Moment description of wave turbulence

V. M. Vorob'ev

Zaporozh'e State University
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A method and a diagrammatic technique are proposed for constructing a formally exact solution in the form of an infinite series for a moment of arbitrary order. The problem of long-time evolution of the normal and anomalous correlation functions of turbulence and the problem of the interaction of a wave packet with initially Gaussian turbulence are considered as examples illustrating the advantages of the method.

1. INTRODUCTION

Several approaches now exist for describing turbulence in nonlinear media.[1] One of the most promising approaches is considered to be Wyld's diagrammatic technique.[2,3] This technique yields for the normal correlation and Green's functions a system of Dyson–Wyld equations which in turn yield, under various simplifying assumptions, such as weak nonlinearity, a system of kinetic equations. However, both systems are nonlinear, and this makes it very difficult to study them.

A different and, in my opinion, simpler method for describing turbulence is presented below. The gist of the method is as follows: For any medium with power-law nonlinearity the dynamics of the moments of the amplitudes of the characteristic modes satisfies an infinite linear system of Karman–Howarth equations[3] whose right-hand sides are triangular matrices. This makes it possible to integrate this system and to obtain, for a moment of arbitrary order, a solution in the form of a series consisting of Green's functions and either the initial amplitudes of the moments of order higher than a given order (in the problem of the evolution of a random initial distribution) or powers of the correlation function of an external Gaussian force (in the problem of stationary turbulence). A diagrammatic technique that makes it easier to obtain an approximate solution is also proposed.

2. MOMENT DESCRIPTION OF TURBULENCE

We shall study the application of this method for describing wave turbulence in a system with a nondecaying spectrum. The dynamical equations for the normal variables of the system $a_{\mathbf{k}}$, $a^*_{-\mathbf{k}}$ are (see Ref. 4; the dot indicates differentiation with respect to time)

$$\dot{a}_{\mathbf{k}}=-i\omega_{\mathbf{k}}a_{\mathbf{k}}-i\int_{-}^{+}\cdots\int_{\infty}^{\infty}d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3$$
$$\times \mathbf{W}(\mathbf{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3)a_{\boldsymbol{\kappa}_1}a_{\boldsymbol{\kappa}_2}a^*_{\boldsymbol{\kappa}_3}\delta(\mathbf{k}-\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3), \quad (1)$$

where $\omega_{\mathbf{k}}$ is the frequency of a mode with wave vector $\mathbf{k}$. The coefficient $W(\mathbf{k},\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3)$, which here is the kernel of an integral equation, describes four-wave interactions satisfying the "resonance conditions"

$$\mathbf{k}+\boldsymbol{\kappa}_3=\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2, \quad \omega_{\mathbf{k}}+\omega_{\boldsymbol{\kappa}_3}=\omega_{\boldsymbol{\kappa}_1}+\omega_{\boldsymbol{\kappa}_2}. \quad (2)$$

We assume next that $a_{\mathbf{k}}$ is a random variable. Averaging over an ensemble of its realizations we obtain for the moments

$$b^{(1)}_{\mathbf{k}_1}=\langle a_{\mathbf{k}_1}\rangle,\ldots,\quad b^{(2n-1)}_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1}}=\langle a_{\mathbf{k}_1}\ldots a_{\mathbf{k}_n}a^*_{\mathbf{k}_{n+1}}\ldots a^*_{\mathbf{k}_{2n-1}}\rangle,$$
$$b^{(2)}_{\mathbf{k}_1,\mathbf{k}_2}=\langle a_{\mathbf{k}_1}a_{\mathbf{k}_2}\rangle,\ldots,\quad b^{(2n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}$$
$$=\langle a_{\mathbf{k}_1}\ldots a_{\mathbf{k}_{n+1}}a^*_{\mathbf{k}_{n+2}}\ldots a^*_{\mathbf{k}_{2n}}\rangle,$$
$$\tilde{b}^{(2)}_{\mathbf{k}_1,\mathbf{k}_2}=\langle a_{\mathbf{k}_1}a^*_{\mathbf{k}_2}\rangle,\ldots,\quad \tilde{b}^{(2n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}=\langle a_{\mathbf{k}_1}\ldots a_{\mathbf{k}_n}a^*_{\mathbf{k}_{n+1}}\ldots a^*_{\mathbf{k}_{2n}}\rangle$$

