A new perturbation theory based on the physical branch of the solution of the renormalization-group equation

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We use the recently found physical branch of the solution of the exact (local) equation of the renormalization group as the zeroth approximation in a perturbation theory that allows for generating nonlocal terms in the free-energy functional. We show that even with these terms the physical branch remains unique, and we discuss the variation in the effective dimensionality of the space caused by nonlocal interactions.

INTRODUCTION

An exact formulation of the transformation of the renormalization group (RG) via an equation in variational derivatives was first suggested by Wilson and Kogut.[1] The exact solution of this equation should, at least in principle, answer all the questions of the theory of critical phenomena (finding fixed points, calculating the critical indices, etc.). In practice, however, because of the exceptional complexity of the suggested equation, one is forced to resort to different variants of perturbation theory that are either based on parameters that are not small in reality[2,3] or presuppose the existence of uncontrollable nonphysical criteria for truncating the systems of coupled equations for the vertices of the free-energy functional.[4,5] Notwithstanding the great progress achieved along these lines of research, the problem of finding the solution to the exact RG equation (or at least establishing a method for finding it) in variational derivatives is still on the agenda. Lately several slightly differing approaches to deriving the exact RG equation have been suggested.[1,6–8] For one thing, the modification of the RG equation suggested in Ref. 8 proved the most suitable for practical applications and at the same time the most rigorous (see also Ref. 9). Following the lines of reasoning of Ref. 8, we write the exact RG equation for an arbitrary Ginzburg–Landau (GL) functional $H_{\text{total}} = H_0 + H$:

$$\dot{H}[\varphi] = \frac{1}{2}\int_{\mathbf{q}} \eta(\mathbf{q})\,[V - G_0^{-1}(\mathbf{q})\,|\varphi(\mathbf{q})|^2] + dV\frac{\partial H}{\partial V} - \int d^d r\left[\frac{(d-2)}{2}\varphi(\mathbf{r}) + \mathbf{r}\nabla_r\varphi(\mathbf{r})\right]\frac{\delta H}{\delta\varphi(\mathbf{r})} + \int_{\mathbf{r}\mathbf{r}'}\left\{h(\mathbf{r}-\mathbf{r}')\left[\frac{\delta^2 H}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')} - \frac{\delta H}{\delta\varphi(\mathbf{r})}\frac{\delta H}{\delta\varphi(\mathbf{r}')}\right] - \frac{1}{2}\eta(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r})\frac{\delta H}{\delta\varphi(\mathbf{r}')}\right\}. \tag{0.1}$$

Here for H_0 we take the Gaussian functional

$$H_0 = \int_{\mathbf{q}} G_0^{-1}(\mathbf{q})\,|\varphi(\mathbf{q})|^2, \qquad \int_{\mathbf{q}} \equiv \int d^d q/(2\pi)^d. \tag{0.2}$$

V is the system volume, G_0 the zeroth-order propagator, and $\mathbf{q}$ the wave vector. In the most general form the functional H can be represented as follows:

$$H = \sum_{k=1}^{\infty} 2^{1-2k}\int_{\{\mathbf{q}_i\mathbf{q}_{i'}\}} (2\pi)^d\delta\Big(\sum_{i=1}^{k}(\mathbf{q}_i + \mathbf{q}_{i'})\Big) g_k\{\mathbf{q}_i,\mathbf{q}_{i'}\}\prod_{i=1}^{k}\varphi_{\mathbf{q}_i}\varphi_{\mathbf{q}_{i'}}. \tag{0.3}$$

