gamma/\Pi Q$ is reasonable. It reduces for $Q=1$ (single-mode laser) to the known value[2] γ/Π that the inversion decreases in multimode lasing ($Q>1$), i.e., the saturation is more complete for stronger pumping, a specific feature of multimode lasing.

A threshold condition of a different type is obtained from the approximate balance equations that follow for the populations from (50)–(53) in the adiabatic approximation, neglecting bilinear field operators of the form $a_{q'}^+$ and $a_{q''}$, in which case we have $n_0=\bar{\Gamma}/\Pi Q(\Pi R/\Gamma_2\gamma-1)$, $R'_{\text{th}}=\gamma\Gamma_2/\Pi$, $\xi'=R/R'_{\text{th}}$. It is easily seen that the obtained threshold pump R_{th} depends on the number of modes participating in the lasing, and is thus shifted at $Q>1$, compared with R'_{th}, toward lower values of the pump needed to have $n_0>0$, whereas at large excesses above threshold, when ξ and ξ' become simultaneously larger than 1, the numbers of photons coincide in both cases. The quantum stationary point (74), (75) obtained from the exact equations differs substantially from the stationary values obtained in the balance approximation ($N'_{10}=\gamma N_0/\Gamma_1, N'_{20}=\gamma N_0/\Gamma_1+\gamma/\Pi$) only near the lasing threshold, when the inversion is comparable with $\gamma N_0/\Gamma_1$. Above the threshold, for γ/Π and $\gamma/\Pi Q\ll\gamma N_0/\Gamma_1$, the two stationary points coincide.

We calculate now the mean stationary values of the diffusion coefficients (60)–(66) obtained above. To this end, recognizing that $[a_{q'},a_{q''}^+]=\delta_{q'q''}$, we shall henceforth assume that $\langle\Sigma_1\rangle$ and $\langle\Sigma_2\rangle$ are c-numbers under stationary conditions. Using (73), we obtain for the following stationary mean values for the diffusion coefficients:

$$\langle 2D_{11}\rangle=\Gamma_1N_{10}+\Pi S_0N_0Q, \tag{78}$$

$$\langle 2D_{22}\rangle=\Gamma_2N_{20}+R+\Pi S_0N_0Q, \tag{79}$$

$$\langle 2D_{12}\rangle=-\Pi S_0N_0Q, \tag{80}$$

$$\langle 2D_{1q}\rangle=-\langle 2D_{2q}\rangle=\Pi S_0N_0, \tag{81}$$

$$\langle 2D_{qq}\rangle=\Pi S_0n_0+\gamma n_0, \tag{82}$$

where $S_0\equiv N_{10}+N_{20}$. We neglect the small terms $Q\Pi N_{20}$ and $Q^2\Pi N_{20}$ at values high above the threshold. We shall use hereafter the diffusion coefficients (78)–(82) to calculate the photon-number fluctuations.

It is easy to verify that the obtained values of the stationary inversion and of all the values that determine the stationary operating point can be obtained by expanding σ and N_j ($j=1$, 2) in the field modes that take part in the lasing. Neglecting frequency pulling, since $\Gamma\gg\gamma_q$, we use expansions of the form

$$\sigma=N_A^{-1}\sum_{n,q}e^{i\omega_A t}\sigma^{(n,q)},\quad N_j=N_A^{-1}\sum_{n,q}\sigma_{jj}^{(n,q)},$$

$$\sigma^{(n,q)}=\exp[i(\omega_q-\omega_A)t](|1\rangle\langle 2|)_{n,q},$$

$$\sigma_{jj}^{(n,q)}=e^{i\omega_q t}(|j\rangle\langle j|)_{n,q}.$$

Substituting these expansions in Eqs. (35)–(37) and using (33) we see readily that $\sigma_{n,q}=(|1\rangle\langle 2|)_{n,q}$ are equal for all q in the time interval $\gamma^{-1},\Gamma_1^{-1},\Gamma_2^{-1}>t\gg\Gamma^{-1}$ of interest to us (the same holds for $\sigma_{jj}^{(n,q)}$) within the framework of the equivalent-modes assumption, in the case $\omega_A\gg\Delta\omega$ where $\Delta\omega=|\omega_1-\omega_A|$ is the multimode-laser frequency range. That is to say, expansion of the microscopic variables $(|1\rangle\langle 2|)_n$ and $(|j\rangle\langle j|)_n$ in terms of the cavity frequencies leads, within the framework of the approximations used, to the slow average variables

$$\sigma\cong\frac{Q}{N_A}\sum_n e^{i\omega_A t}(\widetilde{|1\rangle\langle 2|})_n,\quad N_j\cong\frac{Q}{N_A}\sum_n(\widetilde{|j\rangle\langle j|})_n$$

and therefore does not influence the results of the fluctuation calculations.

4. FLUCTUATIONS OF TOTAL NUMBER OF PHOTONS IN A MULTIMODE-LASER CAVITY

We obtain in this section in an exact expression for the fluctuations of the total (summary) number of photons N_{ph} in the cavity of a multimode laser, in the framework of the employed model and without adiabatic exclusion of the laser-level populations in the quasilinear approximation. By changing to Fourier transforms of the equations of motion for the problem variables n_q, N_{ph}, D, and N_1, and by using the Wiener-Khinchin theorem[1)] for the corresponding random processes we can forgo the adiabatic exclusion of the variables D and N_1 in the equations for the variations, and we can obtain for the fluctuations equations free of constraints imposed by the adiabatic exclusion[16] on the relation between the damping constants.

