Nonlocal Josephson electrodynamics

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The fundamental nonlinear nonlocal integrodifferential equation of Josephson electrodynamics is derived for a tunnel junction between two different superconductors. A general relation for the magnetic field is obtained to determine the magnetic field in superconductors from the Cooper-pair phase difference. The Lagrangian and Hamiltonian of the function are determined. A solution is obtained in the asymptotic nonlocal limit for a kink traveling with constant velocity and carrying, in contrast to the traveling kink of the local theory, two magnetic-flux quanta.

1. INTRODUCTION

A nonlocal nonlinear integrodifferential equation was obtained in Ref. 1 for the phase difference of superconducting pairs on a Josephson junction between two like bulk superconductors. This equation was used there as a basis for nonlocal Josephson electrodynamics. Such an equation generalizes the usual sine-Gordon equation[2] to include the case when the characteristic spatial change of the phase difference φ takes place at distances shorter than the Josephson length. The integrodifferential equation obtained in Ref. 1 was used to investigate the influence of the spatial nonlocality on the propagation of Swihart waves[3] and of weakly nonlinear perturbations. In the latter case it was shown that nonlocality influences the development of modulation instability of weakly linear waves.

An asymptotic solution of the equation proposed in Ref. 1 was obtained by Gurevich[4] in a strongly nonlocal limit, when the Josephson length turns out to be much shorter than the London depth of magnetic-field penetration. Just as in the local theory, such a solution corresponds to a 2π kink. It was shown also that in the limit considered the Josephson vortices correspond to Abrikosov vortices without singularities. Also discussed there was the usefulness of the nonlocal Josephson electrodynamics to a description of the phenomena in latent weak bonds of superconductors, and it was shown that the change of the vortex-core energy decreases the critical magnetic field H_{c1}. The same change of the energy of the vortex core decreases its mass as well as the viscosity. The equation obtained in Ref. 1 was generalized by Gurevich[4] to take into account resistive effects. In Ref. 5 we obtained the first nonstationary resistive solutions and showed how the stationary vortex described in Ref. 4 settles in time. The laws governing vortex decay were adduced.

Even the first investigations point to substantial consequences of nonlocal Josephson electrodynamics. This theory is nonetheless insufficiently complete, and naturally not fully developed. We therefore derive in the present communication the fundamental equation of nonlocal Josephson electrodynamics for a tunnel junction of two different superconductors. This equation goes over into an equation in Ref. 1 for a junction between like superconductors. Various asymptotic forms of the equation are given. The necessary equations are obtained for the description of the magnetic field of the vortices, their energy, and also the Lagrangian and Hamiltonian of a nonstationary vortex (fluxon). We obtain ultimately for the fundamental equation of nonlinear Josephson electrodynamics a first analytic solution describing a vortex traveling with constant velocity. This solution turns out to be a 4π kink moving with a fully defined velocity. Since the magnetic field of such a 4π kink corresponds to an Abrikosov vortex without a singularity, we can state that we present here the first description of a moving regular Abrikosov vortex. The magnetic field of such a vortex is formed here both by the Josephson current and by the displacement current. The latter distinguishes qualitatively the nature of a moving kink from the nature of one at rest.

The magnetic flux carried by a 4π kink comprises two magnetic-flux quanta. This is one of the principal differences between the obtained moving asymptotic kink and the Josephson local-electrodynamics kinks, including moving ones. We emphasize that the stationary kink considered by Gurevich[4] carries only one magnetic-flux quantum, just as the kinks of the local theory.

