Synchrotron radiation of an electron in a strongly focusing magnetic field

O. E. Shishanin

Moscow Automobile Manufacturing Institute
(Submitted 14 April 1992)
Zh. Eksp. Teor. Fiz. **103**, 1118–1127 (April 1993)

The motion of an electron in a periodic magnetic field with a large gradient is discussed. A method is developed for solving the corresponding Lorentz equations. The latter reduce to differential equations with periodic coefficients. Angular distributions of the intensity of the radiation emitted by the electron for a certain frequency, accurate to within the first quantum term, are derived. The amplitude of the vertical oscillations has a strong influence on the spectral and angular characteristics.

INTRODUCTION

Questions concerning the dynamics of charged particles, particularly in beams, are currently of much interest because of the development of ultrahigh-energy, high-intensity accelerators and storage rings.[1,2] Magnetobremsstrahlung, which is unavoidable in an electron synchrotron, has recently found numerous applications,[3–6] and dedicated sources are under construction.

The properties of synchrotron light in uniform and weakly focusing magnetic fields were studied in Refs. 7 and 8, among other places.

In the present paper we examine the radiation by an electron moving in a focusing-defocusing system in which the magnetic field index $n(\varphi)$ alternately takes on the values n_1 and $-n_2$ ($n_2>0$) in the region $\varphi\in[0.2\pi/N]$, where N is the number of periodicity elements on a closed orbit.

To solve the complex problem of the radiation by an electron we need, as in Ref. 8 (for example), expressions which are as simple as possible but still a good approximation of reality to describe the dynamics of the electron in the classical approximation. Those equations which have been found in the accelerator literature involving (for example) the betatron function, are quite unwieldy. Even in this stage, they actually require numerical calculations. In the present paper we wish to propose a slightly different approach to solving the corresponding differential equations.

The function $n(\varphi)$ has first-order discontinuities, so it can be expanded in a Fourier series. By taking this approach, we break up the complex motion of the electron into the sum of simple oscillations in which the first harmonics are predominant. Under certain assumptions, this approach is equivalent to the averaging method of Refs. 10 and 11, as was shown in Ref. 9. That averaging method is more laborious, beginning with the third approximation. The question will be analyzed in its most general form, without consideration of the structural features of the existing devices.

1. THE MOTION PROBLEM

Along our approach, and in the linear approximation, the vertical oscillations are described by the Hill equation

$$\frac{d^2z}{d\tau^2}+\frac{g(\tau)}{N^2}z=0, \qquad (1)$$

where $\tau=N\varphi$, and

$$g(\tau)=\frac{n_1-n_2}{2}+\frac{2(n_1+n_2)}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\tau}{2k+1} \qquad (2)$$

is a periodic function.

We seek a solution of Eq. (1) in the form $z=\varphi_z(\tau)\exp(i\gamma_z\tau)$, where $\varphi_z(\tau+2\pi)=\varphi_z(\tau)$. We assume $\operatorname{Im}\gamma_z=0$; we are thus assuming that we are dealing with a stability region. The equation for $\varphi_z(\tau)$ takes the form

$$\frac{d^2\varphi_z}{d\tau^2}+2i\gamma_z\frac{d\varphi_z}{d\tau}-\gamma_z^2\varphi_z+\frac{g(\tau)}{N^2}\varphi_z=0. \qquad (3)$$

We seek approximate solutions for (3) in the form of asymptotic series:

$$\varphi_z(\tau)=\varphi_0(\tau)+\sum_{k=1}^{\infty}\frac{1}{N^k}\varphi_k(\tau),\qquad \gamma_z=\sum_{k=1}^{\infty}\frac{1}{N^k}\gamma_k.$$