The system of stochastic balance equations for the variables in question is[2)]

$$\frac{dn_q}{dt}=-\gamma n_q+\Pi Dn_q+G_{n_q}, \tag{83}$$

$$\frac{dN_{\text{ph}}}{dt}=-\gamma N_{\text{ph}}+\Pi DN_{\text{ph}}+G_{N_{\text{ph}}}, \tag{84}$$

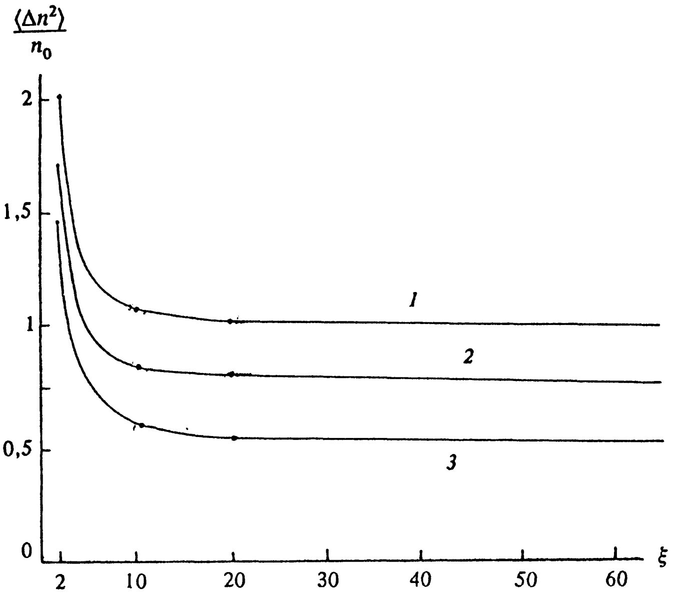

FIG. 1. Ratio of the stationary fluctuations of the number of photons in the cavity of a single-mode laser $\langle \Delta n^2\rangle$ to the mean stationary value, as a function of the excess above threshold ξ ($\Pi=1\ \mathrm{s}^{-1}$, $\gamma=10^6\ \mathrm{s}^{-1}$); *1*— $\Gamma_1=10^9\ \mathrm{s}^{-1}$, $\Gamma_2=10^7\ \mathrm{s}^{-10}$; *2*—regular pumping, $\Gamma_1=\Gamma_2=10^9\ \mathrm{s}^{-1}$; *3*—regular pumping $\Gamma_1\gg\Gamma_2$, $\Gamma_1=10^9\ \mathrm{s}^{-1}$, $\Gamma_2=10^7\ \mathrm{s}^{-1}$.

$$\frac{dD}{dt}=-\Gamma_2 D+(\Gamma_1-\Gamma_2)N_1-2D\Pi N_{\mathrm{ph}}+R+G_2-G_1, \tag{85}$$

$$\frac{dN_1}{dt}=-\Gamma_1 N_1+DN_{\mathrm{ph}}\Pi+G_1, \tag{86}$$

with

$$\Delta n_q=n_q-n_{q0},\quad \Delta N_{\mathrm{ph}}=N_{\mathrm{ph}}-N_{\mathrm{ph}0},$$
$$\Delta D=D-D_0,\quad \Delta N_1=N_1-N_{10}, \tag{87}$$

representing the deviations from the mean stationary values. Discarding the terms nonlinear in the variations, we obtain

$$\frac{d\Delta D}{dt}=-\Gamma_2\Delta D+(\Gamma_1-\Gamma_2)\Delta N_1-2\Pi$$
$$\times(D_0\Delta N_{\mathrm{ph}}+\Delta DN_{\mathrm{ph}0})+G_2-G_1, \tag{88}$$

$$\frac{d\Delta N_1}{dt}=-\Gamma_1\Delta N_2+\Pi(D_0\Delta N_{\mathrm{ph}}+N_{\mathrm{ph}0}\Delta D)+G_1, \tag{89}$$

$$\frac{d\Delta N_{\mathrm{ph}}}{dt}=-\gamma\Delta N_{\mathrm{ph}}+\Pi(D_0\Delta N_{\mathrm{ph}}+N_{\mathrm{ph}0}\Delta D)+G_{N_{\mathrm{ph}}}, \tag{90}$$

$$\frac{d\Delta n_q}{dt}=-\gamma\Delta n_q+\Pi(D_0\Delta n_q+n_0\Delta D)+G_{n_q}. \tag{91}$$

We have left out of the right-hand sides of (90) and (91) the constant terms $-(1-1/Q)\gamma n_0$ and $-(1-1/Q)\gamma N_0$, respectively, since the presence of determinate time-independent constant terms in the stochastic-equation displacement vector will not alter the fluctuations (variances) calculated below, as follows from the properties of the linear transformations of the stochastic quantities $D[aX+b]=a^2D[X]$, where X is a stochastic quantity. Similarly, the constant terms have been left out of (88) and (89).