2. BASIC EQUATIONS

We obtain in this section the basic equations of Josephson electrodynamics for a tunnel junction between different superconductors. We consider a homogeneous tunnel layer of thickness $2d$ and bounded along the x axis by infinite different superconductors parallel to the yz plane. The magnetic field obeys in the superconductors the equations[3]

$$\lambda_-^2\Delta\mathbf{H}-\mathbf{H}=0,\quad x\leqslant -d,\quad \lambda_+^2\Delta\mathbf{H}-\mathbf{H}=0,\quad x\geqslant d,\tag{2.1}$$

where λ_- and λ_+ are the London magnetic-field penetration depths. We have accordingly for the electric field

$$\mathbf{E}=\frac{\lambda_-^2}{c}\operatorname{rot}\frac{\partial\mathbf{H}}{\partial t},\quad x\leqslant -d,\quad \mathbf{E}=\frac{\lambda_+^2}{c}\operatorname{rot}\frac{\partial\mathbf{H}}{\partial t},\quad x\geqslant d.\tag{2.2}$$

Inside the tunnel layer $(-d<x<d)$ we use for a plane geometry with $\mathbf{E}=(E_x,0,E_z)$ and $\mathbf{H}=(0,H_y,0)$ the corollaries of the Maxwell equations

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$$-\frac{\partial H_y}{\partial z}=\frac{4\pi}{c}j_x+\frac{1}{c}\frac{\partial D_x}{\partial t}, \qquad (2.3)$$

$$\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}=\frac{1}{c}\frac{\partial H_y}{\partial t}. \qquad (2.4)$$

Here $D_x=\varepsilon E_x$ is the electric induction and ε the dielectric constant of the layer. For the current density across the tunnel layer we use the standard expression[2]

$$j_x=j_c\sin\varphi+\sigma E_x, \qquad (2.5)$$

where j_c is the critical density of the current through the junction, σ is the conductivity of the tunnel layer, and φ is the phase difference of the Cooper pairs in the superconductors separated by the Josephson junction. Assuming the tunnel-layer thickness to be thin compared with the penetration depth, we regard not only φ but also E_x as independent of the coordinate x inside the layer. We have then for the potential difference V between the junction boundaries

$$V(z,t)=-\int_{-d}^{d}dxE_x\approx-2dE_x(z,t). \qquad (2.6)$$

The nonstationary Josephson relation can accordingly be written in the form

$$\frac{\partial\varphi(z,t)}{\partial t}=\frac{2eV}{\hbar}=-\frac{4ed}{\hbar}E_x(z,t), \qquad (2.7)$$

where $e=-|e|$. Equations (2.5) and (2.6) allow us to rewrite (2.3) in the form

$$\frac{4\pi}{c}j_c\sin\varphi-\frac{4\pi\sigma}{c}\frac{\hbar}{4ed}\frac{\partial\varphi}{\partial t}-\frac{\varepsilon}{c}\frac{\hbar}{4ed}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\partial H_y}{\partial z}. \qquad (2.8)$$

In standard local hydrodynamics of Josephson junctions, the right-hand side of Eq. (2.8) reduces to the second spatial derivative of the phase difference φ. In Ref. 1 it was shown, for the particular case of a junction between like bulky superconductors, how nonlocality arises in Josephson electrodynamics. We derive below a nonlocal integro-differential equation for the general case of a tunnel junction between two different superconductors.

We note first that the z-component of the electric field inside a tunnel layer can be approximately expressed in the form

$$E_z(x,z,t)=\frac{1}{2}[E_z(d,z,t)+E_z(-d,z,t)]+\frac{x}{2d}[E_z(d,z,t)-E_z(-d,z,t)]. \qquad (2.9)$$

This enables us to rewrite Eq. (2.4) for the region inside the junction in the form

$$\frac{1}{2d}[E_z(d,z,t)-E_z(-d,z,t)]+\frac{\hbar}{4ed}\frac{\partial^2\varphi(z,t)}{\partial z\partial t}=\frac{1}{c}\frac{\partial H_y}{\partial t}. \qquad (2.10)$$