In Eqs. (0.1)–(0.3), d is the dimensionality of space, φ is a vector with n components, $h(\mathbf{q}) = \exp(-q^2/2\Lambda^2)$ is the smooth cutoff factor, where Λ is the cutoff momentum, and $\eta(\mathbf{q})$ is the anomalous-dimensionality function. The chief difficulty in applying (0.1) lies in the requirement that a rigorous calculation procedure must employ an essentially nonlocal GL functional.[4,10–13] Even in the case of a purely local initial functional, nonlocal behavior occurs as a result of (0.1). For one thing, this process is closely linked to the problem of excluding redundant solutions of the RG equation.[6] As shown in Ref. 14, the problem can be resolved by requiring that no new nonlocal behavior develop in terms of order $|\varphi(\mathbf{q})|^2$, that is, by fixing $\eta(\mathbf{q}) = (d+2) - 2\Delta\varphi(\varphi)$ in the form

$$\eta(\mathbf{q}) = \eta(0) + [D(\mathbf{q}) - D(0) - \eta(0)\,G_0^{-1}(\mathbf{q})]/[G_0^{-1}(\mathbf{q}) + g_1],$$

$$\eta(0) = dD/d(q^2)\,|_{q=0},$$

$$D(\mathbf{q}) = -g_1^2 h(\mathbf{q}) + \frac{1}{2}\int_{\mathbf{p}} h(\mathbf{p})\,[n g_2(\mathbf{p},-\mathbf{p};\mathbf{q},-\mathbf{q}) + \mathbf{g}_2(\mathbf{p},-\mathbf{q};\mathbf{q},-\mathbf{p})], \tag{0.4}$$

where $\Delta_\varphi(\varphi)$ is the scaling dimensionality of the field φ. The function $\eta(\mathbf{q})$ in turn determines the structure of the physically measurable correlation function $G_q = \langle\varphi_q,\varphi_{-q}\rangle$ at the critical point, and its limit $\eta(q\to 0) = \eta(0)$ coincides with the Fisher index η in the asymptotic forms $G_q \propto q^{-2+\eta(0)}$ at $\tau = 0$ and as $q\to 0$ (see Ref. 13).

If we formally let the cutoff momentum Λ tend to infinity, $h(q)\to\text{const} = h(0) = 1$ and so the function

$$h(\mathbf{r}-\mathbf{r}') = [\Lambda/(2\pi)^{1/2}]^d\exp[-(\mathbf{r}-\mathbf{r}')^2\Lambda^2/2]$$

tends to a delta function, $h(\mathbf{r}-\mathbf{r}')\to\delta(\mathbf{r}-\mathbf{r}')$. Here the nonlocal terms in the free-energy functional are not generated if they were absent in the initial form of the functional. This makes it possible to formally consider the local version of the exact RG equation.[7,15–17] Actually, the initial RG equation is written with the normalization condition $\Lambda = 1$,

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so that the passage to the limit $\Lambda \to \infty$ is unjustified. The use of this formal approach, however, has proved useful in the initial stage.

For H we take the purely local form

$$H=\Phi_0=\int_r f[\varphi(\mathbf{r})]$$

and put $\eta = 0$. In this case the functional equation (0.1) transforms into the "ordinary" equation[7,9,15-17]

$$\dot{f}=df-\frac{d-2}{2}\varphi\nabla_\varphi f+\nabla_\varphi^2 f-(\nabla_\varphi f)^2=\hat{R}f, \tag{0.5}$$

where

$$\nabla_\varphi=\sum_\alpha^n \mathbf{e}^\alpha \frac{\partial}{\partial\varphi^\alpha},$$

here $\mathbf{e}^\alpha$ is the unit vector along the direction specified by φ^α. The equation is still very complicated for analytic study since it is essentially nonlinear and contains the variable φ explicitly. Nevertheless, it can be analyzed by using a combination of numerical and analytical methods.[17] Here we list the properties of the solutions of (0.5) on which we base virtually all our further reasoning.