We carry out next a Fourier transformation of the system (88)–(91)

$$-i\omega\Delta D_\omega=-\Gamma_2\Delta D_\omega+(\Gamma_1-\Gamma_2)\Delta N_{1\omega}-2\Pi(D_0\Delta N_{\mathrm{ph}\omega}$$
$$+\Delta D_\omega N_{\mathrm{ph}0})+G_{2\omega}-G_{1\omega}, \tag{92}$$

$$-i\omega\Delta N_{1\omega}=-\Gamma_1\Delta N_{1\omega}+\Pi(D_0\Delta N_{\mathrm{ph}\omega}+N_{\mathrm{ph}0}\Delta D_\omega)+G_{1\omega}, \tag{93}$$

$$-i\omega\Delta N_{\mathrm{ph}\omega}=-\gamma\Delta N_{\mathrm{ph}\omega}+\Pi(D_0\Delta N_{\mathrm{ph}\omega}+N_{\mathrm{ph}0}\Delta D_\omega)$$
$$+G_{N_{\mathrm{ph}}\omega}, \tag{94}$$

$$-i\omega\Delta n_{q\omega}=-\gamma\Delta n_{q\omega}+\Pi(D_0\Delta n_{q\omega}+n_0\Delta D_\omega)+G_{q\omega}, \tag{95}$$

from (92) and (93) we obtain

$$\Delta D_\omega$$
$$=\frac{(\Gamma_1-\Gamma_2)\Pi D_0\Delta N_{\mathrm{ph}\omega}-z_2G_{1\omega}-2\Pi D_0z_1\Delta N_{\mathrm{ph}\omega}+z_1G_{2\omega}}{\zeta}, \tag{96}$$

$$z_j\equiv-i\omega+\Gamma_j,\quad j=1,2, \tag{97}$$

$$\zeta\equiv z_1z_2+(z_1+z_2)\Pi N_{\mathrm{ph}0}. \tag{98}$$

We substitute (96) in (94)

$$\Delta N_{\mathrm{ph}\omega}=\frac{-\Pi N_{\mathrm{ph}0}z_2G_{1\omega}+\Pi N_{\mathrm{ph}0}z_1G_{2\omega}+G_{N_{\mathrm{ph}}\omega}\zeta}{\zeta_1}, \tag{99}$$

$$\zeta_1\equiv\zeta\Delta+\Pi N_{\mathrm{ph}0}\gamma zQ^{-1}, \tag{100}$$

$$\Delta\equiv-i\omega+\gamma\left(1-\frac{1}{Q}\right),\quad z=z_1+z_2. \tag{101}$$

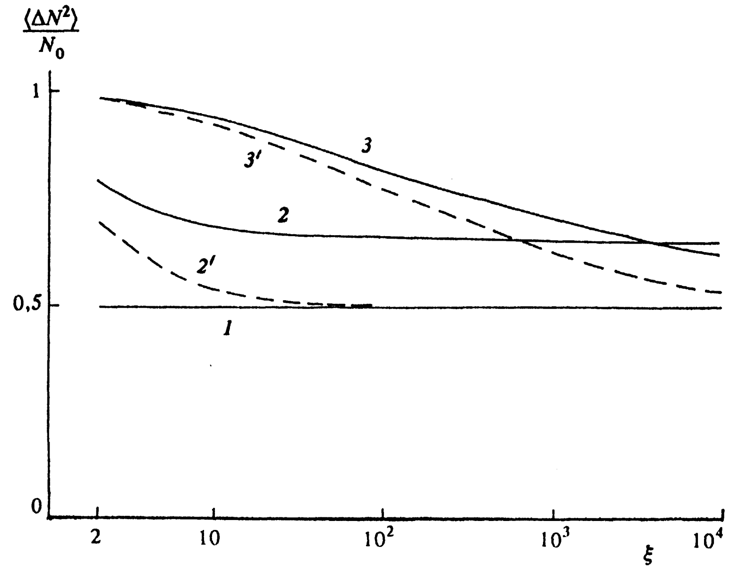

FIG. 2. Ratio of stationary fluctuations of the total number of photons in the cavity of a multimode laser $\langle \Delta N^2 \rangle$ to the mean number of photons N_0 as a function of the excess ξ of the pump above threshold ($\Pi = 1\ \mathrm{s}^{-1}$,) $\Gamma_1 = 10^9\ \mathrm{s}^{-1}$, $\Gamma_2 = 10^7\ \mathrm{s}^{-1}$; *1*—$Q=101$, $\gamma = 10^6$; s^{-1}; *2*—$Q=3$, $\gamma=10^6$; s^{-1}; *3*—$Q=3$, $\gamma = 10^9\ \mathrm{s}^{-1}$. The dashed curves correspond to regular pumping.

Applying the Wigner–Khinchin theorem (see footnote 1) and using (99) we obtain for the stationary photon-number fluctuation

$$\langle \Delta N_{\mathrm{ph}}^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta_1|^{-2} \{ |\Pi N_{\mathrm{ph}0} z_2|^2 \langle 2D_{11} \rangle + |\Pi N_{\mathrm{ph}0} z_1|^2 \langle 2D_{22} \rangle + |\zeta|^2 \langle 2D_{N_{\mathrm{ph}} N_{\mathrm{ph}}} \rangle - \Pi N_{\mathrm{ph}0} 2\,\mathrm{Re}(z_2^* \zeta) \langle 2D_{1N_{\mathrm{ph}}} \rangle + \Pi N_{\mathrm{ph}0} 2\,\mathrm{Re} \times (z_1^* \zeta) \langle 2D_{2N_{\mathrm{ph}}} \rangle - (\Pi N_{\mathrm{ph}0})^2 2\mathrm{Re}\text{-}(z_2^* z_1) \times \langle 2D_{12} \rangle \} d\omega, \langle 2D_{1N_{\mathrm{ph}}} \rangle = Q \langle 2D_{1q} \rangle,$$
$$\langle 2D_{2N_{\mathrm{ph}}} \rangle = Q \langle 2D_{2q} \rangle, \quad \langle 2D_{N_{\mathrm{ph}} N_{\mathrm{ph}}} \rangle = Q \langle 2D_{qq} \rangle. \qquad (102)$$

The diffusion coefficients in (102) are obtained from (60)–(65) by substituting the stationary mean values N_0, n_0, N_{10}, and D_0, equations for which were obtained in Sec. 3.

