Canonical quantization of a Dirac spin particle in an external magnetic field

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We carry out a canonical quantization of a Dirac spin particle in an external magnetic field in a gauge that makes it possible to describe both massive and massless particles in four-dimensional ($D = 4$) space. We derive the coordinates and momenta of the Newton–Wigner type for a particle in an external magnetic field and discuss the relation between this quantization scheme and the Blount picture.

1. This paper uses the classical pseudomechanical action (particle spin is described by elements of a Grassmann algebra) to quantize a relativistic spin particle in an external magnetic field. As in Ref. 1, where we considered the canonical quantization of a free relativistic spin particle, we employ a scheme of quantization in which all additional conditions fixing the gauge are added to the theory at the classical level. This leads to a theory with constraints of only the second kind, to which the Dirac quantization scheme is applied.[2] Here, as in Ref. 1, one of the additional constraints fixing the gauge is chosen in the form $x_0 - \tau\varkappa = 0$, which guarantees that particles and antiparticles can be described simultaneously even in classical theory.[3] Because of the complexity of the Dirac brackets for independent dynamical variables, the operator realization of the theory directly in terms of initial variables appears impossible. Therefore, even to a greater extent than in the case of a free particle,[1] here there emerges the need to go over to new variables, in whose terms the commutation relations would acquire canonical form. Such variables can be found.

Next, by analogy with Ref. 1, we introduce the "classical" spin tensor and the Pauli–Lyubanskiĭ vector in the presence of an external magnetic field–two quantities that are gauge-invariant generalizations of the corresponding quantities for a free particle and whose expressions are found in terms of canonical variables. (In contrast to the case where the particle is free, here the spin tensor and the Pauli–Lyubanskiĭ vector are not supergauge-invariant and are not conserved in time.)

Quantization of the theory is conducted in terms of canonical variables. In the very writing of the quantum analogs of the expressions that link the initial and new variables there arises the question of ordering operators of canonical variables. It appears that if the rules of symmetric (Weyl) quantization are followed, within a certain gauge ($\xi_5 = 0$) we arrive at the theory of a spin particle in an external magnetic field in the Blount picture,[4] as Dirac's theory is obtained in the Foldy–Wouthuysen representation in the case of a free particle.[5]

In Sec. 2, following Dirac's prescription, we establish the complete set of constraints of the theory. In Sec. 3 we give the results of calculating the Dirac brackets for physical variables and perform a transition for the initial variables to the canonical. Section 4 deals with quantization of the theory. Finally, Sec. 5 is devoted to the relation between the given quantization scheme and the Blount picture.

2. We consider the action in the theory describing the behavior of a relativistic spin particle in an external electromagnetic field:[6–8]

$$S = \frac{1}{2}\int d\tau\Big[\frac{(\dot{x}^\mu)^2}{e} + em^2 - i(\xi_\mu\dot{\xi}^\mu - \xi_5\dot{\xi}_5) - \chi\Big(\frac{\xi_\mu\dot{x}^\mu}{e} - m\xi_5\Big) + 2g\dot{x}^\mu A_\mu + igeF_{\mu\nu}\xi^\mu\xi^\nu\Big], \quad (1)$$

where $\mu = 0,1,2,3$, the x^μ are the coordinates of the particle, ξ^μ the Grassmann variables describing the spin degrees of freedom, ξ_5, χ, and e are additional fields (e is the even element of the Grassmann algebra, and ξ_5 and χ are the odd elements), g is the particle charge, A^μ the vector potential of the electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the dot stands for differentiation with respect to τ along the particle's trajectory, and the derivatives with respect to Grassmann variables are left-hand.