The solution of the local RG equation has a unique physical branch. Qualitatively this branch resembles the φ^4 model, but asymptotically it behaves like $\sim\varphi^2$ as $|\varphi|\to\infty$. The series in powers of $(\varphi^2)^k$ for $f(\varphi)$ is conditionally convergent, and a meaningful representation of the solution requires retaining all the powers $(\varphi^2)^k$ in the expansion of f in powers of φ. As the dimensionality of space tends to four $(\varepsilon\to 0)$, the solution approaches the one obtained analytically by means of the ε-expansion.[10,11,16,17]

It is known that the spectrum of Eq. (0.5) linearized near a fixed point, $\{\lambda_k\}$, is real, bounded below, and discrete.[9,18] Numerical calculations for this spectrum with $n = 1,2,3,4,\ldots$ carried out on the basis of the physical solution yield good values of critical indices.

As $d\to 2$, the amplitude of the physical branch of the solution increases, and at $d = 2$ the local equation has no solution corresponding to a real physical situation, with the exception of the so-called high-temperature fixed point.[17,19] The latter stems from the fact that, as shown below, with $d\to 2$ the nonlocalness rejected in deriving Eq. (0.5) play an important role in forming real critical behavior.

In the exactly solvable case, important for theory, in which φ has an infinite number of components, that is, $n\to\infty$, the correponding local equation (0.5) can be solved analytically and yields critical indices coinciding with the well-known results of the spherical model[20,21] (see also Sec. 3).

We believe that the discovery of the uniqueness of a physically meaningful solution is important to the theory of critical phenomena since, on the one hand, it resolves the problem of selecting such a solution that arises in all variants of perturbation theory and, on the other, it allows a new version of such a theory based on a natural small parameter η and the (verifiable) smallness of the nonlocal corrections. Building such a perturbation theory in the form of a gradient expansion[19] and discussing some of its corollaries are the goals of the present paper.

Avoiding the process of truncation of the series in φ saves one the trouble of using perturbation series at the local-equation level. This makes it possible to avoid using expansions in $\varepsilon\geqslant 1$ but requires using the entire body of the numerical data on $f(\varphi)$ in further calculations.

1. THE NONLOCAL RG EQUATION: THE $n=1$ CASE

Even if the initial Hamiltonian H_{total} is purely local, for finite Λ nonlocalness is generated, according to Eq. (0.1), owing to the term

$$-\int_{rr'}\bar{h}(\mathbf{r}-\mathbf{r}')\frac{\delta H}{\delta\varphi(\mathbf{r})}\frac{\delta H}{\delta\varphi(\mathbf{r}')}, \tag{1.1}$$

with $\bar{h}(\mathbf{r}-\mathbf{r}')=h(\mathbf{r}-\mathbf{r}')-\delta(\mathbf{r}-\mathbf{r}')$. The nonlocalness can be allowed for by representing H in the form of the sum $H=\Phi_0+\Phi_1$ of the local part

$$\Phi_0=\int_r f(\varphi(\mathbf{r}))$$

and a nonlocal additional term Φ_1. If, as expected, $\Phi_1\ll\Phi_0$ (in view of the smallness of η), linearizing in Φ_1 yields the following equation:

$$\begin{aligned}\dot{\Phi}_1=&-\frac{1}{2}\int\eta(\nabla\varphi(\mathbf{r}))^2-\int d^dr\left[\frac{(d-2)}{2}\varphi(\mathbf{r})+\mathbf{r}\nabla_r\varphi(\mathbf{r})\right]\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r})}\\&+\int_{rr'}\left\{h(\mathbf{r}-\mathbf{r}')\left[\frac{\delta^2\Phi_1}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')}-2\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r})}\frac{\delta\Phi_0}{\delta\varphi(\mathbf{r}')}\right]\right.\\&\left.-\bar{h}(\mathbf{r}-\mathbf{r}')\frac{\delta\Phi_0}{\delta\varphi(\mathbf{r})}\frac{\delta\Phi_0}{\delta\varphi(\mathbf{r}')}-\frac{1}{2}\eta(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r})\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r}')}\right\}.\end{aligned} \tag{1.2}$$