Calculations using (102) are shown in Fig. 1 for $Q=1$ (single-mode laser) for random pumping with a damping constant $\gamma = 10^6\ \mathrm{s}^{-1}$ and laser-level decay rates (in s^{-1}) $\Gamma_1 = 10^9$ and 10^7 (curve *1*, we omit hereafter the dimensionalities (s^{-1}) of γ, Γ_1, Γ_2, and Π). For regular pumping we consider the two characteristic situations $\Gamma_1 \gg \Gamma_2$ ($\Gamma_1 = 10^9$, $\Gamma_2 = 10^7$) and $\Gamma_1 \approx \Gamma_2$ ($\Gamma_1 = \Gamma_2 = 10^9$). Just as in Ref. 2, we assume for regular pumping that the term responsible for the pump noise in expression (61) for the diffusion coefficient vanishes ($R=0$). The pump-dependent term in Eq. (51) for the population should be conserved, since it determines the stationary operating point of the laser.

In the case of random pumping, the photon-number fluctuations for large ξ reach the Poisson value $\langle \Delta n^2 \rangle = n_0$. With increase of ξ, in accord with the data of Refs. 2–6, sub-Poisson statistics are obtained for regular pumping, and at $\Gamma_1 \gg \Gamma_2$ the fluctuation level reaches a minimum $n_0/2$ equal to the vacuum fluctuations.

The fluctuations of the total number of photons at $Q > 1$ are shown in Fig. 2. Curves *2* and *3* were obtained for $Q=3$, for $\gamma = 10^6$ and 10^9, $\Gamma_1 = 10^9$, and $\Gamma_2 = 10^7$. Curves *2′* and *3′* correspond to regular pumping at the same parameter values. It is seen from the plots that in the multimode case the fluctuations above the threshold are less than the Poisson value, i.e., sub-Poisson statistics obtains, whereas in the case of regular pumping $n_0/2$ reaches a minimum. The vacuum level is not reached for $Q=101$ (curve *1*), but sub-Poisson statistics obtains, and pump noise plays no significant role. Calculation has shown that at large Q ($Q = 101$) the fluctuations are independent of γ. Introduction of regular pumping does not alter the results in this case.

5. PHOTON-NUMBER FLUCTUATIONS IN AN INDIVIDUAL MODE

Let us consider Eq. (95) for the Fourier transforms of the deviations Δn_q of the photon numbers from the stationary mean in a mode of a multimode laser. Using, as in Sec. 4, the equations for $\Delta n_{q\omega}$ and the Wiener-Khinchin theorem, we obtain an expression for the stationary fluctuations $\langle \Delta n_q^2 \rangle$. Since we are using the equivalent-mode approximation, we have $n_0 = n_{q0} = N_0/Q$ and in addition $\Delta N_{\mathrm{ph}\omega} = \Sigma_{q=1}^{Q} \Delta n_{q\omega}$. Substituting expression (96) for the Fourier transformation of population-inversion variation ΔD_ω in (95), we obtain

$$\Delta n_{q\omega} = \frac{\Pi n_0 (-z\gamma \Delta N_{\mathrm{ph}\omega} - z_2 G_{1\omega} + z_1 G_{2\omega}) + \zeta G_{q\omega}}{\zeta \Delta}, \qquad (103)$$

$$\Delta \equiv -i\omega + \left(1 - \frac{1}{Q}\right)\gamma, \qquad (104)$$

$$z = z_1 + z_2 . \qquad (105)$$

Using Eq. (89) for ΔN_{ph} we obtain for $\{\Delta n_{q\omega})$ a set of equations in the form

$$A \Delta n_{q\omega} + B \sum_{q' \neq q} \Delta n_{q'\omega} = C_q , \qquad (106)$$

$$A \equiv \zeta \Delta + z\gamma \Pi n_0 , \qquad (107)$$

$$B = \Pi n_0 \gamma z, \tag{108}$$

$$C_q \equiv -z_2 \Pi n_0 G_{1\omega} + z_1 \Pi n_0 G_{2\omega} + \zeta G_{q\omega}. \tag{109}$$

The quantities ζ, z_1, and z_2 are given by expressions (97) and (98). The solution of this set of algebraic equations for the qth mode is

$$\Delta n_{q\omega} = \frac{C_q + \frac{B}{A-B} \Sigma_{q' \neq q} (C_q - C_{q'})}{A + (Q-1)B}. \tag{110}$$

It is convenient further to rewrite (110) in terms of random sources:

$$\Delta n_{q\omega} = \frac{\alpha G_{1\omega} + \beta G_{2\omega} + \mu G_{q\omega} + \nu \Sigma_{q' \neq q} G_{q'\omega}}{A + (Q-1)B}, \tag{111}$$

$$\alpha \equiv -z_2 \Pi n_0, \tag{112}$$

$$\beta \equiv z_1 \Pi n_0, \tag{113}$$

$$\mu \equiv \frac{\zeta}{A-B} [A + B(Q-2)], \tag{114}$$

$$\nu \equiv -\frac{B\zeta}{A-B}. \tag{115}$$

It is also easy to obtain (see footnote 1) for the photon-number fluctuations in a mode

