Langmuir turbulence at low pumping levels

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We study the stability of the weak plasma-turbulence spectra to the appearance of spontaneous spatial modulations. We present a numerical simulation of the modulational instability of the spectra in the two- and three-dimensional cases. We show that in an isothermal plasma, $T_e \lesssim 3T_i$, the development of the modulational instability leads to self-focusing and collapse of plasma waves. In the opposite case the transfer along the spectrum to the long-wavelength region suppresses the instability. We show that the development of the modulation instability in a magnetized plasma leads to a sharp increase in the Landau damping. As a result the width of the spectrum does not exceed a few times ω_{pi} even well above threshold.

1. INTRODUCTION

If the characteristic times for interactions between waves are significantly longer than their periods one can locally consider the oscillations to be linear with slowly changing parameters. One can in that case consistently change to a description of turbulence using the langauge of kinetic equations for the waves [weak turbulence (WT) theory]. In the framework of weak Langmuir turbulence (WLT) the oscillations are described in quasiparticle (plasmons, phonons, and so on) terms and their interactions constitute decay and scattering of quasiparticles.

The kinetic equations for the waves can be derived in a regular way from the dynamic equations, using the small parameter

$$\gamma_{nl}/\Delta\omega_k \ll 1. \tag{1.1}$$

Here γ_{nl} is a characteristic nonlinear growth rate and $\Delta\omega_k$ is a characteristic frequency difference of the interacting waves.

To derive the kinetic equations for the waves we use two approximations: I. The phases of the individual waves are random and their statistics is Gaussian. This assumption makes it possible to express higher correlation functions in terms of the pair ones. Condition (1.1) guarantees the randomness of the phases. Even if they are initially correlated, the rotation of the individual phases with different frequencies leads to a decay of the correlations.

II. The turbulence is assumed to be uniform. This means that the conditions

$$\langle a_{\mathbf{k}} a_{\mathbf{k}'}^* \rangle = n_k \delta(\mathbf{k}-\mathbf{k}') \tag{1.2}$$

are satisfied. The a_k are here the normal amplitudes of the interacting waves and n_k is the number of interacting waves (occupation number).

WLT theory was created more than 20 years ago (see, e.g., Refs. 1). However, up to the present there do not exist many experimental confirmations of WLT, and, even worse, many results which contradict it.

For instance, in experiments on the parametric excitation of waves both in a magnetized and in an isothermal plasma the width of the turbulent spectra is usually not more than a few ion-sound frequencies whereas the width of the spectrum in WLT must increase linearly when one goes further away from threshold.

Recently there have appeared a number of papers devoted to a numerical simulation of Langmuir turbulence at low levels. It was shown in Ref. 2 that in the one-dimensional problem the dynamic description and the statistical description using a kinetic equation for the waves (there where it is applicable) give identical results. However, the results of a two-dimensional simulation[3–5] demonstrate a number of effects which are not described by WLT theory. Of course, this is first and foremost connected with the important role of the collapse of Langmuir waves, an effect absent in the weak-turbulence description. However, even when one is only just above threshold there may occur a number of deviations from the WLT predictions. In our opinion this can be explained by the following effect. It was shown in Refs. 6 and 7 that the Langmuir turbulence spectra are, as a rule, singular, being arranged on lines and surfaces in k-space or even consisting of a set of quasi-monochromatic waves. Such distributions may be ustable[8,9] to the appearance of spontaneous spatial modulation of the turbulence, i.e., condition II is violated. As a result of the local growth of the field condition I is also violated and there appear coherent electric field bunches, the evolution of which can lead to Langmuir collapse.

The study of this effect is the aim of the present paper.

We discuss in the second section of the paper the basic equations which describe both WLT effects and the development of spatially nonuniform perturbations. Moreover, we use these equations to study the linear stage of the modulation instabilty (MI). Unfortunately, its nonlinear stage can be studied only numerically. The fourth section of the paper is devoted to such a study. We investigate in the fifth section the features of the turbulence of a magnetized plasma. In the Conclusion we discuss experimental manifestations of the results.

2. BASIC EQUATIONS AND STATEMENT OF THE PROBLEM

It is convenient to use the averaged dynamic equations proposed by V. E. Zakharov[10] to describe Langmuir turbulence of an isotropic plasma:

$$\Delta\left(i\psi_t + i\frac{\nu}{2}\psi + \frac{3}{2}\omega_p r_d^2 \Delta\psi \right) + \omega_p \operatorname{div} \frac{\delta n}{2n_0} \nabla\psi = 0, \tag{2.1}$$

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$$\frac{\partial^2\delta n}{\partial t^2}-c_s^2\Delta\delta n+2\Gamma\frac{\partial}{\partial t}\delta n=\frac{1}{16\pi M}\Delta|\nabla\psi|^2. \qquad (2.2)$$

Here Ψ is the complex envelope of the electric potential, $E=\frac{1}{2}(\nabla\psi e^{-i\omega pt}+\text{c.c.})$, δn is the low-frequency change in the density, c_s and Γ are, respectively, the ion-sound speed and its damping rate, and ν is the damping rate of the Langmuir oscillations. If the condition $\gamma_{nl}<\Gamma<c_s k$ is satisfied we can assume in Eq. (2.2) that $\Psi\propto e^{-i\omega_k t}$ where ω_k is the dispersion law for the Langmuir waves. We can then write instead of (2.2) in the k-representation

$$\delta n_\varkappa(t)=\frac{e^2}{4m\omega_p^2T_e}\int G(\varkappa,\omega_{k_1}-\omega_{k_2})$$
$$\times(\mathbf{k}_1\mathbf{k}_2)\psi_{k_1}\psi_{k_2}^*\delta(\varkappa-\mathbf{k}_1+\mathbf{k}_2)\,dk_1\,dk_2, \qquad (2.3)$$

where $G(\varkappa,\Omega)$ is given by the expression

$$G(\varkappa,\Omega)=\frac{\varkappa^2c_s^2}{\Omega^2-\varkappa^2c_s^2+2i\Gamma\Omega}. \qquad (2.4)$$