the following system of equations:[3,5]

$$\dot{b}^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}=-i\sum_{m=1}^{n}\Delta^{(m)}_n\omega_{\mathbf{k}_m}b^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}$$
$$-i\int_{-\infty}^{+\infty}\cdots\int d\boldsymbol{\kappa}_1\ldots d\boldsymbol{\kappa}_{n+2}W^{(n)}_{(\mathbf{k}_1,\ldots,\mathbf{k}_n;\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2})}$$
$$\times b^{(n+2)}_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2}}, \quad n=1,2,\ldots, \quad (3)$$

where

$$\Delta^{(m)}_n=\begin{cases}1, & m\leqslant n\\ -1, & m>n\end{cases}, \quad \Delta^{(m)}_{2n}=\begin{cases}1, & m\leqslant n+1\\ -1, & m>n+1\end{cases},$$

$$\tilde{\Delta}^{(m)}_{2n}=\begin{cases}1, & m\leqslant n\\ -1, & m>n\end{cases},$$

$$W^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2})$$
$$=\sum_{m=1}^{n}\Delta^{(m)}_n W(\mathbf{k}_m,\boldsymbol{\kappa}_{m+1},\boldsymbol{\kappa}_{s_n},\boldsymbol{\kappa}_{\tilde{s}_n})$$
$$\times\delta(\mathbf{k}_m-\boldsymbol{\kappa}_{m+1}-\boldsymbol{\kappa}_{s_n}+\boldsymbol{\kappa}_{\tilde{s}_n})\prod_{\substack{l=1\\ l\neq m}}^{n}\delta(\mathbf{k}_l-\boldsymbol{\kappa}_{l+1}),$$

$$s_n=\begin{cases}1, & \Delta^{(m)}_n=1\\ n+2, & \Delta^{(m)}_n=-1\end{cases}, \quad \tilde{s}_n=\begin{cases}n+2, & \Delta^{(m)}_n=1\\ 1, & \Delta^{(m)}_n=-1\end{cases}.$$

The system of equations for the even moments $\tilde{b}^{(2n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}$ is similar to the system (3) with the substitution $\Delta^{(m)}_{2n}\to\tilde{\Delta}^{(m)}_{2n}$.

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Laplace-transforming in time

$$B^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;p)=\int_0^\infty dt e^{-pt} b^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(t), \quad n=1,2,\ldots, \tag{4}$$

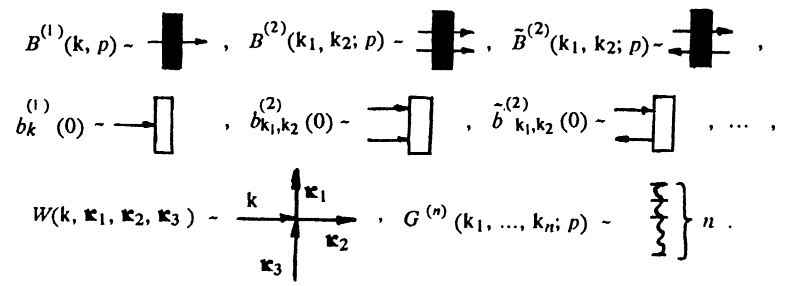
yields from the system (3) a system of algebraic equations, in which the Laplace transforms of moments of order n and $n+2$ are simply related. This fact makes it possible to write down the formally exact solution (in the form of an infinite series) of this system:

$$B^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;p)=G^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;p)\Big\{b^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(0)+\sum_{m=1}^{\infty}(-i)^m\prod_{l=1}^{m}\int_{-\infty}^{+\infty}\ldots\int d\boldsymbol{\kappa}_1^{(2l)}\ldots d\boldsymbol{\kappa}_{n+2m}^{(2l)}G^{(n+2l)}$$
$$\times(\boldsymbol{\kappa}_1^{(2l)},\ldots,\boldsymbol{\kappa}_{n+2l}^{(2l)};p)\,W^{(n+2l-2)}(\boldsymbol{\kappa}_1^{(2l-2)},\ldots,\boldsymbol{\kappa}_{n+2l-2}^{(2l-2)};\boldsymbol{\kappa}_1^{(2l)},\ldots,\boldsymbol{\kappa}_{n+2l}^{(2l)})\,b^{(n+2m)}_{\boldsymbol{\kappa}_1^{(2m)},\ldots,\boldsymbol{\kappa}_{n+2m}^{(2m)}}(0)\Big\}, \tag{5}$$

