A new class of solutions of the magnetostatic Einstein–Maxwell equations

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A new class of asymptotically flat axisymmetric solutions of the magnetostatic Einstein–Maxwell equations is presented. A metric, which reduces to the Schwarzschild one in the absence of the magnetic field, is obtained as a particular case. This solution can be used to describe the external gravitational field of a massive magnetic dipole.

1. INTRODUCTION

The first exact solutions of the magnetostatic Einstein–Maxwell equations were obtained in 1954 by Bonnor.[1] Soon thereafter Tauber[2] made use of the Curzon solution to construct the metric for the gravitational field due to point magnetized sources. In 1966 Bonnor[3] applied his theorem on the correspondence between the stationary vacuum solution and the static Einstein–Maxwell field to the Kerr metric and constructed a metric with two parameters—the mass parameter and the magnetic dipole moment parameter. This metric becomes flat at infinity and the magnetic field tends to the field of a magnetic dipole. However the Bonnor solution does not become the Schwarzschild solution when the magnetism parameter is set equal to zero.

In the late seventies Herlt[4,5] found a class of solutions which included the Bonnor metric. He also developed a procedure for the generalization of the Reissner–Nordström solution. Regrettably, Herlt was unable to construct a magnetostatic solution which could describe the field of a magnetic dipole. Among additional papers devoted to the search for point magnetostatic solutions we note Refs. 6–9. The asymptotically flat metrics constructed therein are interesting in that they all reduce to the Schwarzschild metric in the absence of magnetism.

In the present paper we obtain a new class of solutions of the static Einstein–Maxwell equations and consider the problem of the gravitational field of a magnetic dipole.

2. THE BASIC EQUATION

The Einstein–Maxwell equation has the form

$$R_{ik}=8\pi T_{ik},\quad \frac{\partial}{\partial x^k}\left(\sqrt{-g}F_{lm}g^{il}g^{km}\right)=0,$$
$$F_{ik;l}+F_{kl;i}+F_{li;k}=0, \tag{1}$$

where R_{ik} is the Einstein–Ricci tensor and T_{ik} is the energy-momentum tensor of the electromagnetic field, connected with the electromagnetic field intensity tensor F_{ik} by the relation

$$T_{ik}=\frac{1}{4\pi}\left(F_{il}F^l_{\ k}-\frac{1}{4}F_{lm}F^{lm}g_{ik}\right).$$

It is also customary to introduce the 4-potential of the electromagnetic field

$$F_{ik}=A_{k;i}-A_{i;k}=\frac{\partial A_k}{\partial x^i}-\frac{\partial A_i}{\partial x^k}.$$

The metric of the static axisymmetric gravitational field can be written in the canonical Weyl coordinates in the form

$$dS^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\varphi^2]-fdt^2, \tag{2}$$

where the functions f and γ depend only on ρ and z.

We set $A_i=[0,0,A_3(\rho,z),0]$ and rewrite Eq. (1) explicitly as

$$f\Delta f=(\nabla f)^2+\frac{2f^3}{\rho^2}(\nabla A_3)^2,\quad \nabla\left(\frac{f}{\rho^2}\nabla A_3\right)=0, \tag{3}$$

$$4\frac{\partial\gamma}{\partial\rho}=\frac{\rho}{f^2}\left[\left(\frac{\partial f}{\partial\rho}\right)^2-\left(\frac{\partial f}{\partial z}\right)^2\right]+\frac{4f}{\rho}\left[\left(\frac{\partial A_3}{\partial\rho}\right)^2-\left(\frac{\partial A_3}{\partial z}\right)^2\right],$$
$$2\frac{\partial\gamma}{\partial z}=\frac{\rho}{f^2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z}+4\frac{f}{\rho}\frac{\partial A_3}{\partial\rho}\frac{\partial A_3}{\partial z}. \tag{4}$$

Here

$$\Delta\equiv\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2},\quad \nabla\equiv\boldsymbol{\rho}_0\frac{\partial}{\partial\rho}+\mathbf{z}_0\frac{\partial}{\partial z}$$

($\boldsymbol{\rho}_0$ and $\mathbf{z}_0$ are unit vectors) and $A_3(\rho,z)$ is the magnetic component of the electromagnetic-field 4-potential.