For further progress we must specify the possible form of the nonlocal term Φ_1. This term is sought in the form of a gradient expansion. To this end we expand the factor h in powers of gradients and keep in the first order the lowest nonvanishing contributions of $\nabla\varphi$. Wishing to clarify the general structure of the theory, we consider the simple case of a scalar field $(n = 1)$. Calculating the variational derivatives here is elementary. One can easily verify that the contribution of the generation term to the nonlocal part of the functional has the form $\int_r f_{\varphi\varphi}^2(\nabla\varphi)^2/2$. Bearing this in mind, we seek the solution for the nonlocal part Φ_1 in the form

$$\Phi_1=\int_r\chi(\varphi(\mathbf{r}))(\nabla\varphi)^2. \tag{1.3}$$

According to (1.2), the action of the operator

$$\int h(\mathbf{r}-\mathbf{r}')\frac{\delta^2}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')}$$

on the nonlocal term Φ_1 causes Φ_1 to contribute to the local part of the functional, which has the form $\int d^dr B\chi[\varphi(\mathbf{r})]$. Here the coefficient

$$B=\int_p p^2h(\mathbf{p})=\frac{d}{(2\pi)^{d/2}}\ll 1$$

is numerically small and, hence, assuming that numerically the function $\chi[\varphi(\mathbf{r})]$ is also small, we can ignore the contribution of $\int d^dr B\chi[\varphi(\mathbf{r})]$ to the equation for the local part. On the other hand, the contribution of $\eta(\mathbf{r}-\mathbf{r}')\neq 0$ to $d-2+\eta$ cannot be ignored even though $\eta(\mathbf{r}-\mathbf{r}')\ll 1$, since for dimensionalities of space close to two the given additional term provides the main contribution to the physically significant term $(d-2+\eta)f_\varphi\varphi/2$ (see Ref. 17). It can easily be verified that in this approximation it is suffi-

cient to keep the anomalous-dimensionality function $\eta(\mathbf{r}-\mathbf{r}')$ in the form $\eta(\mathbf{r}-\mathbf{r}')\approx\text{const}=\eta$. Retaining the lowest nonvanishing gradients $\nabla\varphi(\mathbf{r})$, we arrive at the next pair of equations for functions f and χ:

$$\dot{f}=df-\frac{d-2+\eta}{2}\varphi f_{\varphi}+f_{\varphi\varphi}-f_{\varphi}^{2}+B\chi, \tag{1.4a}$$

$$\dot{\chi}=(\eta+4f_{\varphi\varphi})\chi-\left(\frac{d-2+\eta}{2}\varphi+2f_{\varphi}\right)\chi_{\varphi}+\chi_{\varphi\varphi}+\tfrac{1}{2}(f_{\varphi\varphi}^{2}-\eta). \tag{1.4b}$$

At a fixed point we have $\dot{f}=\dot{\chi}=0$ and the system of equations (1.4a) and (1.4b) can be integrated numerically.

The main difficulty with this procedure, however, is that the physical branch of the solution for $f[\varphi(\mathbf{r})]$ corresponding to genuine critical behavior is known only numerically[17] and varies together with η, which enters into (1.4) as a parameter. To get around this difficulty we change variables in equations (1.4), introducing $\varphi=\varphi'(2\Delta_{\varphi})^{1/2}$, where $\Delta_{\varphi}=(d-2+\eta)/2$ is the scaling dimensionality of field φ. We have

$$\frac{d_{eff}f}{d_{eff}-2}-\frac{1}{2}\varphi' f_{\varphi'}+f_{\varphi'\varphi'}-f_{\varphi'}^{2}=0, \tag{1.5a}$$

$$-\left(\frac{\eta}{2\Delta_{\varphi}}+4f_{\varphi'\varphi'}\right)\chi-\left(\frac{1}{2}\varphi'+2f_{\varphi'}\right)\chi_{\varphi}$$

$$+\chi_{\varphi'\varphi'}+\tfrac{1}{2}(f_{\varphi'\varphi'}^{2}-\eta)=0. \tag{1.5b}$$