$$\begin{aligned} \langle \Delta n_q^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} & |A + (Q-1)B|^{-2} \Big\{ |\alpha|^2 \langle 2D_{11} \rangle \\ & + |\beta|^2 \langle 2D_{22} \rangle + |\nu|^2 \sum_{q' \neq q} \langle 2D_{qq'} \rangle + 2\,\mathrm{Re}(\alpha^* \beta) \\ & \times \langle 2D_{12} \rangle + 2\,\mathrm{Re}(\alpha^* \mu) \langle 2D_{1q} \rangle + 2\,\mathrm{Re}(\beta^* \mu) \\ & \times \langle 2D_{2q} \rangle + 2\,\mathrm{Re}(\alpha^* \nu) \sum_{q' \neq q} \langle 2D_{q'1} \rangle \\ & + 2\,\mathrm{Re}(\beta^* \nu) \sum_{q' \neq q} \langle 2D_{q'2} \rangle \Big\} d\omega. \end{aligned} \tag{116}$$

The results of calculations using (116) are shown in Fig. 3 for a radiative damping constant $\gamma = 10^6$. Curves *1, 2,* and *3* correspond to the number of modes $Q = 3$ for $\Gamma_1 = 10^{11}$, $\Gamma_2 = 10^7$, $\Gamma_1 = 10^9$, $\Gamma_2 = 10^7$ and $\Gamma_1 = \Gamma_2 = 10^9$, respectively. It follows from the plots of Fig. 3 that if $\Gamma_1 \gg \Gamma_2$ (curves *1* and *2*) the number of photons in an individual mode is lower than the Poisson value over an appreciable interval of ξ. For very large ξ the fluctuations in cases *2* and *3* increase rapidly and the photons have a super-Poisson statistics, whereas in case *1* ($\Gamma_1 \gg \Gamma_2$) the growth is insignificant. The presence of regular pumping (dashed curves near *1, 2,* and *3*) does not alter the result significantly.

Curves *4, 5,* and *6* were obtained for $Q = 101$ for $\Gamma_1 = 10^{11}$, $\Gamma_2 = 10^7$; $\Gamma_1 = 10^9$, $\Gamma_2 = 10^7$ and $\Gamma_1 = \Gamma_2 = 10^9$. In this case of many modes, the number of fluctuations has a minimum over a large range of ξ, owing to the vacuum fluctuations $n_0/2$, while for $\Gamma_1 \gg \Gamma_2$ (case *4*) the fluctuations reach a minimum everywhere. In cases *5* and *6* the fluctuations increase almost linearly with increase of the pump.

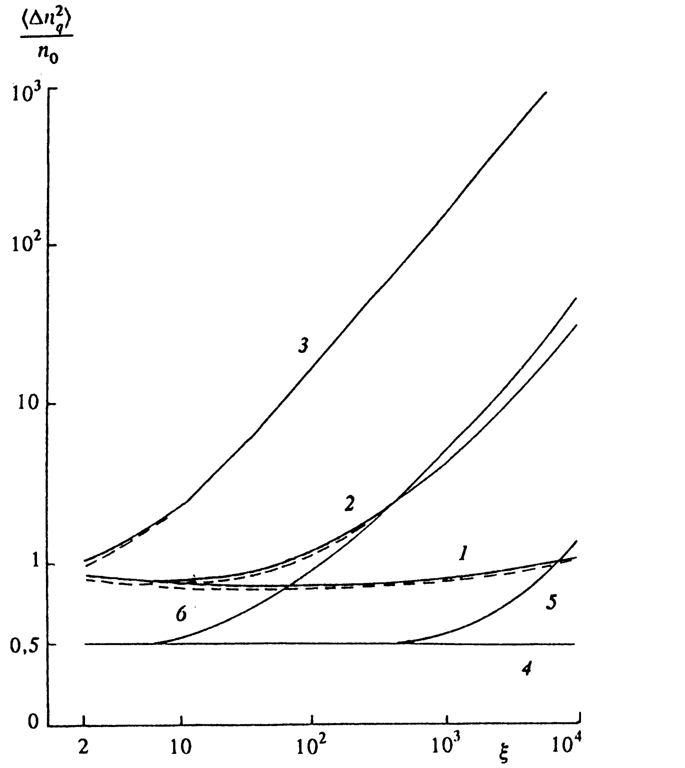

FIG. 3. Ratio of the stationary fluctuations of the number of photons in an individual mode of the cavity of a multimode laser $\langle \Delta n^2 \rangle$ to the mean stationary number n_0 of photons, as a function of the excess ξ of the pump above threshold ($\Pi = 1\ \mathrm{s}^{-1}$, $\gamma = 10^6\ \mathrm{s}^{-1}$); *1*—$Q = 101$, $\Gamma_1 = 10^{11}\ \mathrm{s}^{-1}$, $\Gamma_2 = 10^7\ \mathrm{s}^{-1}$; *2*—$Q = 101$, $\Gamma_1 = 10^9\ \mathrm{s}^{-1}$, $\Gamma_2 = 10^7\ \mathrm{s}^{-1}$; *3*—$Q = 101$, $\Gamma_1 = \Gamma_2 = 10^9\ \mathrm{s}^{-1}$; *4*—$Q = 3$, $\Gamma_1 = 10^{11}\ \mathrm{s}^{-1}$, $\Gamma_2 = 10^7\ \mathrm{s}^{-1}$; *5*—$Q = 3$, $\Gamma_1 = 10^9\ \mathrm{s}^{-1}$, $\Gamma_2 = 10^7\ \mathrm{s}^{-1}$; *6*—$Q = 3$, $\Gamma_1 = \Gamma_2 = 10^9\ \mathrm{s}^{-1}$. The dashed curves correspond to regular pumping

The calculation results for $\gamma = 10^9$ differ little from those for $\gamma = 10^6$. The statistical properties of the pump at $Q = 101$ have likewise no effect on the result.