It follows hence, in particular, that in a thin layer the magnetic field does not vary with the coordinate x. If we use now the corollaries of Eqs. (2.2) for $x=\pm d$

$$E_z(-d,z,t)=\frac{\lambda_-^2}{c}\frac{\partial^2 H_y}{\partial x\partial t}\bigg|_{x=-d},\quad E_z(d,z,t)=\frac{\lambda_+^2}{c}\frac{\partial^2 H_y}{\partial x\partial t}\bigg|_{x=d},$$

and substitute them in Eq. (2.10), we get

$$\lambda_+^2\frac{\partial H_y(x,z,t)}{\partial x}\bigg|_{x=d}-\lambda_-^2\frac{\partial H_y(x,z,t)}{\partial x}\bigg|_{x=-d}+\frac{\hbar c}{2e}\frac{\partial\varphi(z,t)}{\partial z}=2dH_y(z,t). \qquad (2.11)$$

Here $H_y(z,t)$ is the magnetic field inside the tunnel layer. It can be regarded as the limit for magnetic fields inside a superconductor. These fields must be determined, in particular, to be able to express the right-hand side of (2.8) in terms of the phase difference φ. It is convenient then to write the corresponding solution in the form

$$H_y(x,z,t)=\int_{-\infty}^{\infty}dz'H_y(z',t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\{ik(z-z')-(\pm x-d)(k^2+\lambda_\pm^{-2})^{1/2}\}, \qquad (2.12)$$

where the $\pm$ signs correspond to the space regions $x\geqslant d$ and $x\leqslant -d$, respectively. Substitution of (2.12) in (2.11) yields

$$-\int_{-\infty}^{\infty}dz'H_y(z',t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}[\lambda_+^2(k^2+\lambda_+^{-2})^{1/2}+\lambda_-^2(k^2\lambda_-^{-2})^{1/2}]\exp[ik(z-z')]+\frac{\hbar c}{2e}\frac{\partial\varphi(z,t)}{\partial t}=2dH_y(z,t). \qquad (2.13)$$

This integral equation determines the connection between $H_y(z,t)$ and $\varphi(z,t)$, which is obtained by solving this equation and which is given by

$$H_y(z,t)=\frac{\hbar c}{2e}\int_{-\infty}^{\infty}dz'Q(z-z')\frac{\partial\varphi(z',t)}{\partial z'}, \qquad (2.14)$$

where

$$Q(z)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\exp(ikz)}{\lambda_+^2(k^2+\lambda_+^{-2})^{1/2}+\lambda_-^2(k^2+\lambda_-^{-2})^{1/2}+2d}. \qquad (2.15)$$

Equation (2.14) determines the right-hand side of (2.8), which can now be rewritten as

$$\sin\varphi+\frac{\beta}{\omega_J^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_J^2}\frac{\partial^2\varphi}{\partial t^2}=\lambda_0^3\frac{\partial}{\partial z}\int_{-\infty}^{\infty}dz'Q(z-z')\frac{\partial\varphi(z',t)}{\partial z'}, \qquad (2.16)$$

$$\omega_J^2=\frac{16\pi|e|dj_c}{\hbar\varepsilon}=\frac{2|e|j_c}{\hbar C_s},\quad \beta=\frac{4\pi\sigma}{\varepsilon}=\frac{1}{RC_s},\quad \lambda_0^3=\frac{\hbar c^2}{8\pi|e|j_c}, \qquad (2.17)$$

where $R=(2d/\sigma)$, and $C_s=(8\pi d/\varepsilon)^{-1}$ are the resistance and capacitance per unit are of the tunnel junction.

In the local limit, when we can neglect in the denominator of (2.15) the dependence on k, we have

$$Q(z)=\frac{1}{\lambda_{+}+\lambda_{-}+2d}S(z). \tag{2.18}$$

Using then the equation

$$\lambda_J^2=\frac{\lambda_0^3}{\lambda_{+}+\lambda_{-}+2d} \tag{2.19}$$

to determine the Josephson length λ_J, we obtain the usual sine-Gordon equation with damping:[2,3]

$$\sin\varphi+\frac{\beta}{\omega_J^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_J^2}\frac{\partial^2\varphi}{\partial t^2}-\lambda_J^2\frac{\partial^2\varphi}{\partial z^2}=0. \tag{2.20}$$

The local limit is realized here when the characteristic spatial scale of variation of φ is large compared with λ_{+} and λ_{-}.