We see that the expansion is in the parameter $1/N$. Substituting in $\varphi_z(\tau)$ and γ_z from (3), we find the chain of equations

$$\ddot\varphi_0=0,\quad \ddot\varphi_1+2i\gamma_1\dot\varphi_0=0,\quad \ddot\varphi_2-\gamma_1^2\varphi_0+g(\tau)\varphi_0=0,$$
$$\ddot\varphi_3+2i\gamma_1\dot\varphi_2-\gamma_1^2\varphi_1-2\gamma_1\gamma_2\varphi_0+g(\tau)\varphi_1=0,\ldots$$

As in Ref. 10, we solve these equations by eliminating the secular terms. We then find, in succession,

$$\varphi_0=a,\quad \varphi_1=a_1,\quad \gamma_1^2-\frac{n_1-n_2}{2}=0,$$
$$\varphi_2=\frac{2a(n_1+n_2)}{\pi}S_3,$$
$$\gamma_1^2a_1+2\gamma_1\gamma_2a-\frac{n_1-n_2}{2}a_1=0,$$
$$\varphi_3=\frac{4}{\pi}i\gamma_1a(n_1+n_2)C_4+\frac{2}{\pi}a_1(n_1+n_2)S_3,\ldots,$$

where

$$a,\ a_1=\text{const},\qquad S_3=\sum_{k=0}^{\infty}\frac{\sin(2k+1)\tau}{(2k+1)^3},$$
$$C_4=\sum_{k=0}^{\infty}\frac{\cos(2k+1)\tau}{(2k+1)^4}.$$

Hence

$$\gamma_1=\left(\frac{n_1-n_2}{2}\right)^{1/2},\quad \gamma_2=0,\quad \gamma_3=\frac{\pi^2\sqrt2}{96}\frac{(n_1+n_2)^2}{(n_1-n_2)^{1/2}},$$
$$\gamma_4=0,\quad \gamma_5=\frac{\pi^4\sqrt2}{5\cdot96^2}\frac{(n_1+n_2)^2}{(n_1-n_2)^{3/2}}\times[91(n_1^2+n_2^2)-202n_1n_2],\ldots$$

 1063-7761/93/040547-05$10.00

We write the frequency $\nu_z = \gamma_z N$ as

$$\nu_z = \gamma_1 + \frac{\gamma_3}{N^2} + \frac{\gamma_5}{N^4} + \ldots$$

In this approximation we have

$$\nu_z^2 = \frac{n_1 - n_2}{2} + \frac{\pi^2}{N^2}\frac{(n_1 + n_2)^2}{48} + \frac{\pi^4}{N^4}\frac{(n_1 + n_2)^2(n_1 - n_2)}{5\cdot 48}, \quad (4)$$

where the first two terms are the same as Eq. (3.40) of Ref. 12.

We find a general solution in the form

$$z = C\exp(i\gamma_z\tau)\varphi_z(\tau) + \text{c.c.}$$

We introduce some new arbitrary constants:

$$\left(\frac{B}{2}\right)e^{i\psi} = C\left(a + \frac{a_1}{N}\right).$$

We can finally write the solution of the Hill equation in a form accurate to within terms of order $1/N^3$:

$$z = B\cos\left(\frac{\nu_z}{N}\tau + \psi\right)\left[1 + \frac{2(n_1 + n_2)}{\pi N^2}S_3\right] - \frac{4(n_1 + n_2)\gamma_1}{\pi N^3}B\sin\left(\frac{\nu_z}{N}\tau + \psi\right)C_4. \quad (5)$$

The series S_3 and C_4 here can be expressed in terms of Euler polynomials $E_n(z)$. With $n_1 = n_2$ we then find the result found previously in Ref. 13. The higher approximations constitute small increments in the modulated amplitudes of the "fundamental" oscillations.