Action (1) is invariant under reparametrization transformations with parameter u,

$$\delta x^\mu = 2u\mathscr{P}^\mu,\quad \delta\xi^\mu = 2guF^{\mu\nu}\xi_\nu,\quad \delta\xi_5 = 0,$$
$$\delta A_\mu = 2u\mathscr{P}_\nu\partial^\nu A_\mu,\quad \delta e = 2\dot{u},\quad \delta\chi = 0, \quad (2)$$

and supergauge transformations with parameter ε,

$$\delta x^\mu = i\varepsilon\xi^\mu,\quad \delta\xi^\mu = \varepsilon\mathscr{P}^\mu,\quad \delta\xi_5 = \varepsilon m,$$
$$\delta A_\mu = i\varepsilon\xi_\nu\partial^\nu A_\mu,\quad \delta e = i\varepsilon\chi,\quad \delta\chi = 2\dot{\varepsilon}, \quad (3)$$

where we have introduced the notation

$$\mathscr{P}_\mu = \frac{\dot{x}_\mu}{e} - \frac{i\chi}{2e}\xi_\mu. \quad (4)$$

We find the momenta canonically conjugate to the variables x^μ, e, ξ_μ, and χ:

$$P_\mu = \frac{\partial L}{\partial\dot{x}^\mu} = \frac{\dot{x}_\mu}{e} - \frac{i\chi}{2e}\xi_\mu + gA_\mu \equiv \mathscr{P}_\mu + gA_\mu,$$
$$\pi_e = \frac{\partial L}{\partial\dot{e}} = 0,\quad \pi_\mu = \frac{\partial L}{\partial\dot{\xi}^\mu} = \frac{i}{2}\xi_\mu,$$
$$\pi_5 = \frac{\partial L}{\partial\dot{\xi}_5} = -\frac{i}{2}\xi_5,\quad \pi_\chi = \frac{\partial L}{\partial\dot{\chi}} = 0. \quad (5)$$

Equations (5), except the first, serve as primary constraints:

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$$\Phi_\mu=\pi_\mu-\frac{i}{2}\xi_\mu\approx0,\quad \mu=0,1,2,3;\quad \Phi_4=\pi_5+\frac{i}{2}\xi_5\approx0,$$
$$\Phi_9=\pi_e\approx0,\ \Phi_{11}=\pi_\chi\approx0. \tag{6}$$

As in Ref. 1, the numbering of the constraints is chosen from considerations of compactness of the matrix $C_{nn'}=\{\Phi_n,\Phi_{n'}\}$ of the Poisson brackets of all the constraints in the theory. The canonical Hamiltonian H of the theory has the form

$$H=\dot{x}^\mu P_\mu+\dot{\xi}^\mu\pi_\mu+\dot{\xi}_5\pi_5-L=\frac{e}{2}(\mathscr{P}^2-m^2-igF_{\mu\nu}\xi^\mu\xi^\nu)+\frac{i\chi}{2}(\mathscr{P}_\mu\xi^\mu-m\xi_5),$$
$$\mathscr{P}_\mu=P_\mu-gA_\mu. \tag{7}$$

Following Dirac,[2] we set up the total Hamiltonian of the theory,

$$H^*=H+i\lambda^\mu\Phi_\mu+i\lambda_4\Phi_4+\lambda_9\Phi_9+i\lambda_{11}\Phi_{11} \tag{8}$$

(the λ are Lagrange multipliers), and find the secondary constraints

$$\Phi_5=\mathscr{P}_\mu\xi^\mu-m\xi_5\approx0,\ \Phi_7=\mathscr{P}^2-m^2-igF_{\mu\nu}\xi^\mu\xi^\nu\approx0. \tag{9}$$

As in the quantization of a free relativistic spin particle,[1] here there are four constraints of the first kind: in addition to Φ_7, Φ_9, and Φ_{11}, there is one more constraint of the first kind, which is a linear combination of Φ_μ, Φ_4, and Φ_5. Hence, lifting the degeneracy in the theory requires four additional constraints, which we select in the form

$$\Phi_8=x_0-\varkappa\tau\approx0,\ \Phi_6=a\xi_0+b\xi_5\approx0,$$
$$\Phi_{10}=e+1/\omega\approx0,\ \Phi_{12}=\chi\approx0, \tag{10}$$

where

$$\omega=(\mathscr{P}_i^2+m^2+igF_{\mu\nu}\xi^\mu\xi^\nu)^{1/2},\ \mathscr{P}_0=-\varkappa\omega, \tag{11}$$

a and b are parameters that do not vanish simultaneously (for more details concerning the restrictions on parameters a and b see Ref. 1), and $\varkappa=\pm1$. Note that, in contrast to the case of a free particle, $\mathscr{P}_0$ is not a numerical quantity and, actually, $\varkappa$ designates only the sign of the root in Eq. (11). Nevertheless, here too the value $\varkappa=+1$ corresponds to the presence of a particle and $\varkappa=-1$ of an antiparticle.