Another simple limit is realized in the case $\lambda_{+}=\lambda_{-}=\lambda\gg 2d$, when

$$Q(z)=\frac{1}{2\pi\lambda^2}K_0\left(\frac{|z|}{\lambda}\right), \tag{2.21}$$

where $K_0(z)$ is a Macdonald function. Equation (2.16) takes then the form[1,4]

$$\sin\varphi+\frac{\beta}{\omega_J^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_J^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{\lambda_J^2}{\pi\lambda}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}dz'K_0\left(\frac{|z-z'|}{\lambda}\right)\frac{\partial\varphi(z',t)}{\partial z'} \tag{2.22}$$

Here $\lambda_J^2=(\lambda_0^3/2\lambda)$ in accord with Eq. (2.19). Equation (2.22) is realized also in the case $\lambda_{+}=\lambda\gg\lambda_{-}$ and $\lambda_{+}=\lambda\gg 2d$, the only difference being that $\lambda_J^2=\lambda_0^3/\lambda_{+}$, which again corresponds to Eq. (2.19).

To conclude this section we consider the asymptotic limit of (2.16), which corresponds to a case when φ changes abruptly over a length shorter than λ_{+} and λ_{-}. We can use then the approximate expression

$$Q(z)=\frac{1}{\pi(\lambda_{+}^2+\lambda_{-}^2)}\left\{\ln\frac{2}{|z|}-C+f(\lambda_{+},\lambda_{-})\right\}, \tag{2.23}$$

where

$$f(\lambda_{+},\lambda_{-})=(\lambda_{+}^2-\lambda_{-}^2)^{-1}\left\{\lambda_{+}^2\ln\lambda_{+}-\lambda_{-}^2\ln\lambda_{-}+\lambda_{+}\lambda_{-}\operatorname{arctg}\frac{\lambda_{-}^2-\lambda_{+}^2}{2\lambda_{+}\lambda_{-}}\right\}, \tag{2.24}$$

and $C=0.577$ is Euler's constant. Accordingly, Eq. (2.16) takes the form

$$\sin\varphi+\frac{\beta}{\omega_J^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_J^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{\lambda_0^3}{\lambda_{+}^2+\lambda_{-}^2}\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'}. \tag{2.25}$$

The integral in the right-hand side of the last equation is taken in the sense of the Cauchy principal value and corresponds to the usual Hilbert transformation.

3. MAGNETIC FIELD, TUNNEL-JUNCTION ENERGY, LAGRANGIAN, HAMILTONIAN

If Eq. (2.14) describes the magnetic field inside the tunnel layer, the expression for the field in the superconductors can be obtained with the aid of Eq. (2.12), which yields

$$H_y(x,z,t)=\frac{\hbar c}{2e}\int_{-\infty}^{\infty}dz'Q_{\pm}(z-z';\pm x-d)\frac{\partial\varphi(z',t)}{\partial z'}, \tag{3.1}$$

where

$$Q_{\pm}(z,x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\exp\{ikz-x(k^2+\lambda_{\pm}^{-2})^{1/2}\}}{\lambda_{+}^2(k^2+\lambda_{+}^{-2})^{1/2}+\lambda_{-}^2(k^2+\lambda_{-}^{-2})^{1/2}+2d}. \tag{3.2}$$

Here $Q_{\pm}(z,x=0)=Q(z)$. The plus and minus signs in (3.1) correspond respectively to the pace regions $x>d$ and $x<-d$.