The second equation in (3) can be viewed as the condition for the existence of a new potential A_3', connected to A_3 by the relations

$$\frac{\partial A_3'}{\partial\rho}=-\frac{f}{\rho}\frac{\partial A_3}{\partial z},\quad \frac{\partial A_3'}{\partial z}=\frac{f}{\rho}\frac{\partial A_3}{\partial\rho}. \tag{5}$$

In that case Eqs. (3) and (4) can be rewritten as

$$f\Delta f=(\nabla f)^2+2f(\nabla A_3')^2,\quad f\Delta A_3'=\nabla A_3'\cdot\nabla f, \tag{6}$$

$$\frac{4}{\rho}\frac{\partial\gamma}{\partial\rho}=\frac{1}{f^2}\left[\left(\frac{\partial f}{\partial\rho}\right)^2-\left(\frac{\partial f}{\partial z}\right)^2\right]+\frac{4}{f}\left[\left(\frac{\partial A_3'}{\partial z}\right)^2-\left(\frac{\partial A_3'}{\partial\rho}\right)^2\right],$$
$$\frac{2}{\rho}\frac{\partial\gamma}{\partial z}=\frac{1}{f^2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z}-\frac{4}{f}\frac{\partial A_3'}{\partial\rho}\frac{\partial A_3'}{\partial z}. \tag{7}$$

If we introduce further the functions

$$\varepsilon_1=u+A_3',\quad \varepsilon_2=u-A_3',\quad u=\sqrt{f}, \tag{8}$$

we obtain the symmetric magnetostatic equations

$$(\varepsilon_1+\varepsilon_2)\Delta\varepsilon_1=2(\nabla\varepsilon_1)^2,\quad (\varepsilon_1+\varepsilon_2)\Delta\varepsilon_2=2(\nabla\varepsilon_2)^2, \tag{9}$$

$$\frac{(\varepsilon_1+\varepsilon_2)^2}{4\rho}\frac{\partial\gamma}{\partial\rho}=\frac{\partial\varepsilon_1}{\partial\rho}\frac{\partial\varepsilon_2}{\partial\rho}-\frac{\partial\varepsilon_1}{\partial z}\frac{\partial\varepsilon_2}{\partial z},$$

$$\frac{(\varepsilon_1+\varepsilon_2)^2}{4\rho}\frac{\partial\gamma}{\partial z}=\frac{\partial\varepsilon_1}{\partial\rho}\frac{\partial\varepsilon_2}{\partial z}+\frac{\partial\varepsilon_1}{\partial z}\frac{\partial\varepsilon_2}{\partial\rho}. \tag{10}$$

We note certain general properties of Eqs. (9) and (10). Let $(\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)$ denote some solution of Eqs. (9). Then the functions

$$\varepsilon_1=\frac{-a+b\tilde{\varepsilon}_1}{c-d\tilde{\varepsilon}_1},\quad \varepsilon_2=\frac{a+b\tilde{\varepsilon}_2}{c+d\tilde{\varepsilon}_2}, \tag{11}$$

where a, b, c, and d are arbitrary real constants, will also satisfy Eqs. (9).

In the particular case when we set in (11) $c=0$, $a=-2dc_0$, and $b=-d$ we obtain

$$\varepsilon_1=-\frac{2c_0}{\tilde{\varepsilon}_1}+1,\quad \varepsilon_2=-\frac{2c_0}{\tilde{\varepsilon}_2}-1. \tag{12}$$

The analog of transformations (12) for the vacuum stationary gravitational field is given by the inversion transformations [see, e.g., Ref. 10, Eq. 30.27 for $\Phi=0$]. The resultant abbreviated transformation is the Ehlers transformation.

In this way, given a certain solution $(\tilde{f},\tilde{A}_3')$ of Eq. (6), a new solution can be constructed

$$f=4c_0^2\tilde{f}(\tilde{A}_3'^2-\tilde{f}\,)^{-2},\quad A_3'=1-2c_0\tilde{A}_3'(\tilde{A}_3'^2-\tilde{f}\,)^{-1}. \tag{13}$$

Further, it follows from (10) and (12) that $\gamma=\tilde{\gamma}$.