Strictly speaking one must describe the low-frequency motions in an isothermal plasma kinetically. One can show that in that case (see, e.g., Ref. 11) Eq. (2.3) retains its form but the function $G(\varkappa,\Omega)$ is given by

$$G(\varkappa,\Omega)=\frac{\varepsilon_e(\varkappa,\Omega)}{\varepsilon(\varkappa,\Omega)}-1. \qquad (2.5)$$

Here ε is the permittivity of the plasma and ε_e is the contribution to it from the electrons. Expression (2.4) is a rather good approximation of (2.4) for an appropriate choice of Γ. Using (2.3) to eliminate δn in (2.1) we change to the k-representation. Furthermore we introduce the quantity

$$a_k=\frac{k\psi_k}{(4\pi\omega_p)^{1/2}},$$

defined in such a way that the quantity

$$\int\omega_p(1+{}^3/_2k^2r_d^2)\,|a_k|^2\,dk$$

is the same as the total energy of the Langmuir oscillations. Ultimately we obtain a dynamic equation for the a_k

$$\frac{\partial a_k}{\partial t}+\frac{\nu}{2}a_k+i\omega_ka_k=i\int T_{kk_1k_2k_3}a_{k_1}^*a_{k_2}a_{k_3}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,dk_1\,dk_2\,dk_3,$$

$$T_{kk_1k_2k_3}=\frac{\omega_p^2}{4n_0T}\left[\frac{(\mathbf{k}\mathbf{k}_2)(\mathbf{k}_1\mathbf{k}_3)G(k_2-k_3,\omega_{k_1}-\omega_{k_3})+(\mathbf{k}\mathbf{k}_3)(\mathbf{k}_1\mathbf{k}_2)G(k_1-k_2,\omega_{k_1}-\omega_{k_2})}{kk_1k_2k_3}\right]. \qquad (2.6)$$

We easily get from (2.6) for the waves a kinetic equation describing the induced scattering by ions. We multiply (2.6) by a_k^* and add to it its complex conjugate. Assuming the turbulence to be spatially uniform,

$$\langle a_ka_{k'}^*\rangle=n_k\delta(k-k')$$

and breaking up the fourfold correlations in terms of pair correlations we get

$$\frac{\partial n_k}{\partial t}+\nu n_k=n_k\int T_{kk'}n_{k'}\,dk', \qquad (2.7)$$

where

$$T_{kk'}=\text{Im}\,T_{k,k',k,k'}.$$

From the symmetry properties of the permittivity, $\varepsilon_{\varkappa,\Omega}=\varepsilon^*_{\varkappa,-\Omega}=\varepsilon_{-\varkappa,\Omega}$ and from (2.6) and (2.5) it follows that

$$T_{kk'}=-T_{k'k}.$$

The solutions (2.7) have a number of noteworthy features.[6,7] First of all they are strongly anisotropic being concentrated on lines in k-space or even consisting of a set of quasi-monochromatic waves. The structure of the distribution is determined by the source for the excitation of the oscillations and the shape of the matrix element $T_{kk'}$. In particular, in the case of parametric excitation of the waves the plasma oscillations propagate parallel to the external electric field and a one-dimensional model describes the situation rather well.[6,7]

Moreover, the dynamics of the establishment of the stationary solutions is also nontrivial. Equation (2.7) is Hamiltonian and a stationary state is therefore reached only thanks to the presence of small noise terms, neglected in (2.7), and in a number of cases cannot be reached at all.[6,7] Below we shall discuss this problem in detail.

We now describe the transition to the quasidynamic equations[9] which are used to describe the modulation instability (MI) of singular turbulent spectra.

We change to a statistical description, introducing $n_{kk'}=\langle a_ka_{k'}^*\rangle$. The angle brackets indicate here averaging over the phases.

We restrict ourselves to parametric excitation of the waves when the oscillations propagate parallel to the external electric field, the direction of which we choose to be along the z axis. Owing to the large difference between the group velocities along z spatially nonuniform perturbations are suppressed in that direction and the MI can develop only in directions perpendicular to z.[9] Therefore we have

$$n_{kk'}=n_{k_z}(k_\perp,k_\perp')\delta(k_z-k_z').$$

We change in (2.6) to a statistical description. We have already noted above that the breaking up of the fourfold correlators into pair correlators can be done also in a spatially nonuniform situation, and the randomness of the phases is guaranteed by the large width of the packets along k_z. It is convenient to change for the transverse coordinates to the z-representation:

$$n_{k_z}(r_\perp,r_\perp')=\frac{1}{(2\pi)^2}\int n_{k_z}(k_\perp,k_\perp')$$
$$\times\exp\{i(\mathbf{k}_\perp\mathbf{r}_\perp-\mathbf{k}_\perp'\mathbf{r}_\perp')\}\,dk_\perp\,dk_\perp'.$$

Finally we get

$$\left(\frac{\partial}{\partial t}+\nu+i(\tilde\omega_k-\tilde\omega_{k'})+i\frac{3}{2}\omega_pr_d^2(\Delta_r-\Delta_{r'})\right)n_k(r,r'),$$

where

$$\tilde{\omega}_k=\frac{3}{2}\omega_p(kr_d)^2+2\int(F_{kk'}+iT_{kk'})n_{k'}(r,r')dk', \qquad (2.8)$$

$$F_{kk'}=F_{k'k}=\mathrm{Re}\,T_{k_zk_z'k_zk_z'},$$

$$T_{kk'}=-T_{k'k}=\mathrm{Im}\,T_{k_zk_z'k_zk_z'}.$$

We assume that the scale of the spatial modulations is significantly larger than the wavelength and that we can therefore neglect the $k_\perp$ dependence in the matrix elements. To simplify the notation we drop in (2.8) and below the indices of k_z and the index $\perp$ of $r_\perp$. We note that the kernels $F_{kk'}$ and $T_{kk'}$ depend in the one-dimensional model only on $(\omega_k-\omega_{k'})/|k-k'|\simeq(|k-k'|)3/2\omega_p r_d^2$ and that to a good approximation they are difference operators. We can further simplify Eq. (2.8) if we note that it has a solution of the form

$$n_k(r,r')=A_k(r)A_k(r').$$

Here A_k satisfies the equation

$$i\frac{\partial A_k}{\partial t}+i\nu A_k+\frac{3}{2}\omega_p r_d^2\Delta A_k$$
$$+2\int(F_{kk'}+iT_{kk'})|A_{k'}|^2dk'A_k=0. \qquad (2.9)$$