For the corresponding radial oscillations we need to solve the equation

$$\frac{d^2\rho}{d\tau^2} + \frac{1 - g(\tau)}{N^2}\rho = 0, \quad (6)$$

where $g(\tau)$ is expression (2), $\rho = r - R$, and R is the radius of the equilibrium orbit. We seek a solution of (6) in the form $\rho = \exp(i\gamma_x\tau)\varphi_x(\tau)$, and we again expand $\varphi_x(\tau)$ and γ_x in series. Eliminating the secular terms, we find $\gamma_1, \gamma_3, \gamma_5$ ($\gamma_2 = \gamma_4 = 0$), and we construct the frequency as follows:

$$\nu_x^2 = \frac{2 + n_2 - n_1}{2} + \frac{\pi^2}{N^2}\frac{(n_1 + n_2)^2}{48} + \frac{\pi^4}{N^4}\frac{(n_1 + n_2)^2(2 + n_2 - n_1)}{5\cdot 48}, \quad (7)$$

We construct a solution in the form

$$\rho = A\cos\left(\frac{\nu_x}{N}\tau + \chi\right)\left[1 - \frac{2(n_1 + n_2)}{\pi N^2}S_3\right] + \frac{2\sqrt{2}(n_1 + n_2)(2 + n_2 - n_1)^{1/2}}{\pi N^3}A\sin\left(\frac{\nu_x}{N}\tau + \chi\right)C_4, \quad (8)$$

where A and χ are arbitrary constants.

In this model we have $n_2 < n_1 < n_2 + 2$. This condition does not have to be mechanically extended to systems in which there are gaps without fields.

In the asymptotic expressions (5) and (8), the expansion parameter is $4n_1/\pi N^2$ (with $n_1 \approx n_2$). This parameter has values of 0.1–0.2 for existing accelerators, e.g., DESY. The next order makes a correction of less than 1%. The solutions found here are superpositions of bounded functions, in agreement with the strong-focusing principle, which ensures stable motion of the particle in the accelerator. On the next period of the trajectory, the particle runs into similar conditions; a complete revolution occurs in this manner. If we consider a defocusing-focusing structure, there are no fundamental changes in the solutions.

In this problem the corresponding angular velocity is found from

$$\dot\varphi = \omega_0\left\{1 - \frac{\rho}{R} + \frac{3}{2}\frac{\rho^2}{R^2} + \int g(\tau)\left(\frac{z\dot z}{R^2} - \frac{\rho\dot\rho}{R^2}\right)dt\right\}, \quad (9)$$

where $\omega_0 = ceH/E$. The total velocity, averaged over the fast oscillations, is

$$\langle v^2\rangle = R^2\omega_0^2\left(1 + \nu_x^2\frac{A^2}{R^2} + \nu_z^2\frac{B^2}{R^2}\right). \quad (10)$$

The constants A and B can evidently be treated as approximate amplitudes of the "fundamental" slow oscillations (since the solutions for z and ρ yield simple cosine functions when the average is taken), while the quantities ν_x and ν_z are treated as the frequencies of these oscillations. It is apparently a trajectory of this sort which is shown in Fig. 22 in Ref. 14.

The results found in this section of the paper can also be used to study the dynamics of a proton, provided that we change the sign on ω_0 in Eq. (9).

2. THE RADIATION PROBLEM

To find the spectral and angular distributions of the radiation intensity in this periodic field, we use the so-called operator formulation of the semiclassical approximation.[15–17] This method has been used previously by Zhukovskiĭ and the present author.[8]

We put the radiation vector $\mathbf{k} = \omega\mathbf{n}/c$ in the yz plane. We then have $\mathbf{n} = \{0, \sin\theta, \cos\theta\}$, where θ is the spherical angle. We denote by σ the linear component of the radiation, for which the electric vector lies in the plane of the orbit, and we denote by π the component orthogonal to σ. The corresponding polarization vectors are then found from

$$\mathbf{e}_\sigma = \{1, 0, 0\}, \quad \mathbf{e}_\pi = \{0, \cos\theta, -\sin\theta\}.$$