where $G^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;p)=[p+i\Sigma_{m=1}^n\Delta_n^{(m)}\omega_{\mathbf{k}_m}]^{-1}$ is the Green's function for the nth moment in $(p,\mathbf{k})$-space. A similar expression (with the substitution $\Delta_{2n}^{(m)}\to\widetilde{\Delta}_{2n}^{(m)}$) is obtained for the Laplace transforms of the even moments $\widetilde{B}^{(2n)}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n};p)$.

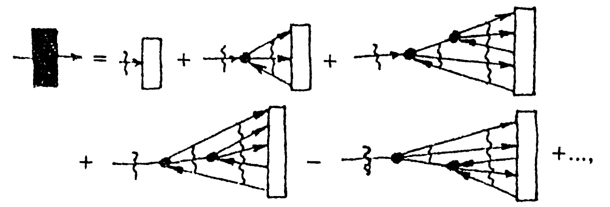
Note that the solution (5) is formally exact and contains the Green's functions and initial values of all moments starting with the $(n+1)$st moment.

We give below a graphical representation of the terms appearing in the solution (5), i.e., we construct a diagrammatic technique for it. For this, we associate with each term of the series (5) the diagram



(6)

The Laplace transform of the first- and second-order moments (of the normal $\widetilde{B}^{(2)}(\mathbf{k}_1,\mathbf{k}_2;p)$ and anomalous $B^{(2)}(\mathbf{k}_1,\mathbf{k}_2;p)$ correlation functions) can then be represented by the following series of diagrams:



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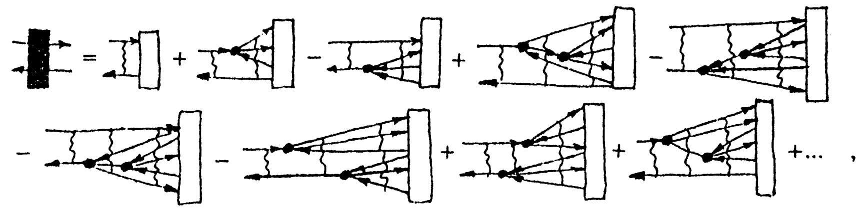

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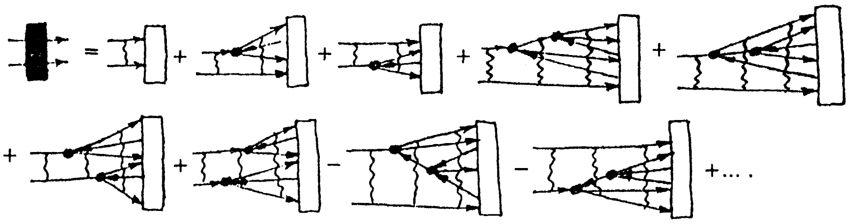


(9)

We now describe the basic topological properties of the diagrams in Eqs. (7)–(9).

1. The tree branches (matrix elements) are "virtual" variables $\boldsymbol{\kappa}_m$, where $m=1,2,\ldots$, i.e., the variables over which integration is performed.

2. Every tree terminates at rectangles representing the initial values of the moments. The number of branches is equal to the order of the moment.

3. A tree trunk corresponds to the actual variables $\mathbf{k}_n$, where $n=1,2,\ldots$, the number of variables being equal to the number of the moment for which the solution is being constructed.

4. The Green's functions $G^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;p)$ of order n connect n branches, including the trunk, and the wavy lines must be drawn vertically through all possible interactions in each term.