In a transparent medium where $T_{kk'}$, $\nu=0$ Eq. (2.9) changes into a many-component nonlinear Schrödinger equation. For a spatially uniform distribution of the oscillations we are led to the one-dimensional Eq. (2.7) by introducing $n_k=|A_k|^2$. In the general case (2.9) describes both the transfer along the spectrum due to induced scattering and the effects of the modulation instability.

It is well known (see, e.g., Refs. 6 and 7) that in the case of parametric excitation of waves the turbulence spectrum consists of a set of narrow peaks, $n_k=\Sigma_s N_s\delta(k-k_0-sk_{\text{diff}})$ which are spaced at distances $k_{\text{diff}}=\frac{2}{3}r_d^{-2}\sqrt{m/M}$ from one another, corresponding to the maximum of the induced scattering. One can in that case simplify (2.7) by changing to the satellite approximation. The equations describing the interaction between the separate peaks have the form

$$\frac{dN_0}{dt}=N_0(\gamma_0-\nu-TN_1),$$
$$\frac{dN_1}{dt}=N_1(-\nu+T(N_0-N_2)),$$
$$\cdots\cdots\cdots\cdots \qquad (2.10)$$
$$\frac{dN_i}{dt}=N_i(-\nu+T(N_{i-1}-N_{i+1})).$$

Here γ_0 is the parametric instability growth rate, ν is the damping rate of the waves, T is the maximum of the matrix element $T_{kk'}=T(k-k')=T(k_{\text{diff}})$. We can similarly simplify Eq. (2.9). Putting

$$A_k(r)=\sum_s\psi_s\delta^{1/2}(k-k_0+sk_{diff}), \qquad (2.11)$$

we get the generalization

$$i\psi_{0t}+{}^3/_2r_d^2\Delta\psi_0+F|\psi_0|^2\psi_0=i(\gamma_0-T|\psi_1|^2)\psi_0,$$
$$i\psi_{1t}+{}^3/_2r_d^2\Delta\psi_1+F|\psi_1|^2\psi_1=i(-\nu+T(|\psi_0|^2-|\psi_2|^2))\psi_1,$$
$$\cdots\cdots\cdots\cdots$$
$$i\psi_{it}+{}^3/_2r_d^2\Delta\psi_i+F|\psi_i|^2\psi_i=i(-\nu+T(|\psi_{i-1}|^2-|\psi_{i+1}|^2))\psi_i$$
$$F=F_{k,k} \qquad (2.12)$$

In obtaining (2.12) we used the fact that $F_{k_i,k_{i+1}}=F(k_i-k_{i+1})=F(k_{\text{diff}})$ is equal to zero, as can be seen from (2.4), while $F(2k_{\text{diff}})$ is small. The number of satellites is, as in homogeneous turbulence, determined by the ratio γ_0/ν. We also note that $F=-(\omega_p^2/4nT)\times G(0)\sim\omega_p^2/4nT$ is a positive quantity. If we consider only one of the coupled nonlinear Schrödinger equations (2.12) the sign of F corresponds to the possibility of the appearance of the MI.

3. MODULATION INSTABILITY OF SINGULAR SPECTRA

Equation (2.9) determines the spatially uniform stationary solutions of the form $A_k=A_k^0\exp(i\Omega_k t)$ where A_k^0 and Ω_0 are determined from the conditions

$$\Omega_k=\int F_{kk'}|A_{k'}^0|^2dk', \quad \nu+\int T_{kk'}|A_{k'}^0|^2dk'=0. \qquad (3.1)$$

We consider the instability of (3.1) under small, spatially nonuniform perturbations

$$A_k=(A_k^0+\delta A_k\exp\{-i\Omega t+i\varkappa\mathbf{r}\})\exp(i\Omega_k t).$$

Linearizing (2.9) we get

$$\Omega\delta A_k-\frac{3}{2}\varkappa^2r_d^2\omega_p\delta A_k$$
$$+A_k^0\int(F_{kk'}+iT_{kk'})A_{k'}^0(\delta A_{k'}+\delta A_{k'}^*)dk'=0,$$
$$-\Omega\delta A_k^*-\frac{3}{2}\varkappa^2r_d^2\omega_p\delta A_k^*$$
$$+A_k^0\int(F_{kk'}-iT_{kk'})A_{k'}^0(\delta A_{k'}+\delta A_{k'}^*)dk'=0.$$

We introduce the quantities

$$u_k=A_k^0(\delta A_k+\delta A_k^*), \quad v_k=A_k^0(\delta A_k-\delta A_k^*)$$

and, eliminating v_k, we obtain

$$\Omega\left(\Omega-2i|A_k^0|^2\int T_{kk'}u_{k'}dk'\right)$$
$$=\frac{3}{2}\omega_p\varkappa^2r_d^2\left(\frac{3}{2}\omega_p\varkappa^2r_d^2+2\int F_{kk'}u_{k'}dk'|A_k^0|^2\right). \qquad (3.2)$$

Notwithstanding the fact that Eq. (2.9) resembles the nonlinear Schrödinger equation qualitatively, the appearance of an MI is completely unexpected. The fact is that the transfer along the spectrum (as we shall show in what follows) can effectively suppress the occurrence of local intensity maxima. It is impossible to study (3.2) in the general case and we restrict ourselves to considering some simple, but the most interesting physical situations.