Taking the first quantum correction into account, we write the corresponding radiation intensities as follows:

$$dW_\sigma = \frac{d^3k}{(2\pi)^3}\frac{ce^2}{R}\frac{\nu'}{\nu}\left|\int dt\, v_x\exp\left[i\frac{\nu'}{\nu}(\omega t - \mathbf{kr})\right]\right|^2,$$
$$dW_\pi = \frac{d^3k}{(2\pi)^3}\frac{ce^2}{R}\frac{\nu'}{\nu}\left|\int dt(v_y\cos\theta - v_z\sin\theta)\times\exp\left[i\frac{\nu'}{\nu}(\omega t - \mathbf{kr})\right]\right|^2, \quad (11)$$

where

$$\nu' = \nu(1 + \hbar\omega/E), \quad \omega = \nu\omega_0.$$

The radiation passes near the plane of the orbit, so we have $\theta \sim \pi/2$. In addition, measuring instruments detect the light in a given direction as the electron traverses a small arc length $\varphi \approx \omega_0 t$. As a result, in the ultrarelativistic case

$$\sqrt{1-\beta^2} = mc^2/E, \qquad \cos\theta, \qquad \tau, \qquad \nu_x A/R, \qquad \nu_z B/R$$

are small quantities of approximately the same order. Let us consider the phase

$$\omega t - \mathbf{kr} = \nu[\omega_0 t - \frac{\omega_0}{c}(R+\rho)\sin\varphi\sin\theta - \frac{\omega_0}{c} z\cos\theta].$$

Expanding to the indicated third-order quantities, retaining terms $\sim 1/N^2$, and switching variables in accordance with

$$\omega_0 t_1 = \omega_0 t + \frac{\nu_x^2 - 1}{\nu_x}\frac{A}{R}\sin\chi + \frac{\pi(n_1+n_2)}{4N}\frac{A}{R}\cos\chi$$

we can write this phase as

$$\nu[\omega_0 t_1 \varepsilon_1/2 + (\omega_0 t_1)^3/6],$$

where

$$\varepsilon_1 = 1 - \beta^2 + \varepsilon_2,$$

$$\varepsilon_2 = \left\{\cos\theta - \frac{B}{R}\left[\frac{\pi(n_1+n_2)}{4N}\cos\psi - \nu_z\sin\psi\right]\right\}^2. \qquad (12)$$

The spectra and angular distributions are no longer dependent on the radial oscillations after this substitution is made.

Nonlinear terms can be taken into account in Eqs. (1) and (6), but the increments in the solutions do not alter $\omega t - \mathbf{kr}$ in our approximation.

Integrating expressions (11), we find equations for the radiation intensity in the first quantum approximation:

$$\frac{dW_\sigma^{\text{quant}}(\nu,\psi)}{d\Omega} = \frac{ce^2\nu\nu'}{6\pi^3 R^2}\varepsilon_1^2 K_{2/3}^2\left(\frac{\nu'}{3}\varepsilon_1^{3/2}\right),$$
$$\frac{dW_\pi^{\text{quant}}(\nu,\psi)}{d\Omega} = \frac{ce^2\nu\nu'}{6\pi^3 R^2}\varepsilon_1\varepsilon_2 K_{1/3}^2\left(\frac{\nu'}{3}\varepsilon_1^{3/2}\right), \qquad (13)$$

where K_i is the modified Bessel function. We also need to take an average over the phase of the axial oscillations, ψ. The small quantity ε_2 can also be written in the form

$$\varepsilon_2 = \left[\cos\theta - \nu_{\text{vert}}\frac{B}{R}\cos(\psi+\psi_0)\right]^2,$$

where

$$\cos\psi_0 = \frac{\pi(n_1+n_2)}{4N\nu_{\text{vert}}},$$

and the vertical frequency is

$$\nu_{\text{vert}} = \left[\frac{n_1-n_2}{2} + \frac{\pi^2(n_1+n_2)^2}{12N^2}\right]^{1/2}.$$

We now transform to Airy functions, which are given in the tables of Ref. 18 as

$$K_{1/3}\left(\frac{2}{3}x_1^{3/2}\right) = \sqrt{\frac{3\pi}{x_1}}V(x_1), \quad K_{2/3}\left(\frac{2}{3}x_1^{3/2}\right) = -\frac{\sqrt{3}\pi}{x_1}V'(x_1).$$