This unexpected behavior of the fluctuations in an individual mode of a multimode laser, much more complicated than that of the fluctuations of a single-mode laser and the fluctuations of the total number of photons in a multimode laser, calls for a more detailed analysis of the multimode case, which is possible in particular within the framework of the quasilinear approximation, using adiabatic exclusion of the atomic variables. The results of such an analysis are given in the next section.

6. CALCULATION OF QUANTUM FLUCTUATIONS IN A QUASILINEAR APPROXIMATION ADIABATIC IN THE ATOMIC VARIABLES

We assume in this section that $\Gamma_1 \gg \gamma_q$ and $\Gamma_2 \gg \gamma_q$. It can be assumed in this case that $d\Delta N_i/dt \ll \Gamma_i \Delta N_i$, $i = 1, 2$ in the equations (88) and (89) for the population deviations from stationary values, and adiabatic exclusion of the variables ΔD and ΔN_1 is possible. In the case of a significant excess above threshold we have for an adiabatic quantities

$$\Delta N_1=\Gamma_1^{-1}\left(\frac{\gamma}{Q}\Delta N_{\mathrm{ph}}+N_0\Delta D\Pi+G_1\right), \tag{117}$$

$$\Delta D\approx-\frac{\Delta N_{\mathrm{ph}}}{N_0}\frac{\gamma}{Q\Pi}+\frac{1}{\Pi N_0(1+\Gamma_2/\Gamma_1)}\left(G_2-\frac{\Gamma_2}{\Gamma_1}G_1\right) \tag{118}$$

(it is assumed in Eq. (118) that $\xi-1\gg1$).

For an individual mode we obtain

$$\frac{d\Delta n_q}{dt}=-\gamma\left(1-\frac{Q-1}{Q^2}\right)\Delta n_q-\sum_{q'\neq q}\frac{\gamma}{Q^2}\Delta n_{q'}+G, \tag{119}$$

$$G=G_q+\frac{1}{Q(1+\Gamma_2/\Gamma_1)}\left(G_2-\frac{\Gamma_2}{\Gamma_1}G_1\right). \tag{120}$$

In the quasilinear approximation used by use, the Langevin equations (119) and (120) for the photon numbers can be written in the following form which is useful for the calculation of second moments of random quantities

$$\frac{d\Delta n_q}{dt}=-\Lambda(\Delta n_1+\ldots+\sigma\Delta n_q+\ldots+\Delta n_Q)+G, \tag{121}$$

$$\Lambda=\frac{\gamma}{Q^2}, \tag{122}$$

$$\sigma=Q^2\left(1-\frac{Q-1}{Q^2}\right), \tag{123}$$

$$\sigma=Q^2\left(\frac{Q^2-Q+1}{Q^2}\right)\approx Q^2\gg1\quad\text{for } Q\gg1. \tag{124}$$

We write the system of stochastic equations for the corresponding Markov processes in a general Langevin form

$$\frac{d\Delta n_q}{dt}=-A_q(t,\Delta n_1,\ldots,\Delta n_Q)+G_q, \tag{125}$$

where A_q is a constant displacement linear in Δn_i and G represents random sources. The following general equation of motion[16] for second moments can be used in Eq. (125):

$$\frac{\partial}{\partial t}\langle\Delta n_q\Delta n_{q'}\rangle=\langle 2D_{qq'}\rangle+\langle\Delta n_qA_{q'}\rangle+\langle A_q\Delta n_{q'}\rangle, \tag{126}$$

and its stationary solution yields the necessary set of the second moments that determine the stationary fluctuations. Setting the derivatives in (126) equal to zero, we obtain

$$\frac{\langle 2D_{qq'}\rangle}{\Lambda}=2\sigma\langle\Delta n_q\Delta n_{q'}\rangle+\sum_{k\neq q}\langle\Delta n_{q'}\Delta n_k\rangle+\sum_{k\neq q'}\langle\Delta n_k\Delta n_q\rangle. \tag{127}$$

Considering diagonal diffusion coefficients ($q=q'$) we get

$$\frac{\langle 2D_{qq}\rangle}{\Lambda}=2\sigma\langle\Delta n_q^2\rangle+2(Q-1)\delta, \tag{128}$$

where δ are nondiagonal correlation functions, which are equal to one another in our equivalent-mode approximation.

If, however, $q\neq q'$, we have

$$\frac{\langle 2D_{qq'}\rangle}{\Lambda}=2\sigma\delta+\frac{\langle 2D_{qq}\rangle}{\sigma\Lambda}-2\,\frac{Q-1}{\sigma}\,\delta+2(Q-2)\delta, \tag{129}$$

whence, taking (128) into account, we have for the stationary value of the nondiagonal correlation function

$$\langle\Delta n_q\Delta n_{q'}\rangle=\frac{\langle 2D_{qq'}\rangle/\Lambda-\langle 2D_{qq}\rangle/\Lambda\sigma}{2\sigma-2\,(Q-1)/\sigma+2(Q-2)}, \tag{130}$$

and for the photon-number variance

$$\langle\Delta n_q^2\rangle=\frac{\langle 2D_{qq}\rangle}{2\sigma\Lambda}-\frac{Q-1}{2}\,\frac{\langle 2D_{qq'}\rangle/\Lambda-\langle 2D_{qq}\rangle/\sigma\Lambda}{\sigma^2+\sigma(Q-2)-Q+1}. \tag{131}$$