In the usual local limit, when φ varies smoothly over space scale of the order of λ_{+} and λ_{-} we have

$$Q_{\pm}(z,x)=\frac{1}{\lambda_{+}+\lambda_{-}+2d}\delta(z)\exp\left(-\frac{|x|}{\lambda_{\pm}}\right). \tag{3.3}$$

The force lines of the local Josephson-vortex magnetic field are therefore stretched along the z axis and crowded toward the boundaries of the superconductors. In the case of two identical superconductors, when $\lambda_{+}=\lambda_{-}=\lambda\gg 2d$, Eq. (3.2) takes the form

$$Q_{\pm}(z,t)=\frac{1}{2\pi\lambda^2}K_0\left(\frac{1}{\lambda}\sqrt{z^2+x^2}\right). \tag{3.4}$$

The expression for the magnetic field agrees here with that of Ref. 4.

If, however, the superconductors on the opposite sides of the junction differ greatly, such that λ_{+} is much larger than λ_{-} and $2d$, one can use the approximate expressions

$$Q_{+}(z,t)=\frac{1}{\pi\lambda_{+}^2}K_0\left(\frac{1}{\lambda_{+}}\sqrt{z^2+x^2}\right), \tag{3.5}$$

$$Q_{-}(z,x)=\frac{1}{\pi\lambda_{+}^2}K_0\left(\frac{|z|}{\lambda_{+}}\right)\exp\left(-\frac{|x|}{\lambda_{-}}\right). \tag{3.6}$$

Finally, in the asymptotic limit when φ as a function of z changes drastically over a length shorter than λ_{+} and λ_{-}, one must use the asymptotic form

$$Q_{\pm}(z,x)=\frac{1}{\pi(\lambda_{+}^2+\lambda_{-}^2)}\left\{\frac{1}{2}\ln\frac{4}{x^2+z^2}-C+f(\lambda_{+},\lambda_{-})\right\}, \tag{3.7}$$

where $f(\lambda_{+},\lambda_{-})$ is given by Eq. (2.24). The asymptote (3.7) holds for x and z much smaller than λ_{+} and λ_{-} when the latter are large compared with $2d$.

Let us use Eq. (2.14) to calculate the magnetic-energy density in a tunnel layer $w=(1/8\pi)\mathbf{H}^2$, and Eq. (3.1) to determine the densities of the magnetic-energy and of the superconducting currents in the superconductors:

$$w_s=\frac{1}{8\pi}[\mathbf{H}^2+\lambda^2(\mathrm{rot}\,\mathbf{H})^2]. \tag{3.8}$$

The sum (per unit length of the y axis) of the energy of the superconducting currents of the superconductors and of the magnetic energy of the entire junction can then be written in the form

$$W_m=\frac{\hbar^2c^2}{32\pi e^2}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dz''Q(z'-z'')\times\frac{\partial\varphi(z',t)}{\partial z'}\frac{\partial\varphi(z'',t)}{\partial z''}. \tag{3.9}$$

The energy (per unit length of the y axis) of the Josephson current through the junction is given by[2]

$$W_J=\frac{\hbar j_c}{2|e|}\int_{-\infty}^{\infty}dz[1-\cos\varphi(z,t)]. \tag{3.10}$$

Thus, the total (per unit length of the y axis) energy of the Josephson junction, which is determined by the phase difference φ, is the sum $W_m+W_J=W$ and takes the form

$$W=\frac{\hbar j_c}{2|e|}\int_{-\infty}^{\infty}dz[1-\cos\varphi]+\frac{\hbar^2c^2}{32\pi e^2}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dz''Q(z'-z'')\frac{\partial\varphi}{\partial z'}\frac{\partial\varphi}{\partial z''}. \tag{3.11}$$

If dissipation is neglected ($\beta=0$) it is natural to use in view of Eq. (2.16) the concept of a Lagrangian

$$\mathscr{L}=\frac{\hbar j_c}{2|e|}\left\{\int_{-\infty}^{\infty}dz\left[\frac{1}{2\omega_J^2}\left(\frac{\partial\varphi}{\partial t}\right)^2-1+\cos\varphi\right]-\frac{1}{2}\lambda_0^3\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dz''Q(z'-z'')\times\frac{\partial\varphi(z',t)}{\partial z'}\frac{\partial\varphi(z'',t)}{\partial z''}\right\}. \tag{3.12}$$

The corresponding Hamiltonian is

$$\mathscr{H}=\frac{\hbar j_c}{2|e|}\left\{\int_{-\infty}^{\infty}dz\left[\frac{1}{2\omega_J^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+1-\cos\varphi\right]+\frac{1}{2}\lambda_0^3\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dz''Q(z'-z'')\times\frac{\partial\varphi(z',t)}{\partial z'}\frac{\partial\varphi(z'',t)}{\partial z''}\right\}. \tag{3.13}$$

The equations for the energy and for the Hamiltonian can be used to obtain approximate solutions.