If the linear damping of the waves is small, one may assume the stationary distribution of the waves to be uniform, $|A_k^0|^2 = \text{const} = N_0$. We have already noted that one can assume the kernels $T_{kk'}$ and $F_{kk'}$ to be difference operators for $k \gg k_{\text{diff}}$,

$$G\left(\frac{\omega_k-\omega_{k'}}{|k-k'|v_{Ti}}\right) \approx G\left(\frac{k-k'}{k_{diff}}\right).$$

In this case we easily get from (3.2) the dispersion equation[9]

$$\Omega_{1,2}=-iT_qN_0\pm\{(T_qN_0)^2+{}^3/_2\omega_p\varkappa^2r_d^2({}^3/_2\omega_p\varkappa^2r_d^2-2F_qN_0)\}^{1/2}$$
$$=-iT_qN_0\pm\{({}^3/_2\varkappa^2r_d^2-F_qN_0)^2+N_0(|T_q|^2-|F_q|^2)\}^{1/2}. \quad (3.3)$$

Here we have

$$T_q=\frac{1}{(2\pi)^{1/2}}\int T(x)e^{iqx}dx, \quad F_q=\frac{1}{(2\pi)^{1/2}}\int F(x)e^{iqx}dx,$$
$$T(x)=T(k-k'), \quad F(x)=F(k-k').$$

Because it is odd $T_{kk'} = T_q$ is a purely imaginary function, $T_q = if(q)$. Similarly, F_q is purely real. It is also clear that as $q \to 0$, we have $T_q \propto q$. We have for uniform perturbations from (3.3) two branches of indifferently-stable perturbations $\Omega = 0$ and $\Omega = 2f(q)N_0$. In the long-wavelength limit one of them changes into second sound. It is clear from (3.3) that spatially nonuniform perturbations are unstable for $|T_q| < |F_q|$. A simple calculation gives for the matrix elements which describe Eq. (2.4)[11]

$$T_q=i\pi T_0\exp\left\{-\frac{\Gamma}{\Omega_s}|q|k_{diff}\right\}\sin qk_{diff},$$
$$F_q=-\pi T_0\exp\left\{-\frac{\Gamma}{\Omega_s}|q|k_{diff}\right\}\sin|q|k_{diff},$$
$$T_0=\frac{\omega_p^2}{4nT}.$$

It is clear that $|T_q| = |F_q|$ and there is no instability that can be eliminated by the induced scattering. We show that this fact is not accidental and is not connected with any actual approximation of the kernel. In the general case (2.5) the function G in the matrix elements can be expressed in terms of the permittivity of the electrons and is analytic in the upper half plane. Hence we have

$$\int_{-\infty}^{\infty} G(x)e^{iqx}dx=0, \quad q>0,$$

whence also follows the relation $|T_q| = |F_q|$.

An instability appears if we take into account the modulation which is always present, of the intensity of the turbulence along the "jet." It is simplest to show this for spectra consisting of a set of identical satellites. We consider perturbations which involve only even or only odd peaks. It is clear from the system (2.12) that then there is no interaction between neighboring peaks and there appears separately a modulation instability of each excited peak.

The nonlinear stage of the instability can be studied only numerically. In reality the situation is even more complicated. As we have already mentioned, the reaching of a stationary state in the WLT framework always takes a long time and may not occur at all. It is therefore unclear how much a study of the stability of stationary spectra reflects reality.

In the case of parametric excitation of waves, and for weak damping of waves $\gamma_0 \gg \nu$, there occurs a periodic splitting of pulses propagating in the small-k range which is the range where the energy is absorbed owing to the Langmuir collapse.[6,7] The propagation of a pulse in the inertial range is described by an equation following from (2.10)

$$\frac{dN_i}{dt}=N_iT(N_{i-1}-N_{i+1}). \quad (3.4)$$

It has a travelling soliton solution[6]

$$N_n=F\left(Tt-\frac{n}{s}-\tau_0\right),$$
$$F(x)=N_0\left(1+\frac{a}{1-b+b\,\text{ch}\,\delta x}\right). \quad (3.5)$$

If the soliton amplitude is much larger than the noise level N_0, i.e., $a \gg 1$, we have for the parameters of (3.5)

$$\delta=aN_0, \quad b^2=1/2a, \quad s=\delta/\ln a.$$

For the amplitude of a soliton excited with a growth rate γ_0 the estimate

$$N_0a\approx\frac{1}{2}\frac{\gamma_0}{T}\ln\frac{\gamma}{2TN_0}$$

was found in Ref. 6. This soliton propagates with a velocity

$$v\approx\frac{1}{2}\gamma_0k_{diff}\frac{\ln(\gamma_0/2TN_0)}{\ln[\ln(\gamma_0/2TN_0)\cdot(\gamma_0/2TN_0)]},$$

and (as is clear from (3.5)) only 2 to 3 peaks are excited at each time. The propagation velocity does not depend critically on the noise level, $v \sim \frac{1}{2}\gamma_0 k_{\text{diff}} \sim TNk_{\text{diff}}$. A modulation instability with a characteristic growth rate FN must develop after the time it takes the soliton to traverse a distance k_{diff}. We see thus that the condition $FN < k_{\text{diff}}/v$, or $F < T$ is necessary for the suppression of the instability. It is clear that the possibility for the development of a modulation instability and the subsequent collapse depends on the ratio F/T, i.e., on the ratio of the electron to the ion temperature. An elucidation of the actual value of this ratio, of the nature of the development of the instability, and so on, can only be given after a numerical simulation to which the next section is devoted.