We define the diffusion coefficients $\langle 2D_{qq}\rangle$ and $\langle 2D_{qq'}\rangle$ in accordance with

$$\langle G_q(t)G_q(t')\rangle=\langle 2D_{qq}\rangle\delta(t-t'),$$
$$\langle G_q(t)G_{q'}(t')\rangle=\langle 2D_{qq'}\rangle\delta(t-t'), \tag{132}$$

where the fluctuation sources in the correlators are given by (120). Using Eqs. (78)–(82) for the stationary diffusion, as well as expressions (72) and (74)–(77), we obtain

$$\langle 2D_{qq}\rangle=\gamma n_0+\frac{\gamma n_0}{Q}+2\left(\frac{\bar{\Gamma}}{Q\Gamma_2}\right)^2\frac{\gamma\Gamma_2}{\Pi Q}\approx\gamma n_0+\frac{\gamma n_0}{Q} \tag{133}$$

for $Q>1$. We denote here

$$\eta\equiv\frac{\Gamma_2}{\Gamma_1+\Gamma_2}(\xi-1)=\frac{\Pi Q^2}{\Pi_1}n_0. \tag{134}$$

For the off-diagonal diffusion coefficient we have

$$\langle 2D_{qq'}\rangle=\langle 2D_{qq}\rangle-\gamma n_0-\Pi n_0S_0. \tag{135}$$

Using (133), we find that above the threshold

$$\langle 2D_{qq'}\rangle=\frac{\gamma n_0}{Q}-\Pi S_0n_0=-2\gamma\frac{\Pi Q}{\Gamma_1}n_0^2=-2\gamma\frac{\eta}{Q}n_0. \tag{136}$$

Substituting (133) and (136) in (131), we arrive at the following expression for photon-number fluctuations in a mode above the lasing threshold, for arbitrary $Q>1$,

$$\langle(\Delta n_q)^2\rangle=\frac{(Q+1)(Q^3+1)}{Q^2(Q^2-Q+1)}\frac{n_0}{2}+\frac{\eta}{Q^2}n_0. \tag{137}$$

If the number of modes is large ($Q\gg1$), Eq. (137) reduces to

$$\langle(\Delta n_q)^2\rangle=\frac{n_0}{2}+n_0\frac{\eta}{Q^2}=\frac{n_0}{2}+\frac{\Pi}{\Gamma_1}n_0^2. \tag{138}$$

If $\Gamma_1\gg1$ and $\eta\to0$, the fluctuations reach a minimum value

$$\langle\Delta n_q^2\rangle\to\frac{n_0}{2}, \tag{139}$$

but if $\Gamma_1\approx\Gamma_2$ and $\eta\to(\xi-1)/2$, the fluctuations at low pumps are close to vacuum fluctuations, but increase linearly for large ξ in accordance with the equation

$$\langle\Delta n_q^2\rangle\to\frac{n_0}{2}+\frac{\xi-1}{2Q^2}n_0. \tag{140}$$

Consider now the second term in expression (137) for the fluctuations. Using (134) with $Q \gg 1$ and multiplying the numerator and denominator of this term by N_{10}, we obtain at $\xi \gg 1$

$$\frac{\Pi N_0 N_{10}}{\Gamma_1 N_{10}} \frac{n_0}{Q} \approx \frac{R}{Q^2 R_{\text{th}}} \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} n_0.$$

Evidently $\Pi N_0 N_{10}/\Gamma_1 N_{10}$ is the ratio of the stimulated absorption to the spontaneous decay of the lower laser level. It can therefore be concluded that the level of the vacuum fluctuations is reached in a laser mode at the minimum ratio of these quantities, and in particular in the case of an empty lower laser level, $\Gamma_1 \to \infty$.

7. DISCUSSION OF RESULTS AND CONCLUSION

The quantum calculations of the present paper have shown that the photon statistics in the cavity of a multimode laser above the lasing threshold differ substantially from the single-mode case. The total number of photons in the cavity of a multimode laser becomes stabilized, just as in a single-mode one, but in the multimode case one observes a fundamental dependence of the photon-number fluctuation on the number of modes participating in the lasing, on the laser-level relaxation parameters, and on the radiative damping constants of the field modes. The fluctuation is below the Poisson level at large excess of the pump above the threshold, whereas for many modes ($Q = 101$) and for $\Gamma_1 \gg \Gamma_2$ a minimum fluctuation level $n_0/2$ due to vacuum observations is observed in the entire range $2 < \xi < 10$, while for few modes ($Q=3$) the fluctuation level is higher, but the field still retains sub-Poisson statistics at $10^6 \leqslant \gamma \leqslant 10^9$. In the presence of regular pumping, the fluctuations for large ξ tend to the vacuum-fluctuations level for any number of modes.

Calculations for a separately taken mode have shown that the number of photons in a mode is also stabilized at a moderate excess above threshold, while under the conditions $\Gamma_1 \gg \Gamma_2$ and $Q \gg 1$ the photon-number fluctuations reach the minimum vacuum level in the entire range of ξ. If Γ_1 and Γ_2 are comparable or equal, the fluctuations increase with ξ and the photons have a super-Poisson statistics at $\xi \gg 1$. If the number of modes is small ($Q=3$) a super-Poisson statistics is observed in the entire range of pump excess above threshold $\xi > 2$.

It is interesting to note that in contrast to fluctuations of the total number of photons, the photon-number fluctuations in an individual mode are practically independent of the radiative-damping constant.