The collection of constraints (6), (9), and (10) represents a complete set of constraints of the theory, all constraints of the second kind.

3. As already mentioned in item 1, further study will be done for a potential $A_\mu(x)=(0,A_i(x))$, which corresponds to a magnetic field that is constant in time.

We perform the canonical transformation from variables x^μ and P_μ to variables x'^μ and P'_μ via the relations

$$x_0'=x_0-\varkappa\tau,\ x'^i=x^i,\ P_\mu'=P_\mu \tag{12}$$

(the corresponding generating function has the form $W=x^\mu P'_\mu-\tau\varkappa P'_0$). In terms of the new variables the constraint Φ_8 acquires the form $x'_0\approx0$; the other constraints remain unspecified with this choice of the potential. Thus the system of constraints consisting of Eqs. (6), (9), and (10) no longer depends explicitly on time. This enables us to make direct use of standard "Dirac" quantization with the second-order constraints. As a result of this canonical transformation the system Hamiltonian on the constraint surface takes the form

$$H_\Phi=(\mathscr{P}_i^2+m^2+igF_{ik}\xi_i\xi_k)^{1/2}=\tilde{\omega}. \tag{13}$$

Calculating the Dirac brackets over the complete system of constraints (6), (9), and (10) for the independent variables of the theory, for which we have taken x^i, P_i, and ξ^i, we obtain

$$\{x^i,x^j\}_D=\frac{i\gamma}{\alpha^2}\left[\xi^i\xi^j+\frac{b}{\beta P_0}(\mathscr{P}^h\xi^h)(\xi^i\mathscr{P}^j-\xi^j\mathscr{P}^i)\right],$$

$$\{x^i,P_j\}_D=-\delta_j^i-\frac{i\gamma}{\alpha^2}g(\partial_jA_k)\times\left[\xi^i\xi_k+\frac{b}{\beta P_0}(\mathscr{P}^n\xi^n)(\xi^i\mathscr{P}_k-\xi_k\mathscr{P}^i)\right],$$

$$\{x^i,\xi^j\}_D=\frac{\gamma}{\alpha^2}\left[\xi^i\mathscr{P}^j-\frac{b}{\beta P_0}(\mathscr{P}^n\xi^n)\mathscr{P}^i\mathscr{P}^j+ig\frac{b}{\beta P_0}(\mathscr{P}^n\xi^n)F^{jhk}\xi^k\xi^i\right], \tag{14}$$

$$\{P_i,P_j\}_D=ig^2\frac{\gamma}{\alpha^2}(\partial_iA_k)(\partial_jA_n)\times\left[\xi_k\xi_n+\frac{b}{\beta P_0}(\mathscr{P}^n\xi^n)(\xi_k\mathscr{P}_n-\xi_n\mathscr{P}_k)\right],$$

$$\{\xi^i,P_j\}_D=\frac{g\gamma}{\alpha^2}\left[\mathscr{P}^i(\partial_jA_k)\left(\xi_k-\frac{b}{\beta P_0}(\mathscr{P}^n\xi^n)\mathscr{P}_k\right)+\frac{ib}{\beta P_0}(\mathscr{P}^n\xi^n)\xi^l\xi^m\left(gF^{il}\partial_jA^n+\frac{1}{2}\mathscr{P}^i\partial_jF^{ln}\right)\right],$$

$$\{\xi^i,\xi^j\}_D=-i\left(\delta^{ij}-\frac{\mathscr{P}^i\mathscr{P}^j}{\alpha^2}\gamma\right)+\frac{b\gamma g}{\beta P_0\alpha^2}\xi^l(\mathscr{P}^n\xi^n)(F^{il}\mathscr{P}^j+F^{jl}\mathscr{P}^i),$$

where $\alpha=a\mathscr{P}_0+bm$, $\beta=am+b\mathscr{P}_0$, $\gamma=a^2-b^2$, and

$$P_0=\mathscr{P}_0=-\varkappa(\mathscr{P}_i^2+m^2+igF_{ik}\xi_i\xi_k)^{1/2}=-\varkappa\tilde{\omega}.$$