4. NUMERICAL EXPERIMENT

To simulate the effect of the development of the modulation instability on turbulence spectra we used the system (2.12) written in dimensionless variables

$$\gamma_0t\to t, \quad k\to k_{diff}k, \quad F|\psi_i|^2\to\gamma_0|\psi_i|^2,$$
$$\nu/\gamma_0\to\nu, \quad \frac{2}{3}\frac{r^2}{r_d^2}\frac{\gamma_0}{\omega_p}\to r^2,$$
$$i\psi_{0t}+\Delta\psi_0+|\psi_0|^2\psi_0=i\psi_0(1-\nu-\tilde{T}|\psi_1|^2),$$
$$\cdots\cdots\cdots\cdots$$
$$i\psi_{it}+\Delta\psi_i+|\psi_i|^2\psi_i=i\psi_i(-\nu+\tilde{T}(|\psi_{i-1}|^2-|\psi_{i+1}|^2)). \quad (4.1)$$

Here $\tilde{T} = T/F = \Omega/2\Gamma$. We shall show in what follows that this ratio determines whether it is possible that a modulation instability can develop.

Under actual physical conditions the pumping range is small, and is not more than ten steps and in the small k region there is an energy source caused by the Langmuir collapse. To simulate it we used in our numerical calculations a running boundary condition for the last of the peaks,

$$\psi_n = \psi_{n-1}.$$

We varied the number of peaks up to ten, using noisy initial conditions. For the sake of simplicity we considered only axially symmetric distributions, $\nabla^2\psi_i = 1/r\partial/\partial r\, r\,\partial\psi_i/\partial r$. The size L of the computing region in r was $L = 10$, much larger than the typical size of the MI, $L \sim 1$, for $|\psi|^2 \sim \gamma_0/F$. We used as boundary condition $\partial\psi/\partial r = 0$.

Of most interest are the cases where one is well above threshold; we therefore started with calculations for $\nu = 0$. The evolution of the oscillations depended in substantially on the parameter $\tilde{T}$. We show in Fig. 1. the evolution in time of the maximum amplitude of the separate peaks. The appearance of collapse is clear and the collapse occurs not typically in the directly excited peak, but in the scattered ones.

The first scattered peak collapses at $\tilde{T} = 1$. When $\tilde{T}$ increases the oscillations are transferred along the spectrum and the collapse occurs after multiple scattering. Figure 1 corresponds to $\tilde{T} = 1.4$. The spatial distribution of the intensities of the oscillations in the various satellites is shown in Fig. 2. When the intensity of the collapsing satellite starts to exceed the intensity of the neighboring ones significantly, one can neglect the interaction with them and the growth of the field is described by the well studied nonlinear Schrödinger equation. This means that in the collapse a finite energy is absorbed[12] and it serves as an efficient dissipation mechanism.

The number of transfer steps $kr_d\sqrt{m/M}$ can in practice not be larger than ten. This means that already for $\tilde{T} \sim 3$, which corresponds to $T_e = 3T_i$ our calculations show that the modulational instability does not manage to develop and the transfer to the small k region, the collapse region, is described by weak turbulence theory.[6,7] It has the nature of a periodic splitting of oscillation pulses (solitons), as is very clearly seen in numerical experiments.[7] We note if we consider an initial stationary uniform distribution of peaks, our calculations show it to be stable for $\tilde{T} > 3$ in the range of about 10 peaks. In that case the perturbations manage to be carried to the boundary of the interval before the collapse manages to occur.

The following question is interesting: Does the collapse effect occur at all in Eqs. (4.1) in the case of a large $\tilde{T}$ for a large inertial range? Unfortunately we did not manage to answer this analytically. To check this numerically we performed the following experiment. Initially we considered the excitation of a pulse and its propagation over a distance of several transfer steps. After this we made the beginning and the end of the interval the same and the pulse propagated along a ring. The calculations show that collapse occurs even for large values of $\tilde{T}$, after a sufficiently long time. However, it is impossible to say with confidence that this is not connected with the periodicity of the problem due to the build-up of the perturbations when one goes many times around the ring.

At finite excesses above threshold the spectra consist of $n \sim \gamma_0/\nu$ satellites. Numerical calculations show that if the peak with number n_c in which the collapse develops is closer than $(\gamma_0/\nu)k_{\text{diff}}$ to the source of the excitation, the process looks the same as for $\nu = 0$. If, on the other hand, $n_c \gtrsim \gamma_0/\nu$ the collapse does not occur. The dynamics of the peaks is nonstationary, in agreement with Refs. 6 and 7, and changes from being periodic just above threshold to rather entagled.[13]

So far we have considered the two-dimensional Eqs. (3.6), which corresponds to a study of a real three-dimensional turbulence. Recently a detailed numerical simulation of parametrically excited turbulence has been reported in a

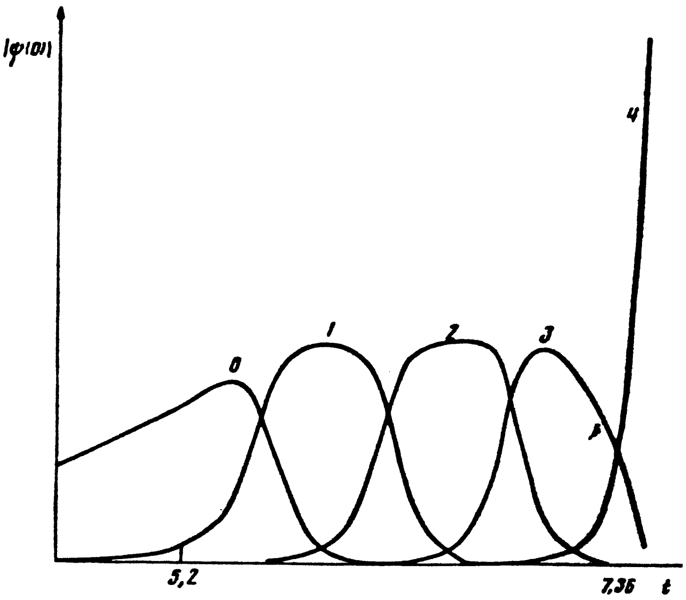

FIG. 1. Temporal evolution of the amplitudes of the different modes in the center of the packets for $\tilde{T} = 1.4$. One can see a stage-by-stage energy transfer along the modes corresponding to the weak-turbulence description. A field singularity develops in the fourth peak after a finite time.