3. A NEW CLASS OF SOLUTIONS

We rewrite Eqs. 6 in the form

$$f\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{\partial^2 f}{\partial z^2}\right]=\left(\frac{\partial f}{\partial\rho}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2+2f\left[\left(\frac{\partial A_3'}{\partial\rho}\right)^2+\left(\frac{\partial A_3'}{\partial z}\right)^2\right],$$

$$f\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_3'}{\partial\rho}\right)+\frac{\partial^2 A_3'}{\partial z^2}\right]=\frac{\partial f}{\partial\rho}\frac{\partial A_3'}{\partial\rho}+\frac{\partial f}{\partial z}\frac{\partial A_3'}{\partial z}, \tag{14}$$

and introduce into consideration the functions ψ and χ satisfying the equations

$$\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{\partial^2\psi}{\partial z^2}=0,\quad \frac{\partial^2\chi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}+\frac{\partial^2\chi}{\partial z^2}=0, \tag{15}$$

and also connected to each other by the relations

$$\frac{\partial\psi}{\partial\rho}=\frac{1}{\rho}\frac{\partial\chi}{\partial z},\quad \frac{\partial\psi}{\partial z}=-\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}. \tag{16}$$

It is not hard to show that we can take as the solutions $(\tilde{A}_3',\tilde{f}\,)$ of Eqs. (14) the following functions

$$\tilde{A}_3'=\chi,\quad \tilde{f}=\rho^2F(\psi), \tag{17}$$

where $F(\psi)$ satisfies the ordinary differential equation

$$F\frac{d^2F}{d\psi^2}=\left(\frac{dF}{d\psi}\right)^2+2F. \tag{18}$$

The solution of this equation has the form

$$F(\psi)=\frac{1}{b_0}\left(\frac{e^{\psi}+b_0e^{-\psi}}{2}\right)^2,\quad b_0=\text{const}. \tag{19}$$

For $b_0=1$, i.e., $F=\cosh^2\psi$, the solution of Eqs. (14) in accordance with (13) can be written in the form

$$f=\frac{4c_0^2\rho^2\cosh^2\psi}{(\chi^2-\rho^2\cosh^2\psi)^2},\quad A_3'=1-\frac{2c_0\chi}{\chi^2-\rho^2\cosh^2\psi}. \tag{20}$$

On the other hand Eqs. (7) reduce in that case to the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2\gamma-2\ln\cosh^2\psi-\ln\frac{\rho^2}{k_0^2}\right)=-2\left[\left(\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}\right)^2-\left(\frac{1}{\rho}\frac{\partial\chi}{\partial z}\right)^2\right],$$

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(2\gamma-2\ln\cosh^2\psi-\ln\frac{\rho^2}{k_0^2}\right)=-\frac{4}{\rho^2}\frac{\partial\chi}{\partial\rho}\frac{\partial\chi}{\partial z}. \tag{21}$$

With the help of (20) we can also rewrite Eqs. (5)

$$\frac{\partial}{\partial\rho}(2c_0A_3+\chi^2\tanh\psi)=\rho\frac{\partial\chi}{\partial z},$$

$$\frac{\partial}{\partial z}(2c_0A_3+\chi^2\tanh\psi)=-\rho\frac{\partial\chi}{\partial\rho}+2\chi. \tag{22}$$

It is readily seen that the integrability condition for both (21) and (22) is ensured by the second of Eqs. (15). Formulas (20), (21), and (22) define a new class of solutions of the magnetostatic Einstein–Maxwell equations.

We show now how the Schwarzschild solution can be obtained from (20). To this end it is convenient to go over to prolate ellipsoidal coordinates (x,y), which are connected to the Weyl coordinates by the relations

$$\rho=k_0\sqrt{(x^2-1)(1-y^2)},\quad z=k_0xy \tag{23}$$

(k_0—a real constant).

The operators $\partial/\partial\rho$ and $\partial/\partial z$ are then given by the expressions

$$k_0\frac{\partial}{\partial\rho}\equiv\sqrt{(x^2-1)(1-y^2)}(x^2-y^2)^{-1}\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right),$$

$$k_0\frac{\partial}{\partial z}\equiv(x^2-y^2)^{-1}\left[(x^2-1)y\frac{\partial}{\partial x}+(1-y^2)x\frac{\partial}{\partial y}\right]. \tag{24}$$

In terms of the coordinates (x,y) the metric (2) is rewritten in the form

$$dS^2=k_0^2f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]-fdt^2, \tag{25}$$

and the solution (20) and the Eqs. (15) and (16) take on the form

$$f=\frac{4c_0^2k_0^2(x^2-1)(1-y^2)\cosh^2\psi}{[\chi^2-k_0^2(x^2-1)(1-y^2)\cosh^2\psi]^2},$$

$$A_3'=1-\frac{2c_0\chi}{\chi^2-k_0^2(x^2-1)(1-y^2)\cosh^2\psi}, \qquad (26)$$

$$\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial\psi}{\partial x}\right]+\frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial\psi}{\partial y}\right]=0,$$

$$(x^2-1)\frac{\partial^2\chi}{\partial x^2}+(1-y^2)\frac{\partial^2\chi}{\partial y^2}=0, \qquad (27)$$

$$k_0\frac{\partial\psi}{\partial x}=-\frac{1}{x^2-1}\frac{\partial\chi}{\partial y},\quad k_0\frac{\partial\psi}{\partial y}=\frac{1}{1-y^2}\frac{\partial\chi}{\partial x}. \qquad (28)$$