Comparison of these formulas with similar ones obtained for a free particle[1] shows that in this case the expression for the Dirac brackets of independent variables is much more complicated. Hence, there is still more reason to go over from the variables x^i, P_i, and ξ_i to a new set variables q^i, Π_i, and ψ^i for which the Dirac brackets are canonical:

$$\{q^i,q^j\}_D=0,\ \{\psi^i,\psi^j\}_D=-i\delta^{ij},\ \{q^i,\Pi_j\}_D=\delta_j^i,$$
$$\{q^i,\psi^j\}_D=\{\Pi_i,\Pi_j\}_D=\{\Pi_i,\psi^j\}_D=0. \tag{15}$$

Such variables can be found, and their relation to the old variables is specified by the following equations:

$$q^i=x^i-i\xi^i\frac{(a+b\varkappa)(\mathscr{P}^l\xi^l)}{\beta(m+\tilde{\omega})},$$
$$\Pi_i=P_i+ig(\partial_iA_m)\xi_m\frac{(a+b\varkappa)(\mathscr{P}^l\xi^l)}{\beta(m+\tilde{\omega})},$$
$$\psi^i=\xi^i+\mathscr{P}^i\frac{(a+b\varkappa)(\mathscr{P}^l\xi^l)}{\beta(m+\tilde{\omega})}, \tag{16}$$

where $\beta = b\mathcal{P}_0 + am$.

Note that if the formulas for q^i and ψ^i are gauge-invariant generalizations of the respective formulas for a free particle, in the expression for the canonical momentum there appears a new term proportional to g, a term that reflects the fact that the Dirac bracket $\{P_i,P_j\}_D$ is nonzero in an external field.

Equations (16) yield the inverse formulas that link the variables x^i, $\mathcal{P}_i = P_i - gA_i(x)$, and ξ^i with the new variables q^i, $\pi_i = \Pi_i - gA_i(q,\tau)$, and ψ^i:

$$x^i = q^i - i\psi^i \frac{(a\varkappa+b)(\pi^k\psi^k)}{(m+\Omega)(bm-a\varkappa\Omega)},$$
$$\mathcal{P}_i = \pi_i + igF_{il}\psi_l \frac{(a\varkappa+b)(\pi^k\psi^k)}{(m+\Omega)(bm-a\varkappa\Omega)},$$
$$\xi^i = \psi^i + \pi^i \frac{(a\varkappa+b)(\pi^k\psi^k)}{(m+\Omega)(bm-a\varkappa\Omega)}, \qquad (17)$$

where

$$\Omega = [\pi_i^2 + m^2 + igF_{ij}(q,\tau)\psi_i\psi_j]^{1/2}. \qquad (18)$$

In terms of the new variables the Hamiltonian of the theory assumes the form

$$H_\Phi = \Omega. \qquad (19)$$

When the spin variables for the case of a free particle were described in Ref. 1, we introduced the quantity $\tilde\xi^\mu = \xi^\mu - (P^\mu/m)\xi_5$ in terms of which the spin tensor $S^{\mu\nu} = i\tilde\xi^\mu\tilde\xi^\nu$ was expressed. Being a supergauge invariant, this quantity is also conserved in time (on the equations of motion). As a consequence, $S^{\mu\nu}$ is also conserved in time. Note that the total angular momentum $J^{\mu\nu}$ of a free particle possesses the same property. But in an external field, obviously, neither the total angular momentum nor the spin of the particle is conserved in time. Nevertheless, in this case as well it proves expedient to introduce the quantity

$$S^{\mu\nu}_{(A)} = i\tilde\xi^\mu_{(A)}\tilde\xi^\nu_{(A)}, \quad \tilde\xi^\mu_{(A)} = \xi^\mu - \frac{\mathcal{P}^\mu}{m}\xi_5, \qquad (20)$$

which is a gauge-invariant generalization of tensor $S^{\mu\nu}$ (the index "A" indicates that there is an external field). In terms of the variables q, π, and ψ the quantity $\tilde\xi^\mu_{(A)}$ assumes the form

$$\tilde\xi^0_{(A)} = -\frac{\varkappa}{m}(\pi^j\psi^j), \quad \tilde\xi^i_{(A)} = \psi^i + \frac{\pi^i(\pi^j\psi^j)}{m(m+\Omega)}. \qquad (21)$$