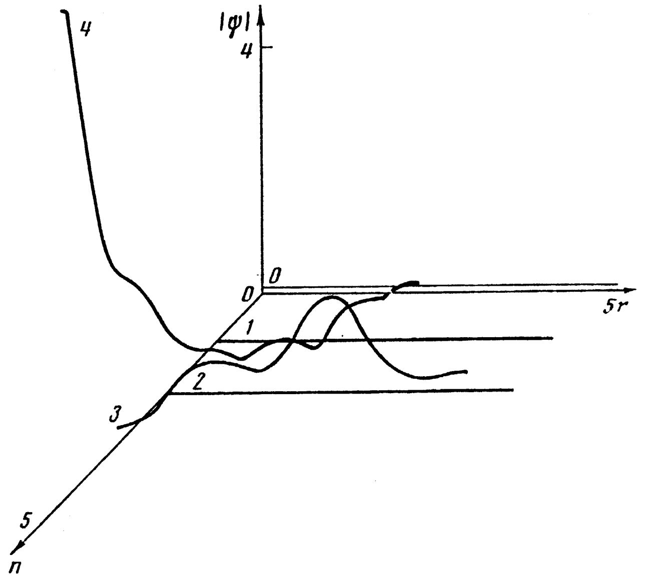

FIG. 2. Spatial distribution of the amplitudes of the different modes at the moment immediately before the collapse for the calculations represented in Fig. 1.

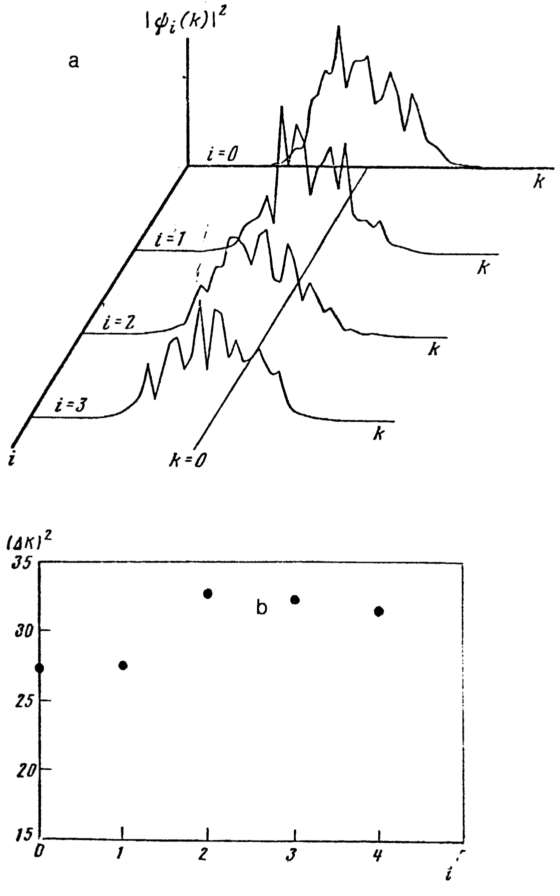

FIG. 3. a) Time-average spectra of the various satellites for two-dimensional turbulence, $\tilde{T}=1$, $\gamma_0/\nu=30$. b) Spectral widths of the various satellites corresponding to Fig. 3a.

number of papers[3–5,14] in the framework of the two-dimensional dynamical equations. They correspond in our model to the one-dimensional Schrödinger equation. The development of the modulation instability does not lead in that case to the development of collapse, but only to the formation of soliton structures. Therefore, weak-turbulence transfer should be observed also for $\tilde{T}<3$. The development of the MI leads to a transverse broadening of the spectrum, $(\Delta k/k)^2 \sim FN$. This broadening is reached after a few steps. We show in Fig. 3 the picture of the spectra, averaged over the time. The width Δk was determined from the relation

$$(\Delta k_i)^2 = \frac{\int k^2 |\psi_i|^2 dk}{\int |\psi_i|^2 dk}.$$

In dimensional variables we get

$$^3/_2\omega_p r_d^2 (\Delta k)^2 \approx FN \sim \gamma_0/\tilde{T}.$$

These results correspond to the numerical calculations in the framework of the dynamical equations,[3–5] where a pulsed energy transfer was observed to small k and a transverse broadening, less for the first peak and appoximately equal for the later ones.

The development of the modulation instability can be interpreted as the scattering of Langmuir plasmons by quasi-static density fluctuations which change after a nonlinear time and which occur under the action of the ponderomotive forces. The scattering in that case takes place along a constant frequency surface. In the approximation used by us this is transverse broadening. More precisely, this is a $k^2 = \text{const}$ surface, as can clearly be seen from the figures given in Ref. 3. Correspondingly, the sound perturbations propagate mainly in a transverse direction.

It was noted in Ref. 5 that the results of the simulation using the dynamical equations were not the same as those of the one-dimensional WT equations and that the number of satellites in the first case was smaller. In our opinion this does not mean that WT theory is inapplicable. The transverse broadening of the spectra indicates that one can no longer replace the matrix elements by their maximum values and use the one-dimensional description. An increase in the width of the spectrum in the two-dimensional simulation as compared to the one-dimensional model was observed in Ref. 7.