To obtain the Schwarzschild solution it is sufficient to set in (27) and (28)

$$\chi=k_0(x+1),\quad \psi=\frac{1}{2}\ln\frac{1+y}{1-y}.$$

In that case, setting $c_0=k_0$, we obtain from formula (26)

$$f=\frac{x-1}{x+1},\quad A_3'=0.$$

In terms of the curvature coordinates r and ϑ [$x=(r-m_0)/k_0$, $y=\cos\vartheta$, k_0 and m_0—constants], and further setting $k_0=m_0$, we obtain

$$f=1-\frac{2m_0}{r},\quad A_3'=0,\quad e^{2\gamma}=\frac{r^2-2m_0r}{r^2-2m_0r+m_0^2\sin^2\vartheta},$$

i.e., the Schwarzschild solution.

4. THE GRAVITATIONAL FIELD OF THE MAGNETIC DIPOLE

The class of solutions (26) obtained by us permits a generalization of the Schwarzschild solution to the case when the gravitating center has magnetic properties. Indeed, if we choose as the solution of the equation for χ in (27) the function

$$\chi=k_0(x+p_0+q_0y),\quad q_0=\text{const},\quad p_0=\text{const}, \qquad (29)$$

we obtain for ψ in (28) the expression

$$\psi=\frac{1}{2}\ln\left[\frac{1+y}{1-y}\left(\frac{x+1}{x-1}\right)^{q_0}\right]. \qquad (30)$$

In such a case

$$\cosh^2\psi=\frac{M^2}{1-y^2},$$

$$M=\frac{1+y}{2}\left(\frac{x+1}{x-1}\right)^{q_0/2}+\frac{1-y}{2}\left(\frac{x-1}{x+1}\right)^{q_0/2}, \qquad (31)$$

and the functions ε_1 and ε_2, determined by formulas (8)

$$\varepsilon_1=-\frac{2c_0}{u+\chi}+1,\quad \varepsilon_2=-\frac{2c_0}{u-\chi}-1, \qquad (32)$$

become for $c_0=m_0$

$$\varepsilon_1=-\frac{2m_0}{k_0}\frac{1}{M\sqrt{x^2-1}+(x+p_0+q_0y)}+1,$$

$$\varepsilon_2=-\frac{2m_0}{k_0}\frac{1}{M\sqrt{x^2-1}-(x+p_0+q_0y)}-1. \qquad (33)$$

Therefore we obtain for the potentials

$$f=\left(\frac{2m_0u}{\chi^2-u^2}\right)^2,\quad A_3'=1-\frac{2m_0\chi}{\chi^2-u^2}$$

the expressions

$$f=\frac{4m_0^2(x^2-1)M^2}{k_0^2[(x+p_0+q_0y)^2-(x^2-1)M^2]^2},$$

$$A_3'=1-\frac{2m_0(x+p_0+q_0y)}{k_0[(x+p_0+q_0y)^2-(x^2-1)M^2]}. \qquad (34)$$

We find the metric coefficient γ from Eqs. (21), which have in terms of the coordinates (x,y) the form

$$k_0^2(x^2-y^2)\frac{\partial\Gamma}{\partial x}=x\left(\frac{\partial\chi}{\partial x}\right)^2-\frac{x(1-y^2)}{x^2-1}\left(\frac{\partial\chi}{\partial y}\right)^2-2y\frac{\partial\chi}{\partial x}\frac{\partial\chi}{\partial y},$$

$$k_0^2(x^2-y^2)\frac{\partial\Gamma}{\partial y}=\frac{y(x^2-1)}{1-y^2}\left(\frac{\partial\chi}{\partial x}\right)^2-y\left(\frac{\partial\chi}{\partial y}\right)^2+2x\frac{\partial\chi}{\partial x}\frac{\partial\chi}{\partial y}, \qquad (35)$$

where

$$\Gamma=2\ln\cosh\psi+\tfrac{1}{2}\ln[(x^2-1)(1-y^2)]-\gamma.$$

The calculations give

$$(x+y)^{(1+q_0)^2}(x-y)^{(1-q_0)^2}e^{2\gamma}=M^4(x^2-1)^{1+q_0^2}. \qquad (36)$$