Our results contradict those obtained by semiclassical calculations[11] where, in general, a value indicative of random blackbody radiation, $\langle \Delta n_q^2 \rangle \sim n_0^2$ was obtained for mode intensity fluctuations of a multimode laser in the high-pumping limit. Note the model-like character of the computations of Ref. 11, using approximate semiclassical Van der Pol equations that are correct only near the lasing threshold,[1,16] and a model-dependent rigid intermode coupling was considered.

In view of the above contradiction, we must mention the results of a study[12] where, using a substantially more accurate semiclassical theory than in Ref. 11, a numerical simulation has shown for the actual case of a dye laser that even near the threshold (at an excess $\lesssim 10\%$) the observed chaotic behavior of the intensity in the mode is due not to quantum noise but to the presence, in the equations, of a term responsible for four-wave interaction. Our present results lead to the conclusion that nonstationary field pulsations in the modes of a multimode laser are, under our conditions, the consequence of a dynamic interaction between modes, whereas above the threshold the contribution of quantum fluctuations is insignificant.

It was stated in Refs. 17 and 18 that there is no intensity stabilization at all in a mode of a multimode laser. This was deduced from simplified balance equations with random sources, without allowance for the fluctuations of the laser-level population, and under the assumption that the lowest laser level is empty. This is completely contradicted by our present results, which are based on a consistent application of the quantum Langevin formalism, as follows from the foregoing. It is noteworthy that the fluctuation obtained in Ref. 18 for the total photon number in the case of a threefold or larger excess above the threshold becomes smaller than the minimum value of the vacuum fluctuations, thus contradicting also the data of Ref. 11.

We note in conclusion that we have considered in our laser model a case when the spontaneous decay of the laser levels takes place directly in the lower atomic state of a three-level system, i.e., it has been assumed that the rate of the spontaneous decay of the upper laser level to the lower one is much lower than the rate of its decay in the lower level. Allowance, in the laser model, for spontaneous transitions between the laser levels can apparently alter somewhat the calculated photon-number fluctuations.

The author thanks A. F. Suchkov for calling attention to the question of quantum noise in a multimode laser.

[1)] According to the Wiener–Khinchin theorem, the mean square d_t over an ensemble of realizations of a random process is connected with the (real) spectral density $\langle f_\omega f^*_{\omega'} \rangle = G(\omega)\delta(\omega - \omega')$ by the relation $\langle f_t^2 \rangle = 1/2\pi \int_{-\infty}^{\infty} G(\omega) d\omega$ where f_ω and f_t are mutual Fourier transforms. Assuming that $f_\omega = A(\omega) F_\omega$, $f^*_{\omega'} = A^*(\omega') F^*_{\omega'}$, and also, since the random process is Markovian, that $\langle F_{t'} F_t^* \rangle = 2D\delta(t - t')$, where $2D$ is the stationary value of the diffusion coefficient, we obtain $\langle f_t^2 \rangle = D/2\pi \int_{-\infty}^{\infty} A(\omega) A^*(\omega) d\omega$.

[2)] Here and below we omit from the balance equations, in contrast to Eqs. (50)–(52), the terms responsible for the spontaneous emission and contained in the constant Langevin displacments, since these terms are small compared with the others.

[1] F. T. Arecchi, M. O. Scully, H. Haken, and W. Weidlich, in *Quantum Optics*, R. J. Glauber, ed., Academic, 1969.

[2] T. A. B. Kennedy and D. F. Walls, Phys. Rev. A **40**, 6366 (1989).

[3] Yu. M. Golubev and I. V. Sokolov, Zh. Eksp. Teor. Fiz. **87**, 408 (1984) [Sov. Phys. JETP **60**, 234 (1984)].

[4] C. Benkert, M. O. Scully, J. Bergou, L. Davidovich, M. Hillery, and M. Orszag, Phys. Rev. A **41**, 2756 (1990).

[5] C. Benkert and M. O. Scully, *ibid.* A **42**, 2817 (1990).

[6] Y. Yamamoto, S. Mashida, and O. Nilsson, *ibid.* **34**, 4025 (1986).

[7] D. E. McCumber, Phys. Rev. **141**, 306 (1966).

[8] M. M-Tehrani and L. Mandel, Phys. Rev. A **17**, No. 2, 677 (1978).

[9] M. M-Tehrani and L. Mandel, *ibid.* p. 694 (1978).

[10]P. Lett and L. Mandel, J. Opt. Soc. Am. B **2**, 1615 (1985).
[11]F. T. Hioe, J. Math. Phys. **19**, 1307 (1978).
[12]I. McMackin, C. Radzewicz, M. Beck, and M. G. Reymer, Phys. Rev. A **38**, 820 (1988).
[13]W. Brunner, R. Fisher, and H. Paul, J. Opt. Soc. Am. B **2**, 202 (1985).
[14]M. Lax, Phys. Rev. **145**, 110 (1966).
[15]M. Lax and W. H. Louisell, ibid. **185**, 568 (1969).
[16]M. Lax, in *Fluctuations and Coherence Phenomena*, Gordon and Breach, 1968, pp. 271–478.
[17]V. R. Mironenko and V. N. Yudson, Zh. Eksp. Teor. Fiz. **79**, 1174 (1980) [Sov. Phys. JETP **52**, 594 (1980)].
[18]S. A. Kovalenko, Kvant. Elektron. (Moscow) **8**, 1271 (1981) [Sov. J. Quant. Electron. **11**, 759 (1981)].

Translated by J. G. Adashko