Equation (1.5a) is universal in the sense that the physical branch of the solution $f[\varphi'(\mathbf{r})]$ corresponding to critical behavior is a unique function of $d_{\text{eff}}=2d/(2-\eta)>d$, which in this context can be identified with the effective dimensionality of space. Actually, however, conditions (0.4) do not fully determine the numerical value of η since in the given order it reduces to the boundary conditions $\chi(0)=\chi_{\varphi}(0)=0$. With these restrictions and for a fixed branch of the solution, $\dot{f}(\varphi)=0$, the equation for a fixed point, $\dot{\chi}=0$, has a one-parameter family of solutions $\chi(\varphi;\eta)$. This creates the problem of selecting the physical branch of the solution for $\chi(\varphi;\eta)$, just as it does in the purely local case with $f[\varphi(\mathbf{r})]$ (see Ref. 17). This fixes the value $\eta\neq 0$. Consider Eq. (1.5b). Bearing in mind that for $\varphi\gg 1$ the function $f(\varphi)$ assumes its asymptotic form $f(\varphi)\approx\varphi^{2}/2+\ldots$ (see Ref. 17), we get

$$\chi_{\varphi\varphi}-(4+\eta)\chi-{}^{1}/_{2}(4+2\Delta_{\varphi})\varphi\chi_{\varphi}+{}^{1}/_{2}(1-\eta)=0 \tag{1.6}$$

or for $\varphi\gg 1$ (where χ, χ_{φ}, and $\chi_{\varphi\varphi}$ tend to $\pm\infty$)

$$z\chi_{zz}+\left(\frac{1}{2}-z\right)\chi_{z}-\frac{(4+\eta)}{(4+2\Delta_{\varphi})}\chi=0, \tag{1.7}$$

with $z=(4+2\Delta_{\varphi})\varphi^{2}/4$ a new variable. Equation (1.7) is the well-known Kummer equation[22] and has the asymptotic form

$$\chi(\varphi)=c(\eta)\varphi^{2\gamma}\exp[(4+2\Delta_{\varphi})\varphi^{2}/4], \tag{E}$$

where $\gamma=(4-d-\eta)/[2(2+d+\eta)]>0$. For the given boundary conditions $\chi(0)=\chi_{\varphi}(0)=0$ the constant $c(\eta)$ is determined by the parameter η. Note, however, that if $|\varphi|$ is sufficiently high, χ is much larger than f. Hence, these solutions for χ contradict the way in which Eq. (1.5b) is chosen (i.e., the requirement that the nonlocalness be small, $\chi\ll f$) and represent the "redundant" solution branches. The only possibility of satisfying the inequality $\chi\ll f$ is to assume that $\chi\to\text{const}=(1-\eta)/2(4+\eta)$ as $|\varphi|\to\infty$. This restriction uniquely determines the Fisher index $\eta=\eta(d_{\text{eff}})=\eta[d_{\text{eff}}(d)]$.

The results of calculating the functions f and χ numerically for the given case of $n=1$ qualitatively resemble those depicted for the more general multicomponent case in Figs. 1 and 2, respectively. The physical branches of the solution at the initial stages of variation of φ' for these functions are depicted by solid curves. It is obvious that we have $\chi(\varphi)\propto 10^{-2}f(\varphi)\ll f(\varphi)$, so that the calculation procedure developed here is self-consistent. Table I lists the values of parameters η, $f(\varphi=0)$, and d found numerically for given values of d_{eff}.

When the dimensionality d of space is greater than 3.75 the amplitude of the solution's physical branch is small and the effectiveness of the numerical methods employed lowers considerably. At the same time, in this limit, that is, as $\varepsilon\to 0$, the ε-expansion is applicable. In the opposite limiting case, $d\to 2$, higher-order gradients become important and the accuracy of the present calculation diminishes. As a result the numerical value of $\eta(d=2;\ n=1)$ of approximately 0.28 proves somewhat overestimated in comparison with the well-known exact result for the Ising model, $\eta(d=2;\ n=1)=1/4$ (see Ref. 23).