It is not hard to also rewrite in terms of the new coordinates Eqs. (22), serving to determine the magnetic potential A_3

$$2m_0\frac{\partial A_3}{\partial x}=-\frac{1}{k_0}\frac{\partial}{\partial x}(\chi^2\tanh\psi)+2y\chi+(1-y^2)\frac{\partial\chi}{\partial y},$$

$$2m_0\frac{\partial A_3}{\partial y}=-\frac{1}{k_0}\frac{\partial}{\partial y}(\chi^2\tanh\psi)+2x\chi-(x^2-1)\frac{\partial\chi}{\partial x}. \qquad (37)$$

In our case $\chi=k_0(x+p_0+q_0y)$, $\tanh\psi=2\{1+[(1-y)\times(1+y)][(x-1)(x+1)]^{q_0}\}^{-1}-1$ and integration of Eqs. (37) gives

$$\frac{2m_0}{k_0^2}A_3=-(x+p_0+q_0y)^2\tanh\psi+y(x^2+1)$$
$$+q_0x(y^2+1)+2p_0xy+D_0, \quad (38)$$

where D_0 is an integration constant.

The coordinate transformation $x=(r-m_0)/k_0$, $y=\cos\vartheta$, as well as the following choice and renaming of the constants:

$$D_0=2q_0p_0,\quad p_0^2+q_0^2=1,\quad k_0=m_0(1-q_0^2)^{-1/2},$$
$$\sigma_0=q_0m_0(1-q_0^2)^{-1/2}$$

bring the metric (1), f and A_3 to their final form

$$dS^2=f^{-1}M^4N(dr^2+Kd\vartheta^2)+f^{-1}K\sin^2\vartheta\, d\varphi^2-fdt^2, \quad (39)$$

$$f=4m_0^2KM^2[(r+\sigma_0\cos\vartheta)^2-(r^2+2m_0r-\sigma_0^2)M^2]^{-2}, \quad (40)$$

$$m_0A_3=(\cos\vartheta-\tanh\psi)(r+\sigma_0\cos\vartheta)^2+\sigma_0(r+\sigma_0\times\cos\vartheta+m_0)\sin^2\vartheta, \quad (41)$$

where we have set:

$$K=r^2-2m_0r-q_0^2m_0^2(1-q_0^2)^{-1},$$

$$M=\sin^2\frac{\vartheta}{2}\left[\frac{\sqrt{1-q_0^2}(r-m_0)-m_0}{\sqrt{1-q_0^2}(r-m_0)+m_0}\right]^{q_0/2}+\cos^2\frac{\vartheta}{2}\left[\frac{\sqrt{1-q_0^2}(r-m_0)+m_0}{\sqrt{1-q_0^2}(r-m_0)-m_0}\right]^{q_0/2},$$

$$N=\left[\frac{(1-q_0^2)K}{(1-q_0^2)+K+m_0^2\sin^2\vartheta}\right]^{q_0^2}\times\left[\frac{\sqrt{1-q_0^2}(r-m_0)-m_0\cos\vartheta}{\sqrt{1-q_0^2}(r-m_0)+m_0\cos\vartheta}\right]^{2q_0}.$$

We note that for $q_0=0$ the metric (9) reduces to the Schwarzschild metric in standard notation. Moreover, it is clear from the asymptotic behavior $f^{-1}M^4N\to[1+(2m_0/r)]$, $f^{-1}KM^4N\to r^2$ of the various metric coefficients that our metric approaches the Schwarzschild metric as $r\to\infty$:

$$dS^2\approx\left(1+\frac{2m_0}{r}\right)dr^2+r^2d\vartheta^2+r^2\sin^2\vartheta d\varphi^2-\left(1-\frac{2m_0}{r}\right)dt^2.$$

Whence we conclude that the metric (9) is asymptotically flat, and the parameter m_0 is to be identified with the mass of the gravitational center. It follows from the asymptotic behavior of the function $A_3[A_3\to-(2/3)m_0\sigma_0(1/2)\sin^2\partial]$, which determines the magnetic properties of the gravitational center, that $\sigma_0 = q_0m_0/\sqrt{1-q_0^2}$ should be identified with the specific magnetic dipole moment (i.e., the moment of a unit mass).

In this manner the metric (39) obtained in this work can be used to describe the gravitational field of a star endowed with a magnetic dipole moment.

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