It was shown in Ref. 14 that the results of the two-dimensional dynamical calculations were the same as the solution of the WLT equations, which is more exact even than their satellite approximation (2.10) when one is not too far above threshold. The authors of Ref. 14 were not able to make the comparison when one is well above threshold because of the limited computer resources. We show in Fig. 4 the results of the temporal evolution of the oscillations in the framework of (2.10) and (4.1). In the first curve we show the evolution of the intensity of the peaks in (2.10) and in the second one the evolution of the integral intensity $\int|\psi_i|^2dx$. It is clear that when the first few pulses split the results of the calculations are practically the same. However, when time goes on the gaps between the temporal maxima start to become less distinct and a stationary solution is reached. This is not surprising since the system (4.1) is not Hamiltonian. Equations (4.1) are local and the transfer rates in different points along x are different. The nonlinear interaction correlates x but since the growth rates of the transfer and of the MI are comparable the total intensity of the peaks does not

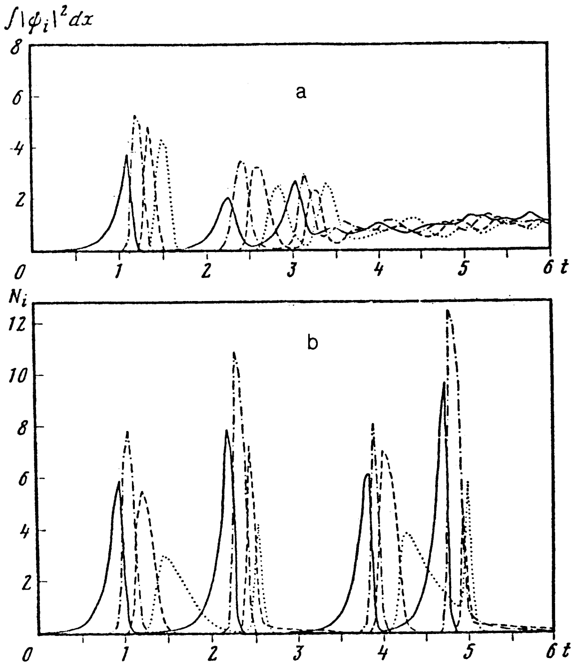

FIG. 4. a) Evolution of the integral intensity $\int |\psi_i|^2 dx$ for two-dimensional turbulence when one is well above threshold. We show in the figure the intensities of the first four peaks; $\widetilde{T} = 3.7$, $\gamma_0/\nu = 13$. b) Evolution of the peak intensities in the satellite approximation [Eqs. (2.10)] for the parameters corresponding to Fig. 4a.

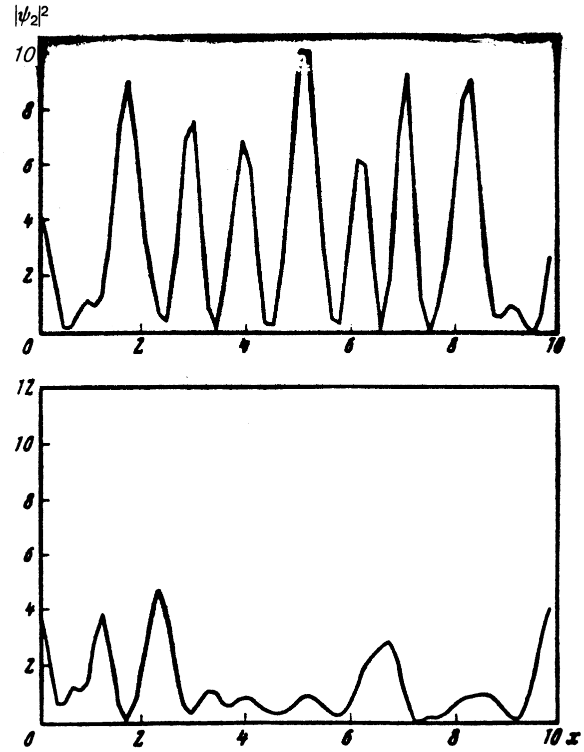

FIG. 5. Spatial distribution of the intensity of one of the peaks (the second) for the parameters corresponding to Fig. 4. The upper figure gives the intensity profile at the time of the field maximum ($t = 1.35$ in Fig. 4). The lower gives the field distribution in the quasistationary state ($t = 5$ in Fig. 4).

drop to zero and stationarity is reached. There occurs then an appraciable deformation of the spatial distribution of the waves as compared to the initial stage of the process. It is well known[6] that the establishment of a stationary state is connected with taking noise into account in (2.7), which is primarily non-thermal but caused by the terms neglected in the derivation of (2.7).

One usually[6] considers effects connected with the higher nonlinear effects, which are neglected in (2.7), such as the four-wave interaction.

Our calculations show that the establishment of a stationary state is caused by the development of the MI. We show in Fig. 5 the spatial distribution of the amplitude of one of the peaks at the time when the deviations from the satellite model become noticeable, and at later times when on the average a stationary state is being established. It is clear that the deviation from the satellite model is caused by deep amplitude modulations which are smoothed out with time.

We note also that in our model we replaced the matrix elements by their maximum values. This may lead to an overestimate of the role of the short-wavelength transverse modulations as compared to the simulation of the dynamic Eqs. (2.1) and (2.2).

5. TURBULENCE OF A MAGNETIZED PLASMA

A weak magnetic field ($\omega_H \ll \omega_p$) affects the above description of turbulent spectra qualitatively little. The situation changes suddenly for $\omega_H > \omega_p$ when the dispersion law of the magnetized Langmuir waves has the form $\omega_k = \omega_p |\cos\theta|$ where θ is the angle between the wave vector and the magnetic field. In that case in an isothermal plasma when there is induced scattering by the ions,

$$\omega_k = \omega_{k'} + |k - k'| v_{Ti} \tag{5.1}$$

energy losses of the plasmons in the scattering process do not lead necessarily to a decrease of the wave vector; it can also increase in the transfer process. A detailed study of the WLT spectra has shown:[15,16] the spectra have a jet character, $n_k = n(\theta)\delta(k - k_0)$ and they are isotropic in the axial angle φ. The transfer leads to condensation of the oscillations in the large k region which is stopped only by the Landau damping, while k_0 is determined by the condition $\gamma_L(k_0) \approx (k_0 r_d)^2 \nu_{ei}$ where is ν_{ei} is the collisional damping rate.

In the present case the jet is two-dimensional and extends along θ and φ. The modulation instability (MI) develops only across the jet, i.e., it is essentially one-dimensional. As we have already said earlier, in the one-dimensional case the MI does not lead to collapse but only to a broadening of the spectrum. However, even a small broadening of the spectrum, $\Delta k/k \sim (k_0 r_d)^2$, leads to a steep increase in the Landau damping so that the MI can lead to a significant growth in the dissipation of the plasmons also in that case.[17] An analytical description of the problem is extremely difficult and to elucidate the general physical picture we used a numerical simulation.