The preceding equation can be represented as

$$\mathscr{H}=W+T, \tag{3.14}$$

where W is the energy (3.11) of the interacting Josephson vortices, and the second term, which we can call the kinetic energy, is

$$T=\int_{-d}^{d}dx\int_{-\infty}^{\infty}dz\frac{\varepsilon E_x^2}{8\pi}=\frac{\hbar^2C_s}{8e^2}\int_{-\infty}^{\infty}dz\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{3.15}$$

Allowance for the finite conductivity of the tunnel junction leads to Joule dissipation. The specific losses correspond to σE_x^2, so that by using (2.7) we can write

$$\frac{d\mathscr{H}}{dt}=-\int_{-d}^{d}dx\int_{-\infty}^{\infty}dz\sigma E_x^2=-\frac{\hbar^2}{4e^2R}\int_{-\infty}^{\infty}dz\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{3.16}$$

In the last two equations, the considered nonlocality of Josephson electrodynamics can become manifested only on account of new dependences of the phase difference φ on the coordinates and on the time. The ratio of $d\mathscr{H}/dt$ (3.16) and T (3.15) does not depend on the form of φ and is equal to -2β, as in the usual local electrodynamics. It reflects the ratio of the second and third terms of the left-hand side of Eq. (2.16).

4. TRAVELING VORTEX

In the usual local electrodynamics of Josephson junctions, with dissipation neglected, Eq. (2.20) has a known solution corresponding to a traveling vortex (fluxon) of the 2π-kink type:[2]

$$\varphi(z,t)=4\,\mathrm{arctg}\left\{\exp\left[\pm\frac{1}{(1-u^2)^{1/2}}\left(\frac{z}{\lambda_J}-u\omega_Jt\right)\right]\right\}, \tag{4.1}$$

where $|u|<1$. The $\pm$ signs correspond to the kink (soliton) and antikink (antisoliton), respectively. The velocity of such a vortex is determined in local electrodynamics by the sign and magnitude of the parameter u:

$$v_L=u\omega_J\lambda_J. \tag{4.2}$$

We consider in this section, neglecting dissipation, the solution of Eq. (2.25) corresponding to a traveling vortex (fluxon) of the 4π-kink type:

$$\varphi(z,t)=\phi\left(\frac{\lambda_+^3+\lambda_-^2}{\lambda_0^3}[z-vt]\right)\equiv\phi(\xi). \tag{4.3}$$

The function ξ satisfies here the equation

$$\alpha^2\frac{d^2\phi}{d\xi^2}+\sin\phi=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{d\phi(\xi')}{d\xi'}, \tag{4.4}$$

where

$$\alpha^2=\left(\frac{v}{\omega_J}\right)^2\left(\frac{\lambda_+^2+\lambda_-^2}{\lambda_0^3}\right)^2,$$

and the integral in the right-hand side of (4.4) is taken in the Cauchy sense of principal value.

It is easily seen that a solution of (4.4) is[1)]

$$\phi(\xi)=4\,\mathrm{arctg}(\pm\xi). \tag{4.5}$$

The velocity of such a nonlocal kink (antikink) is determined then by the relation

$$v^2=\frac{\lambda_0^6\omega_J^2}{(\lambda_+^2+\lambda_-^2)^2}=(\lambda_J\omega_J)^2\left[\frac{\lambda_J(\lambda_++\lambda_-)}{\lambda_+^2+\lambda_-^2}\right]^2. \quad (4.7)$$

The two signs of the solution of the last equation correspond to the two possible directions of vortex motion.