5. Each diagram contains a factor $(-i)^{m_1}$, where m_1 is the number of matrix elements in the diagram.

The evolution of the moments $b^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(t)$ is determined by the poles of the Green's functions in the expression (5). For conservative media ($\operatorname{Im}\omega_{\mathbf{k}}=0$) the poles lie on the imaginary axis of the complex p-plane. If the poles are simple, the dynamics of the moments are oscillatory. If the poles are degenerate, which happens under the decomposition conditions

$$\sum_{m=1}^{n_1}\Delta_{n_1}^{(m)}\omega_{\mathbf{k}_m}=\sum_{m=n_1+1}^{n}\Delta_{n}^{(m)}\omega_{\mathbf{k}_m}\quad\text{or}$$

$$\sum_{m=1}^{n_2}\tilde{\Delta}_{n_2}^{(m)}\omega_{\mathbf{k}_m}=\sum_{m=n_2+1}^{2n}\tilde{\Delta}_{2n}^{(m)}\omega_{\mathbf{k}_m}\tag{10}$$

(here $n_{1,2}$ are arbitrary integers), the time dependence of the moments takes the form $t^{\alpha}\exp(-i\tilde{\omega}_{\mathbf{k}}t)$, where α is the multiplicity of the poles. It is easy to see that as the parameter m on the right-hand side of Eq. (5) increases, the multiplicity also increases. The two cases can be combined if the solution is written as follows:

$$b^{(n)}_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n}(t)=b^{(n)}_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n}(0)\exp\left\{-i\sum_{m=1}^{n}\Delta_n^{(m)}\omega_{\mathbf{k}_m}t+\Gamma_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n}(t)t\right\},\tag{11}$$

where $\Gamma_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n}(t)$ is interpreted as the effective (nonlinear) turbulent growth (decay) rate and is represented by a Taylor series in the time:

$$\Gamma_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n}(t)=\sum_{m=0}^{\infty}(-i)^m\Gamma^{(m)}(\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n)\frac{t^m}{m!},\tag{12}$$

where the first terms have the form

$$\Gamma^{(0)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)=\int_{-\infty}^{+\infty}\!\!\cdots\!\int d\boldsymbol{\kappa}_1\ldots d\boldsymbol{\kappa}_{n+2}W^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2})\frac{b^{(n+2)}_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2}}(0)}{b^{n}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(0)},\quad \Gamma^{(1)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$$

$$=-\frac{1}{2}[\Gamma^{(0)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)]^2+\frac{1}{2}\int_{-\infty}^{+\infty}\!\!\cdots\!\int d\boldsymbol{\kappa}_1\ldots d\boldsymbol{\kappa}_{n+2}\frac{W^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2})}{b^{(n)}_{\mathbf{k}_1,\ldots,\mathbf{k}_n}(0)}\Big[\Lambda^{(n)}$$

$$\times(\mathbf{k}_1,\ldots,\mathbf{k}_n;\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2})b^{(n+2)}_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2}}(0)+\int_{-\infty}^{+\infty}\!\!\cdots\!\int d\boldsymbol{\kappa}'_1\ldots d\boldsymbol{\kappa}'_{n+4}W^{(n+2)}$$

$$\times(\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_{n+2};\boldsymbol{\kappa}'_1,\ldots,\boldsymbol{\kappa}'_{n+4})b^{(n+4)}_{\boldsymbol{\kappa}'_1,\ldots,\boldsymbol{\kappa}'_{n+4}}(0)\Big],$$