Note that $\tilde\xi^\mu_{(A)}$ and, hence, $S^{\mu\nu}_{(A)}$ are independent of the parameters a and b in the fermion gauge of Φ_6 (although now they are not supergauge invariants). If by analogy with the case of a free particle we introduce the vectors

$$S_i^{\tilde\xi(A)} = -\frac{i}{2}\varepsilon_{ijk}\tilde\xi_j^{(A)}\tilde\xi_k^{(A)}, \quad S_i^\psi = -\frac{i}{2}\varepsilon_{ijk}\psi_j\psi_k \qquad (22)$$

(S_i^ψ is the spin vector in terms of the variables ψ that describes the particle spin in the rest frame), we can find the formula that links them,

$$S_i^{\tilde\xi(A)} = \frac{\Omega}{m}S_i^\psi - \frac{\pi_i(\pi_j S_j^\psi)}{m(m+\Omega)}, \qquad (23)$$

which is a generalization of the corresponding formula for a free particle[1] to the case involving an external field. Note that in deriving Eq. (23) we employed the fact that the vector ψ_i is three-dimensional (terms containing the product of four and more ψ were set to zero). In terms of the variables S_i^ψ Eqs. (17) become

$$x^i = q^i - \frac{(a\varkappa+b)\varepsilon_{ijk}S_j^\psi\pi_k}{(m+\Omega)(bm-a\varkappa\Omega)},$$
$$\mathcal{P}_i = \pi_i - g\frac{(a\varkappa+b)[S_i^\psi(B_k\pi_k) - \pi_i(S_k^\psi B_k)]}{(m+\Omega)(bm-a\varkappa\Omega)}, \qquad (24)$$
$$\Omega = (\pi_i^2 + m^2 - 2gS_k^\psi B_k)^{1/2},$$

where B_i is the magnetic induction vector: $B_i(q) = \frac{1}{2}\varepsilon_{ijk}F_{jk}(q)$.

Finally, let us introduce the analog of the "classical" Pauli-Lyubanskiĭ vector in the presence of an external field:

$$W_\mu^{(A)} = -\frac{i}{2}\varepsilon_{\mu\nu\lambda\sigma}\mathcal{P}^\nu\tilde\xi^\lambda_{(A)}\tilde\xi^\sigma_{(A)} = -\frac{i}{2}\varepsilon_{\mu\nu\lambda\sigma}\mathcal{P}^\nu\xi^\lambda_{(A)}\xi^\sigma_{(A)}. \qquad (25)$$

The independence of $\tilde\xi^{(A)}$ from the fermion-gauge parameters implies the independence of vector $W_\mu^{(A)}$ from the same parameters. Using Eqs. (21), we can easily find the expression for $W_\mu^{(A)}$ in terms of the variables q, π, and ψ:

$$W_0^{(A)} = \pi_i S_i^\psi, \quad W_i^{(A)} = -\varkappa\left[mS_i^\psi + \frac{\pi_i(\pi_k S_k^\psi)}{(m+\Omega)}\right], \qquad (26)$$

which is also a gauge-invariant generalization of the respective expressions for a free particle.[1]

4. Obviously, quantization of the theory is most conveniently done in terms of the canonical variables q, Π, and ψ, whose Dirac brackets are specified in (15). Introducing the corresponding operators $\hat q$, $\hat\Pi$, and $\hat\psi$, we can write the commutation relations for them by following the rule that $[\dots,\dots] = i\hbar\{\dots,\dots\}_D$:

$$[\hat q^i,\hat q^j]_- = [\hat\Pi_i,\hat\Pi_j]_- = [\hat q^i,\hat\psi^j]_- = [\hat\Pi_i,\hat\psi^j]_- = 0,$$
$$[\hat q^i,\hat\Pi_j]_- = i\hbar\delta_j^i, \quad [\hat\psi^i,\hat\psi^j]_+ = \hbar\delta^{ij}, \qquad (27)$$

where the last formula specifies a Clifford algebra in three-dimensional space. The unique finite-dimensional irreducible representation of the operators $\hat\psi^i$, as is known, is given by the Pauli matrices σ^i:

$$\hat\psi^i = \pm\left(\frac{\hbar}{2}\right)^{1/2}\sigma^i = \varkappa\left(\frac{\hbar}{2}\right)^{1/2}\sigma^i, \quad i = 1,2,3. \qquad (28)$$