The magnetic field of the traveling kink (4.6) is given by[2)]

$$H_y(x,z,t)=-\frac{2\hbar c}{e(\lambda_+^2+\lambda_-^2)}\left\{f(\lambda_+\lambda_-)-C-\frac{1}{2}\times\ln\left[\frac{1}{4}\left(\left(|x\mp d|+\frac{\lambda_0^3}{\lambda_+^2+\lambda_-^2}\right)^2+(z-vt)^2\right)\right]\right\}. \quad (4.8)$$

In accordance with (3.14), the Hamiltonian can be represented as a sum $\mathscr{H}=W+T$. We have then for the kink (4.6)

$$T=\frac{\hbar^2c^2}{4e^2}\frac{1}{\lambda_+^2+\lambda_-^2}, \quad (4.9)$$

$$W=\frac{\hbar^2c^2}{2e^2(\lambda_+^2+\lambda_-^2)}\left\{\frac{1}{2}-C+\ln\left[(\lambda_+^2+\lambda_-^2)\times\left(\frac{\lambda_+\lambda_-}{\lambda_0^6}\right)^{1/2}\right]+\frac{1}{\lambda_+^2-\lambda_-^2}\left[\frac{1}{2}(\lambda_+^2+\lambda_-^2)\ln\frac{\lambda_+}{\lambda_-}-\lambda_+\lambda_-\operatorname{arctg}\frac{\lambda_+^2-\lambda_-^2}{2\lambda_+\lambda_-}\right]\right\}. \quad (4.10)$$

The total energy carried by the 4π kink is

$$\mathscr{H}=\frac{\Phi_0^2}{2\pi^2(\lambda_+^2+\lambda_-^2)}\left\{1-C+\ln\left[(\lambda_+^2+\lambda_-^2)\times\frac{(\lambda_+\lambda_-)^{1/2}}{\lambda_0^3}\right]+\frac{1}{\lambda_+^2-\lambda_-^2}\left[\frac{1}{2}(\lambda_+^2+\lambda_-^2)\times\ln\frac{\lambda_+}{\lambda_-}-\lambda_+\lambda_-\operatorname{arctg}\frac{\lambda_+^2-\lambda_-^2}{2\lambda_+\lambda_-}\right]\right\}, \quad (4.11)$$

where $\Phi_0=\pi\hbar c/|e|$ is the magnetic-flux quantum.

Our solution (4.6) of Eq. (4.4) corresponds to the condition $\alpha^2=1$. Let us compare this solution with the known solution of Ref. 4, the form of which, for our case of two different superconductors, is in accordance with (4.3)

$$\phi(\xi)=\pi+2\operatorname{arctg}\xi, \quad (4.12)$$

where $\xi=z(\lambda_+^2+\lambda_-^2)\lambda_0^{-3}$. Expression (4.12) is the solution of Eq. (4.6) for $\alpha^2=0$. A 2π kink described by (4.12) corresponds to a magnetic field

$$H_y(x,z)=\frac{\hbar c}{|e|(\lambda_+^2+\lambda_-^2)}\left\{-\frac{1}{2}\ln\left[\frac{1}{4}\left(\left(|x\mp d|+\frac{\lambda_0^3}{\lambda_+^2+\lambda_-^2}\right)^2+z^2\right)\right]-C+f(\lambda_+\lambda_-)\right\}. \quad (4.13)$$