We performed the calculations in the framework of the one-dimensional system (4.1), used already above, with $\nabla^2\psi_i = \psi_{xxi}$ and with in each of the equations an additional term $\hat{\gamma}_k \psi_i$, simulating the Landau damping. In the k-representation the operator $\hat{\gamma}_k$ is equal to

$$\hat{\gamma}_k=\begin{cases}\alpha k^2, & k>0,\\ 0, & k<0.\end{cases} \qquad (5.2)$$

The magnitude of α was chosen such that for a modulation broadening Δk corresponding to being above threshold by an amount or order unity, $\alpha(\Delta k)^2$ was of the order ν. Numerical experiments showed that the nature of the evolution of the system and its integral characteristics were not sensitive to α. Before going over to a description of the numerical calculations we discuss what information we hope to get from them. The assumption of a strong dissipation occurring thanks to the simultaneous action of the MH and of the damping (5.2) is not obvious. It is possible, in principle, that there are situations when together with a broadening there is a significant shift to the region of negative γ and the effective absorption is small. Our calculations showed that although such a shift does occur, a significant part of the energy is contained in the "Landau damping" region and that its role increases when the pumping growth rate increases.

As a good indicator of the efficiency of the damping we can use the width $\Delta\omega$ of the spectrum. In our discrete model its role is played by the effective number, $\bar{n}$, of excited peaks,

$$\bar{n}=\frac{\sum_k k\int_{-\infty}^{\infty}|\psi_k|^2\,dx}{\sum_k \int_{-\infty}^{\infty}|\psi_k|^2\,dx}. \qquad (5.3)$$

In the framework of the uniform model (2.10) $\bar{n}$ increases when the damping ν decreases, on average as $\bar{n}\sim 1/\nu$. We show in Fig. 6 the results of evaluating $\bar{n}$ for the model (2.10). Including the damping (5.2) leads to the fact that magnitude of $\bar{n}$ ceases to increase and reaches saturation. The effective decrease in the width of the spectrum is very clear also in Fig. 4 where the spectral densities of the intensity of the different satellites, averaged over time, are shown. This result is reasonable. As to order of magnitude, the effective damping must be equal to the growth rate of the modulation instability, $\gamma_{\rm mod}\sim F|\psi|^2\sim FN$. Since we have for the number of peaks $\bar{n}\sim\gamma_p/\nu_{\rm eff}\sim\gamma_p/(\nu+\gamma_k)\sim\gamma_p/F|\psi|^2$, it is clear that when the growth rate increases, $\bar{n}$ reaches a constant value. More precisely, we can write for the stationary state

$$T(N_{k-1}-N_{k+1})=FN_k.$$

Assuming the change in intensity from peak to peak to be small,

$$N_{k-1}-N_{k+1}=-2dN_k/dk,$$

we get $N_k=N_0e^{-(F/2T)k}$, $N_0=\gamma_p/T$, i.e., the width of the spectrum is $\bar{k}\sim 2T/F$, or, in dimensional variables, $\Delta\omega=(2T/F)kc_s$, and is independent of the magnitude of the growth rate. This corresponds to the experimental results mentioned in the Introduction.

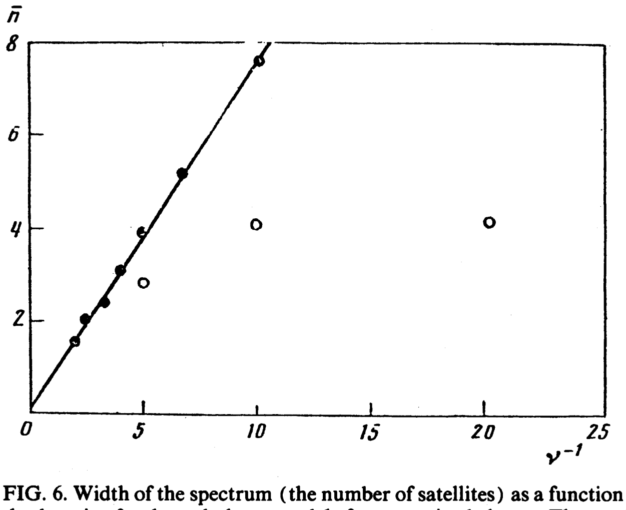

FIG. 6. Width of the spectrum (the number of satellites) as a function of the damping for the turbulence model of a magnetized plasma. The results of the calculations carried out using Eq. (5.3) are indicated by crosses. The asterisks show the results obtained when Landau damping is included.

CONCLUSION

We have thus shown that the weak-turbulence description of Langmuir turbulence is valid only when one is not too far above threshold. In an isotropic nonisothermal plasma there occurs a regime of weak-turbulence energy transfer to the long-wavelength part of the spectrum with subsequent collapse. When the ion and electron temperatures are comparable, self-focusing of the oscillations, a local increase of the field, and collapse occur in the spectral transfer process.

In a magnetized plasma the MI leads to a significant increase in the Landau damping and a modification of the turbulent spectra. We have shown that in an isothermal plasma, both in an isotropic and in a magnetized plasma, the width of the turbulence spectrum is not larger than a few ion-sound frequencies. In an isotropic plasma the development of the MI and collapse lead to energy absorption already after a few transfer steps. In a magnetized plasma the development of the MI increases the Landau damping and the spectral transfer is halted.

Even at low levels of plasma turbulence the WLT theory is inapplicable for its description, but its modification, the introduction of an effective damping, makes it possible adequately to describe the experimental situation.

Experimentally the modification of WLT manifests itself in that the width of the turbulence spectra in an isothermal plasma is not more than a few spectral transfer steps both in the isotropic situation and also when a magnetic field is present, even when one is well above threshold.

We have shown that the Landau damping is an effective energy absorption mechanism for the waves. Therefore, even at low pumping levels, part of the absorbed energy goes not into heating the plasma but in accelerating a small group of electrons—an effect well known to experiments.

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