$$
\begin{aligned}
\Gamma^{(2)}(\mathbf{k}_1,...,\mathbf{k}_n) = & -\frac{1}{3}\,[\Gamma^{(0)}(\mathbf{k}_1,...,\mathbf{k}_n)]^3 - 2\Gamma^{(0)}(\mathbf{k}_1,...,\mathbf{k}_n)\Gamma^{(1)}(\mathbf{k}_1,...,\mathbf{k}_n) \\
& + \int_{-\infty}^{+\infty}\!\!\cdots\int d\boldsymbol{\kappa}_1 ... d\boldsymbol{\kappa}_{n+2}\, \frac{W^{(n)}(\mathbf{k}_1,...,\mathbf{k}_n;\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2})}{b^{(n)}_{\mathbf{k}_1,...\mathbf{k}_n}(0)} \Big\{ [\Lambda^{(n)}(\mathbf{k}_1,...,\mathbf{k}_n;\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2})]^2 b^{(n+2)}_{\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2}}(0) \\
& - \int_{-\infty}^{+\infty}\!\!\cdots\int d\boldsymbol{\kappa}'_1 ... d\boldsymbol{\kappa}'_{n+4} W^{(n+2)}(\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2};\boldsymbol{\kappa}'_1,...,\boldsymbol{\kappa}'_{n+4}) \Big[[\Lambda^{(n)}(\mathbf{k}_1,...,\mathbf{k}_n;\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2}) + \Lambda^{(n+2)} \\
& \times (\mathbf{k}_1,...,\mathbf{k}_n;\boldsymbol{\kappa}'_1,...,\boldsymbol{\kappa}'_{n+4})]\, b^{(n+4)}_{\boldsymbol{\kappa}'_1,...,\boldsymbol{\kappa}'_{n+4}}(0) + \int_{-\infty}^{+\infty}\!\!\cdots\int d\boldsymbol{\kappa}''_1 ... d\boldsymbol{\kappa}''_{n+6} W^{(n+4)} \\
& \times (\boldsymbol{\kappa}'_1,...,\boldsymbol{\kappa}'_{n+4};\boldsymbol{\kappa}''_1,...,\boldsymbol{\kappa}''_{n+6})\, b^{(n+6)}_{\boldsymbol{\kappa}''_1,...,\boldsymbol{\kappa}''_{n+6}}(0) \Big]\Big\}. \qquad (13)
\end{aligned}
$$

Here

$$\Lambda^{(n)}(\mathbf{k}_1,...,\mathbf{k}_n;\boldsymbol{\kappa}_1,...,\boldsymbol{\kappa}_{n+2}) = \sum_{m=1}^{n} \Delta_n^{(m)}\omega_{\mathbf{k}_m} - \sum_{m=1}^{n+2} \Delta_n^{(m)}\omega_{\boldsymbol{\kappa}_m}$$

is the frequency detuning in the decay process

$$\sum_{m=1}^{n} \mathbf{k}_m = \sum_{m=1}^{n+2} \boldsymbol{\kappa}_m .$$

3. DYNAMICS OF INITIALLY GAUSSIAN TURBULENCE

As an illustration of the proposed method we consider the problem of the evolution of homogeneous Gaussian turbulence created in a nonlinear medium at $t=0$. These initial characteristics of the turbulence mean that in the series (8) and (9) terms other than those containing the initial values of the even moments can be dropped. In the retained terms the branches must be "glued together" in pairs in all possible ways. Next, we introduce a graphical representation of the initial values of the normal n_k and anomalous σ_k correlation functions:

$n_{\mathbf{k}}$ ~  , $\sigma_{\mathbf{k}}$ ~  .

Then the Laplace transforms of the normal and anomalous correlation functions are described by the following series:

$$
\begin{aligned}
\widetilde{B}^{(2)}(\mathbf{k}_1,\mathbf{k}_2;p) = & \,\delta(\mathbf{k}_1-\mathbf{k}_2)p^{-1}\Big\{ n_{\mathbf{k}_1} \\
& + 2\,\mathrm{Im}\Big[\sigma^*_{\mathbf{k}_1} \int_{-\infty}^{+\infty} d\boldsymbol{\kappa}\sigma_{\boldsymbol{\kappa}} W(\mathbf{k}_1,\boldsymbol{\kappa},-\boldsymbol{\kappa}, \\
& -\mathbf{k}_1)[(p+i\Delta(\mathbf{k}_1,\boldsymbol{\kappa}))^{-1} + i(\Omega_{\boldsymbol{\kappa}} \\
& -\Omega_{\mathbf{k}_1})(p+i\Delta(\mathbf{k},\boldsymbol{\kappa}))^{-2} + ...]\Big] + ...\Big\}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
B^{(2)}(\mathbf{k}_1,\mathbf{k}_2;p) = & \,\delta(\mathbf{k}_1+\mathbf{k}_2)[p+i(\omega_{\mathbf{k}_1}+\omega_{-\mathbf{k}_1})]^{-1}\Big\{ \sigma_{\mathbf{k}_1} \\
& - i n_{\mathbf{k}_1} \int_{-\infty}^{+\infty} d\boldsymbol{\kappa}\sigma_{\boldsymbol{\kappa}} W(\mathbf{k}_2,\boldsymbol{\kappa},-\boldsymbol{\kappa}, \\
& -\mathbf{k}_2)[p+i(\omega_{\mathbf{k}_1}+\omega-\mathbf{k}_1)]^{-1} \\
& + (\mathbf{k}_1 \rightleftarrows \mathbf{k}_2) - 2i[\sigma_{\mathbf{k}_1}\Omega_{\mathbf{k}_2} + \sigma_{\mathbf{k}_2}\Omega_{\mathbf{k}_1}] \\
& \times [p+i(\omega_{\mathbf{k}_1}+\omega_{-\mathbf{k}_1})]^{-1} \\
& + (\mathbf{k}_1 \rightleftarrows \mathbf{k}_2) + ...\Big\}, \qquad (15)
\end{aligned}
$$