The transition to the operators $\hat x$, $\hat P$, and $\hat\xi$, which correspond to the initial variables x, P, and ξ of the theory, with employment of Eqs. (17) is complicated, in contrast to the case of a free particle, by the problem of ordering the operators $\hat q$, $\hat\Pi$, and $\hat\psi$ in the respective quantum expressions. Here we follow the rules of symmetric, or Weyl, quantization. Symmetric quantization for the elements ψ of a Grassmann algebra is defined in the following manner:[6,9] a classical function $f(\psi)$ is expanded in a power series in ψ thus:

$$f(\psi)=\sum_{\nu=0}^{n}\sum_{(i)} f_{i_1\ldots i_\nu}\psi^{i_1}\ldots\psi^{i_\nu} \tag{29}$$

(in view of the Grassmann nature of ψ, the series contains a finite number of terms; in the case at hand $n=3$). The transition to the quantum analog in (29) is performed by supplying the variables ψ with operator carets only after the expansion coefficients in (29) have been antisymmetrized over all the indices.

Allowing for the fact that $\hat{S}_i^\psi=(\hbar/2)\sigma_i$, we arrive at the following expressions for the quantum operators $\hat{x}$, $\hat{\mathscr{P}}$, $\hat{S}^{\xi(A)}$ and $\hat{H}_\Phi$

$$\begin{aligned}
\hat{x}_i &\rightleftharpoons q_i-\frac{\hbar}{2}\frac{(a\varkappa+b)\varepsilon_{ijk}\sigma_j\pi_k}{(bm-a\varkappa\tilde{\Omega})(m+\tilde{\Omega})},\\
\hat{\mathscr{P}}_i &\rightleftharpoons \pi_i-g\frac{\hbar}{2}\frac{(a\varkappa+b)[\sigma_i(\pi_kB_k)-\pi_i(\sigma_kB_k)]}{(bm-a\varkappa\tilde{\Omega})(m+\tilde{\Omega})},\\
\hat{S}^{\xi(A)} &\rightleftharpoons \frac{\hbar}{2}\frac{\tilde{\Omega}}{m}\sigma_i-\frac{\hbar}{2}\frac{\pi_i(\pi_k\sigma_k)}{m(m+\tilde{\Omega})},\\
\hat{H}_\Phi &\rightleftharpoons \tilde{\Omega}-g\frac{\hbar}{2}\frac{(B_k\sigma_k)}{\tilde{\Omega}},
\end{aligned} \tag{30}$$

where the symbol $\rightleftharpoons$ designates the correspondence between an operator and its Weyl transformation,[10] and $\tilde{\Omega}=(\pi_i^2+m^2)^{1/2}$. The presence of $\tilde{\Omega}$ in (30) instead of Ω (cf. Eqs. (23) and (24)) is due to the fact that terms with the fourth and higher powers of ψ have been discarded.

5. To compare our results with those from the literature obtained by other quantization schemes we write the expressions for $\hat{x}_i$ and $\hat{\mathscr{P}}_i$ in the $\xi_5\approx 0$ gauge ($a=0$):

$$\begin{aligned}
\hat{x}_i &\rightleftharpoons q_i-\frac{\hbar}{2}\frac{\varepsilon_{ijk}\sigma_j\pi_k}{m(m+\tilde{\Omega})},\\
\hat{\mathscr{P}}_i &\rightleftharpoons \pi_i-g\frac{\hbar}{2}\frac{[\sigma_i(\pi_kB_k)-\pi_i(\sigma_kB_k)]}{m(m+\tilde{\Omega})}.
\end{aligned} \tag{31}$$