Restricting the discussion to the classical case, we write the basic result of this study as follows:

$$\frac{dW_\sigma^{\text{cl}}(\nu)}{d\Omega} = W_1\frac{1}{2\pi}\int_0^{2\pi} V'^2(x_1)d\psi,$$
$$\frac{dW_\pi^{\text{cl}}(\nu)}{d\Omega} = W_1\left(\frac{\nu}{2}\right)^{2/3}\frac{1}{2\pi}\int_0^{2\pi}\varepsilon_2 V^2(x_1)d\psi, \qquad (14)$$

where

$$W_1 = 2^{1/3}ce^2\nu^{2/3}/\pi^2R^2, \qquad x_1 = (\nu/2)^{2/3}\varepsilon_1.$$

For $\theta \sim \pi/2$, the intervals in (14) can be evaluated through series expansion of the Airy functions.[18] As we move away from the orbital plane, we can carry out an expansion in the quantity $(B^2/R^2)/\varepsilon$, where $\varepsilon = 1 - \beta^2\sin^2\theta$.

In this section of the paper we have not taken an average over the fast oscillations. In other words, we are taking into account in this radiation problem all components of the complex motion of the electron: the orbital revolution and the fundamental sinusoidal oscillations along the vertical and in the orbital plane, on which fast oscillations are superimposed. The frequencies of the average motion are determined by expansions (4) and (7). Those expansions agree with existing expressions which have been derived for $\cos\mu$ ($\mu = 2\pi\nu/N$) by a matrix method in accelerator theory.

There have been no experiments on the spectral and angular properties in strongly focusing fields, in contrast with the experiments involving weakly focusing fields. In constructing theoretical curves we must accordingly bear in mind that Eqs. (13) and (14) are very sensitive to changes in the electron energy, the frequency of the emitted light, and the amplitude of the vertical oscillations. The shapes of the curves may thus change substantially. In particular, it follows from (13) and (14) that the π component of the radiation does not vanish at $\theta = \pi/2$. In the long-wave region of the light, the intensity of the π component is nearly proportional to the square amplitude of the "fundamental" vertical oscillations.

For this asymmetric model we have

$$\frac{dW_\pi^{\text{cl}}(\nu)}{d\Omega} = W_1\left(\frac{\nu}{2}\right)^{2/3}\frac{B^2}{R^2}\frac{1}{2\pi}\int_0^{2\pi}\nu_\psi^2 V^2(x_1)d\psi,$$

where

$$x_1 = \left(\frac{\nu}{2}\right)^{2/3}(1 - \beta^2 + \frac{B^2}{R^2}\nu_\psi^2),$$

$$\nu_\psi = \frac{\pi(n_1+n_2)}{4N}\cos\psi - \nu_z\sin\psi.$$

We see that the magnetobremsstrahlung is polarized best when the axial oscillations are at a minimal level.

The curve for the σ component is broader and lower than the corresponding curve for a uniform magnetic field (at the same energy). For both components the difference depends on the ratio