where $\Omega_{\mathbf{k}} = \int_{-\infty}^{+\infty} d\boldsymbol{\kappa} n_{\boldsymbol{\kappa}} W(\mathbf{k},\mathbf{k},\boldsymbol{\kappa},\boldsymbol{\kappa})$ is the nonlinear frequency shift.

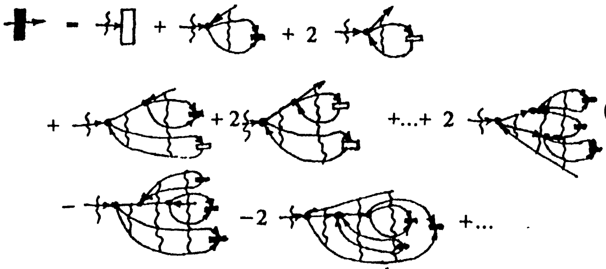
We note that in the single-point approximation ($\mathbf{k}_1=\mathbf{k}_2=\mathbf{k}$) the normal correlation function $\widetilde{b}^{(2)}_{\mathbf{k}_1,\mathbf{k}_n}(t)$ is the spectral density of the turbulence energy. For this reason, we now consider in greater detail the expression (14). Taking the inverse Laplace transform, we find that this quantity contains oscillations with the combination frequencies $\omega_{\boldsymbol{\kappa}}-\omega_{\mathbf{k}}$ and $\Omega_{\boldsymbol{\kappa}}-\Omega_{\mathbf{k}}$.

Thus the evolution of initially homogeneous Gaussian turbulence can be interpreted in terms of four-wave interactions of the characteristic modes of a system whose frequencies satisfy the conditions

$$\omega_{\mathbf{k}}+\omega_{-\mathbf{k}} = \omega_{\boldsymbol{\kappa}}+\omega_{-\boldsymbol{\kappa}}, \quad \mathbf{k}-\mathbf{k} = \boldsymbol{\kappa}-\boldsymbol{\kappa}. \qquad (16)$$

As the next example we consider the interaction of homogeneous Gaussian turbulence and a wave packet whose amplitude at $t=0$ is described by the function $w_{\mathbf{k}}$.[1)] We neglect below for simplicity the intrinsic nonlinearity of the wave. This means that in the series (7) that determines the dynamics of a wave packet all terms other than those containing the initial values of the odd moments can be dropped, and in the retained terms the branches must be "glued together" in pairs in all possible ways. The remaining free branch will correspond to the initial amplitude of the packet.

The expression for the Laplace transform of the packet amplitude has then the form



(17)

In order to obtain the growth (decay) rate of the instability of the wave packet, which determines the dynamics of the wave packet in a turbulent medium, the diagrams containing the Green's functions with frequencies satisfying the condition (16) must be summed in Eq. (17).[2)] Then we obtain from Eq. (17)

$$\begin{aligned} w_{\mathbf{k}}(p) &\equiv B^{(1)}(\mathbf{k},p) \\ &= w^*_{-\mathbf{k}} \int_{-\infty}^{+\infty} d\boldsymbol{\kappa} P(\mathbf{k},\boldsymbol{\kappa}) \{(-i)(p+i\omega_{\mathbf{k}})^{-2} \\ &\quad + (-i)^2 (p+i\omega_{\mathbf{k}})^{-3} [\Delta(\mathbf{k},\boldsymbol{\kappa}) + 4\Omega_{\boldsymbol{\kappa}}] \\ &\quad - (-i)^3 \Lambda^2(\mathbf{k},\boldsymbol{\kappa}) (p+i\omega_{\mathbf{k}})^{-4} + \ldots\}, \end{aligned} \tag{18}$$