We specify, in addition, the velocity operator $\hat{v}_i=d\hat{x}_i/dx_0$. Using the relations

$$v^i=\frac{dx^i}{dx_0}=\varkappa\frac{dx^i}{d\tau}=\varkappa\{x^i,H_\Phi\}_D=-\varkappa\frac{\mathscr{P}_i}{\Omega}, \tag{32}$$

and the expression for $\mathscr{P}_i$ in the $\xi_5\approx 0$ gauge, we arrive at the following formula for the operator $\hat{v}_i$:

$$\begin{aligned}
\hat{v}^i=\frac{d\hat{x}^i}{dx_0} &\rightleftharpoons -\varkappa\frac{\pi_i}{\tilde{\Omega}}-g\varkappa\frac{\pi_i(B_kS_k^\psi)(m^2+m\tilde{\Omega}+\tilde{\Omega}^2)}{m(m+\tilde{\Omega})\tilde{\Omega}^3}\\
&+g\varkappa\frac{S_i^\psi(\pi_kB_k)}{m\tilde{\Omega}(m+\tilde{\Omega})}.
\end{aligned} \tag{33}$$

Equations (31) and (32) and the expressions for $\hat{S}^{\xi(A)}$ in (30) taken in the first approximation in the coupling constant g coincide fully with analogous formulas in Ref. 10 (in the absence of an electric field and of an anomalous magnetic moment on the particle) if we replace π_i with $-\pi_i$ in view of the difference in the signs in the definition of the canonically conjugate momentum P_i. These equations express the operators of position, momentum, velocity, and spin in the Blount picture.[4,10] It is important to note, however, that our formulas are valid in all orders in g and have been obtained without any restrictions imposed on the potentials.

It is also easy to find the equation for the spin in this picture:

$$\frac{d\hat{S}_i^{\xi(A)}}{dx_0}\rightleftharpoons\varkappa\frac{g}{\tilde{\Omega}}\varepsilon_{ijk}S_j^{\xi(A)}B_k \tag{34}$$

(cf. the respective equation in Ref. 10). Finally, we write the quantum analogs of relations (26) for the Pauli–Lyubanskiĭ vector:

$$\begin{aligned}
\hat{W}_0^{(A)} &\rightleftharpoons \frac{\hbar}{2}(\pi_k\sigma_k),\\
\hat{W}_i^{(A)} &\rightleftharpoons -\frac{\hbar}{2}\varkappa\left[m\sigma_i+\frac{\pi_i(\pi_k\sigma_k)}{m+\tilde{\Omega}}\right].
\end{aligned} \tag{35}$$

Note the following. In the case of a free particle[1] it was found that to the two classical objects $m\xi_\mu$ and W_μ there corresponds, to within a constant factor, a single quantum operator. To clarify the situation in the present case, we also write the quantum analogs of Eqs. (21):

$$\begin{aligned}
\hat{\xi}^0_{(A)} &\rightleftharpoons -\frac{1}{m}\left(\frac{\hbar}{2}\right)^{1/2}(\pi^j\psi^j),\\
\hat{\xi}^i_{(A)} &\rightleftharpoons \varkappa\left(\frac{\hbar}{2}\right)^{1/2}\left[\sigma^i+\frac{\pi_i(\pi_k\sigma_k)}{m(m+\tilde{\Omega})}+g\frac{\hbar}{2}\frac{\pi_i(\pi_kB_k)}{m(m+\tilde{\Omega})^2\tilde{\Omega}}\right].
\end{aligned} \tag{36}$$

The presence of the last term in $\hat{\xi}^i_{(A)}$ is due to the term proportional to ψ^3 in the expansion of the denominator in (21) in powers of ψ. Comparing (36) with (35) shows that in an external field the operators $m\hat{\xi}^\mu_{(A)}$ and $W^\mu_{(A)}$ are distinct (the additional term in (36) is proportional to the coupling constant and has the form of a quantum correction).

In conclusion we add that, as Eqs. (16) imply, in an external magnetic field there exists a gauge $a+b\varkappa$ in which the canonical variables coincide with the initial variables of the theory and the Dirac quantization of the theory in terms of these variables coincides with ordinary canonical quantization.

Thus, the above discussion within the framework of the pseudomechanics of the canonical quantization of a relativistic spin particle in an external time-constant magnetic field (in the $\xi_5=0$ gauge) brings us to the so-called Blount picture,[4,10] just as quantization of a free particle leads to the Dirac picture in the Foldy-Wouthuysen representation.

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