This expression differs from (4.8) first because it corresponds to the condition $v=0$, and second by a factor 2, as is evident from a comparison of (4.6) and (4.12). Equation (4.13) makes it possible to calculate to energy of a 2π kink at rest, which reduces only to the energy W of the interacting vortical field (3.11). The absence of kinetic energy notwithstanding ($T=0$), the energy of the kink at rest is equal to

$$W=\mathscr{H}=\frac{\Phi_0^2}{8\pi^2(\lambda_+^2+\lambda_-^2)}\left\{1-C+\ln\left[(\lambda_+^2+\lambda_-^2)\times\frac{(\lambda_+\lambda_-)^{1/2}}{\lambda_0^3}\right]+\frac{1}{\lambda_+^2-\lambda_-^2}\left[\frac{1}{2}(\lambda_+^2+\lambda_-^2)\times\ln\frac{\lambda_+}{\lambda_-}-\lambda_-\lambda_+\operatorname{arctg}\frac{\lambda_+^2-\lambda_-^2}{2\lambda_+\lambda_-}\right]\right\}. \quad (4.14)$$

According to this equation, the energy of a kink at rest is exactly one-fourth the energy (4.11) of the traveling nonlocal 4π kink (4.3).

We consider now the value of the magnetic flux that carries the moving kink. According to (3.1), the magnetic flux of a kink is given by the relation

$$\Phi=\frac{1}{2\pi}\Phi_0[\varphi(z=-\infty,t)-\varphi(z=+\infty,t)].$$

Therefore both the 2π kink of the local Josephson electrodynamics (4.1) and the immobile asymptotic 2π kink (4.12) of the non-local electrodynamics correspond to vertices with one magnetic-flux quantum. On the contrary, the moving 4π kink of nonlocal Josephson electrodynamics, obtained in the present paper on the basis of the asymptotic equation (2.25), describes a vortex with a magnetic flux equal to two quanta.

Comparing solutions (4.3) and (4.12) for a traveling and immobile nonlocal kink, we verify that they cannot be obtained by using the simple transformation that makes it possible to obtain the 2π kinks of the sine-Gordon equation. The reason is that the sine-Gordon equations are not covariant with respect to a Lorentz transformation with a maximum velocity $\lambda_J\omega_J$. Incidentally, the velocity v given by Eq. (4.7) is low compared with $\lambda_J\omega_J$, since Eq. (2.25) and its solution (4.3) hold under the condition $\lambda_J^2\ll\lambda_+^2+\lambda_-^2$. The kink mass m can be determined from the equation $mv^2/2=T$ (cf. Ref. 4). Using (4.9) and (4.7), we obtain

$$m=\frac{2\pi\hbar^2C_s}{e^2}\frac{\lambda_+^2+\lambda_-^2}{\lambda_0^3}.$$

Comparing this equation with that obtained in Ref. 4 for $\lambda_+=\lambda_-=\lambda$, we see that the mass obtained by us is four times larger. The same holds also for the estimate of the viscosity coefficient $\eta_0=m/RC_s$ (Ref. 4). It must be emphasized here once more that the 2π kink in Ref. 4 is immobile and the estimates of the vortex mass and of the viscosity coefficients were made in that reference under the tacit assumption that a Lorentzian (or at least Galilean) covariance is possible, which is patently not the case for

Eq. (2.16). In contrast to the kink considered in Ref. 4, the 4π kink (4.5) is a traveling one. In our case therefore the estimate of the mass of the moving vortex is more rightful, since the corresponding nonstationary solution of Eq. (4.4), as shown in the present paper, does exist.[3)]

It should be noted in conclusion that the steps undertaken to find solutions of the integro-differential equation (2.16), and particularly its asymptotic form (2.25), show that these solution can as yet not be obtained by a regular approach, a development of which is exceedingly desirable (see Ref. 6) also, primarily, to establish the existence of the solutions.

[1)]The first results of a numerical investigation of the sine-Hilbert equation (2.25) give grounds for hoping that the 4π kink is stable (private communication from G. L. Alfimov, to whom we are most indebted).

[2)]$f(\lambda,\lambda)=\ln\lambda$ in the limit of two identical superconductors.

[3)]It cannot be considered at present that the question of how to excite a 4π kink has been answered. It would be possible to answer this question, in particular, by solving the Cauchy problem for the sine-Hilbert equation (2.25). One might think that a 4π kink could be excited in a tunnel junction of finite length by a nonstationary surface current from one of the end faces of the junction.

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