where $\Delta(\mathbf{k},\boldsymbol{\kappa}) = \omega_{\boldsymbol{\kappa}} + \omega_{-\boldsymbol{\kappa}} - \omega_{\mathbf{k}} - \omega_{-\mathbf{k}}$ is the detuning and $\Gamma_{\mathbf{k}} = \int_{-\infty}^{+\infty} d\boldsymbol{\kappa} \sigma_{\boldsymbol{\kappa}} W(\mathbf{k},\boldsymbol{\kappa},-\boldsymbol{\kappa},-\mathbf{k})$, $\Lambda^2(\mathbf{k},\boldsymbol{\kappa}) = |\Gamma_{\mathbf{k}}|^2 + \Omega_{\mathbf{k}}^2 - [\Delta(\boldsymbol{\kappa},\mathbf{k}) + \Omega_{\mathbf{k}}]^2$, and $P(\mathbf{k},\boldsymbol{\kappa}) = \sigma_{\boldsymbol{\kappa}} W(\mathbf{k},\boldsymbol{\kappa},-\boldsymbol{\kappa},-\mathbf{k}) / \Lambda(\mathbf{k},\boldsymbol{\kappa})$.

Performing the inverse Laplace transform and summing the Taylor time series obtained from Eq. (18), we represent the packet amplitude as follows:

$$w_{\mathbf{k}}(t) = -i\omega^*_{-\mathbf{k}} \int_{-\infty}^{+\infty} d\boldsymbol{\kappa} P(\mathbf{k},\boldsymbol{\kappa}) \operatorname{sh}\{\Lambda(\mathbf{k},\boldsymbol{\kappa})t\}, \tag{19}$$

whence it follows that turbulence can intensify the wave packet in a wave vector region determined by the inequality

$$\Lambda(\mathbf{k},\boldsymbol{\kappa}) \geqslant 0, \quad \text{i.e.,} \quad |\Delta(\mathbf{k},\boldsymbol{\kappa}) + \Omega_{\mathbf{k}}| \leqslant \sqrt{|\Gamma_{\mathbf{k}}|^2 + \Omega_{\mathbf{k}}^2}. \tag{20}$$

According to Eq. (20), the maximum turbulence-induced growth rate of the packet, equal to

$$\Gamma_{\max}(\mathbf{k}) = |\Gamma_{\mathbf{k}}|, \tag{21}$$

is reached for

$$\Delta(\mathbf{k},\boldsymbol{\kappa}) = -2\Omega_{\mathbf{k}}. \tag{22}$$

Making the formal substitution $\sigma_{\mathbf{k}} \to A_0 \delta(\mathbf{k})$ it is easy to see that the expressions (20)–(22) are analogous to the corresponding expressions describing the modulation instability of a regular monochromatic pump wave with amplitude A_0 and wave vector $\mathbf{k}_0$ (see, for example, Ref. 7).

It should be noted that when the intrinsic nonlinearity of the wave packet is taken into account, additional terms describing interaction of the modes of the packet with one another appear in Eq. (20). Owing to breakdown of the matching conditions (16), this in turn limits the growth of instability.

4. CONCLUSIONS

We now discuss the region of applicability of our results. The proposed method, like the earlier theories, makes it possible to solve the problem in the form of an infinite diagrammatic series. However, in problems dealing with evolution of a random initial state this series can be put into the form of a Taylor series whose coefficients contain a finite number of diagrams that are of lowest order in the interaction. This suggests that the solution will converge at least in the initial stage of the process, especially since for sufficiently simple initial conditions (Gaussian and homogeneous turbulence) this series can be summed and an analytic expression can be obtained for the solution (19).

In the general case, in the regime of developed turbulence, the series obtained above are actually divergent and the question of whether or not the solution remains valid when a series is replaced by its finite sum remains open and is an interesting question for future investigations.

[1)] A similar problem was previously studied in Ref. 6 for a medium with decay.
[2)] These will be the so-called weakly coupled diagrams.

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