$$\frac{\pi n}{\sqrt{3}N}\frac{B}{R}/\sqrt{1-\beta^2}.$$

For the symmetric model, with $n_1 = n_2$, we need to set

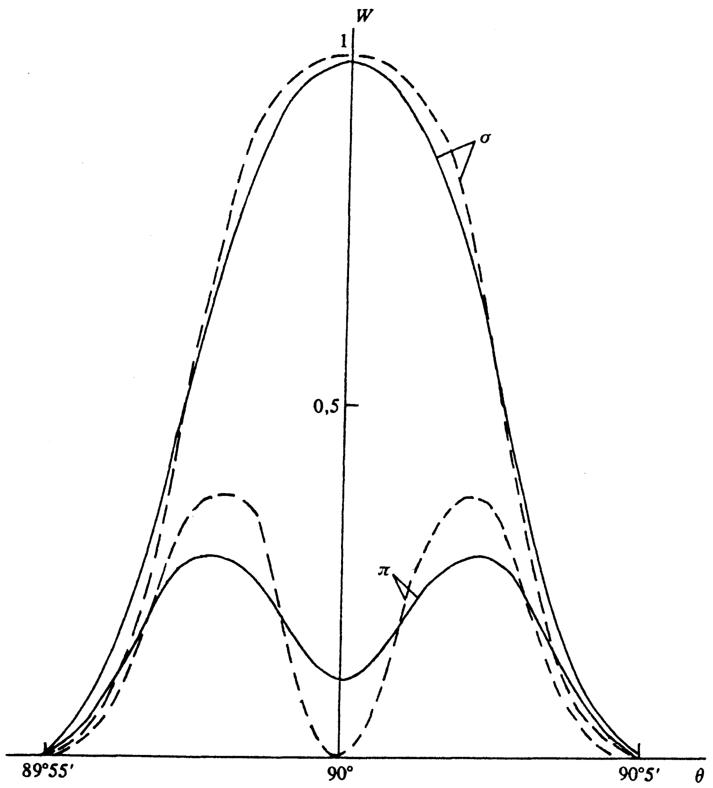

FIG. 1. Spectral and angular distributions of the σ and π components of the linear polarization of the radiation. The dashed lines correspond to a uniform magnetic field.

$$\varepsilon_2 = \left[\cos\theta - 2\nu_{\rm symm}\frac{B}{R}\cos\left(\psi + \frac{\pi}{6}\right)\right]^2,$$

in Eqs. (12)–(14). Here $\nu_{\rm symm} = \pi n/2\sqrt{3}N$. The increment of $\pi/6$ in the phase here is interesting.

We introduce $\alpha = 2\nu_{\rm symm}B/R$. Under the condition $\alpha^2/\varepsilon \ll 1$ we then find the following result from (14) in the first approximation (with $n_1 = n_2$):

$$\frac{dW_\sigma(\nu)}{d\Omega} = W_1[V'^2 + 2x_2^2 p(2x_2 gU + (1+2g)VV'],$$
$$\frac{dW_\pi(\nu)}{d\Omega} = W_1 x_2[(g+p)V^2 + 2x_2 gp(2x_2 gU + 5VV')], \quad (15)$$

where the arguments of V and V' are $x_2 = (\nu/2)^{2/3}\varepsilon$ and $p = \alpha^2/2\varepsilon$, $g = \cos^2\theta/\varepsilon$, and $U = x_2V^2 + V'^2$. The first terms in (15) correspond to a uniform magnetic field.

In the orbital plane it is better to use series expansions of V and V' (for small values of x_2). The square brackets in (15) must then be replaced by, respectively,

$$V'^2(0) + V(0)V'(0)x_2^2(1 + 2p + 4pq + \frac{3}{2}p^2)$$

and

$$V^2(0)(p+g) + V(0)V'(0)x_2(2p + 3p^2 + 2g + 10pg),$$

where the constants are $V(0) = 0.629\ 27$ and $V'(0) = -0.458\ 75$.

Using these equations, and allowing for the smaller corrections to them, we can plot spectral-angular distributions of the σ and π components (Fig. 1). In the model selected here, the energy of the electron is $E = 5$ GeV, the amplitude of the average oscillations is $B = 2$ mm, and the wavelength of the emitted radiation is $\lambda = 1000$ Å ($R = 30$ m, $N = 24$, $n = 70$).

We see from this figure that the greatest deviation from the corresponding curves for a uniform magnetic field (for the same values of E and λ) comes from the π component of the radiation, especially at $\theta = 90°$.

In this case, in contrast with the case of weak focusing,[8] instead of the parameter $\sqrt{n}\,B/R$ we find $(\pi n/\sqrt{3}N)B/R$. In the latter case, however, B is only the amplitude of the average motion, not the overall motion. Instead of the frequency $\sqrt{n}$ we have the quantity $\pi n/2\sqrt{3}N$, which describes a stable motion along the vertical in the focusing and defocusing regions.

In many publications the spherical angle of the radiation θ is replaced by the angular deviation from the plane of motion of the particle, ψ, expressed in radians. To go over to this parameter in the results found above, we would have to use $\cos\theta \approx \psi$.

CONCLUSION

The most fundamental aspects of this paper are the expansion of the magnetic field gradient in a Fourier series (this approach may be physically justified for many periodic structures), the derivation of a constant total velocity when an average is taken over oscillations, and the incorporation of all possible oscillations in a study of the properties of the radiation.

However, the main conclusion reached here is that the properties of the radiation are specifically determined by the average motion of the particle.

This analysis was carried out for the focusing-defocusing model. The methods used here have revealed only the major effects. In real situations, there will also be some stimulated oscillations, because of displacements of magnets, which will influence the straight sections, etc.

[1] *Proceedings of the Thirteenth International Conference on High-Energy Particle Accelerators,* Vols. 1 and 2, Nauka, Novosibirsk, 1987.
[2] *Particle Accelerators (Proceedings of the Fourteenth International Conference on High Energy Accelerators,* Tsukuba, Japan, 1989), Vols. 26–33, 1990.
[3] C. Kunz (editor), *Synchrotron Radiation: Techniques and Applications,* Springer-Verlag, N.Y., 1979.
[4] E. -E. Koch, D. E. Eastman, and Y. Farge, *Handbook on Synchrotron Radiation*, Vol. 1, North-Holland, Amsterdam, 1983.
[5] *AIP Conference Proceedings*, Vol. 184. *Physics of Particle Accelerators,* Vol. 2, New York, 1989.
[6] Nucl. Instrum. Methods A **266**, No. 1–3 (1990); A **291**, No. 1, 2 (1990); A **308**, No. 1, 2 (1991).
[7] A. A. Sokolov and I. M. Ternov, Zh. Eksp. Teor. Fiz. **31**, 473 (1956) [Sov. Phys. JETP **4**, 396 (1957)].
[8] V. Ch. Zhukovskiĭ and O. E. Shishanin, Zh. Eksp. Teor. Fiz. **61**, 1371 (1971) [Sov. Phys. JETP **34**, 729 (1972)].
[9] O. E. Shishanin, Ukr. Mat. Zh. **44**, 142 (1992).
[10] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, *Asymptotic Methods in the Theory of Nonlinear Oscillations,* Nauka, Moscow, 1974 (an earlier edition of this book was published in English translation by Gordon and Breach, New York).
[11] A. A. Kolomenskiĭ, A. A. Lebedev, and A. N. Teoriya, *Theory of Cyclic Accelerators*, Fizmatgiz, Moscow, 1962.
[12] E. Courant and H. Snyder, Ann. Phys. (Leipzig) **3**, 1 (1958).
[13] O. E. Shishanin, in *Abstracts, Twelfth All-Union Conference on Particle Accelerators,* ITEF, Moscow, 1990, p. 156.
[14] A. A. Kolomenskiĭ, *Physics of Methods for Accelerating Charged Particles*, Izd. MGU, Moscow, 1980.
[15] J. Schwinger, Proc. Natl. Acad. Sci. USA **40**, 132 (1954).
[16] V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. **53**, 1478 (1967) [Sov. Phys. JETP **26**, 854 (1967)]; **55**, 1542 (1968) [Sov. Phys. JETP **28**, 807 (1969)].
[17] V. B. Berestetskiĭ, E. M. Lifshitz, and P. Pitaevskiĭ, *Quantum Electrodynamics*, Nauka, Moscow, 1989, §90 (an earlier edition of this book was published in English translation by Pergamon, Oxford).
[18] G. D. Yakovleva, *Tables of Airy Functions and Their Derivatives*, Nauka, Moscow, 1969.

Translated by D. Parsons