In reality, the small parameter of the theory is $f(0)$, which, as shown in Appendix 2, falls off exponentially as d increases: $f(0)\propto\exp[-a(d-2+\eta)]$, where a is positive and of the order of unity. On the order hand, as mentioned above, for $d\to 4$ the quantity $f(0)\sim\varepsilon$ is small and one must use the ε-expansion. As a result, this approach is most effective for $d\approx 3$, that is, where this is most desirable.

To conclude this section we note that allowing for nonlocal behavior leads to the appearance of corrections to the "local" approximation equation that are taken into account self-consistently in the solution of this equation. In view of this the quantity η in $(d-2+\eta)/2$, far from being a correction to the difference $d-2$, which becomes small as $d\to 2$, increases the effective dimensionality d_{eff}. As a result, the expansion is satisfactory even for $d\to 2$. The increase in the effective dimensionality in comparison to $d=2$ in turn

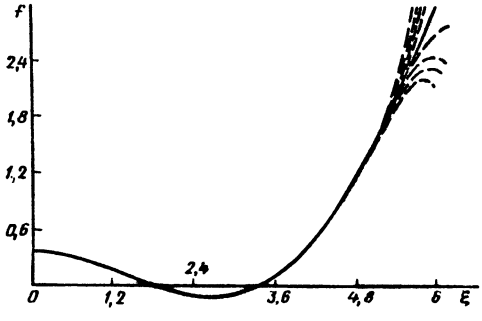

FIG. 1. The typical structure of function f (the case $d_{\text{eff}}=3$ and $\eta=3$). The dashed curves depict the secondary branches of the solution close to f.

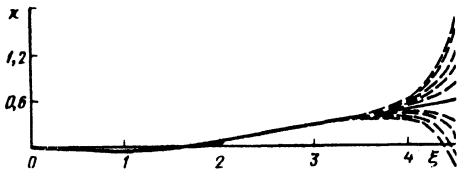

FIG. 2. The same as in Fig. 1 for function χ.

leads to a situation in which the critical behavior of a planar system can qualitatively be the same as for $d > 2$. This problem is discussed in greater detail in Appendix 1.

2. THE NONLOCAL RG EQUATION: THE $n > 1$ CASE

Let us now consider the general case of a multicomponent order parameter. Following the previous example, we start by establishing the structure of the generation term (1.1) in the general case. Calculating the variational derivatives and performing straightforward transformations we find

$$\sum_{\gamma\sigma}\int_{\mathbf{r}\mathbf{r}'}\bar{h}^{\gamma\sigma}(\mathbf{r}-\mathbf{r}')\frac{\delta\Phi_0}{\delta\varphi^{\tau}(\mathbf{r})}\frac{\delta\Phi_0}{\delta\varphi^{\sigma}(\mathbf{r}')}=-\frac{1}{2}\sum_{\gamma}\int_{\mathbf{q}}q^2 f_q^{\tau}f_q^{\tau}$$
$$=-\frac{1}{2}\sum_{\tau}\int_{\mathbf{r}}(\nabla f^{\tau})(\nabla f^{\tau})=-\frac{1}{2}\sum_{\tau\alpha\beta}\int_{\mathbf{r}}f^{\tau\alpha}f^{\tau\beta}\nabla\varphi^{\alpha}\nabla\varphi^{\beta}, \tag{2.1}$$

where for the sake of brevity we have introduced the notation $f^{\gamma}=\partial f/\partial\varphi^{\gamma}$, with $(f^{\gamma})_q$ the Fourier transform of f^{γ}. Thus, for $n\neq 1$ the nonlocal behavior generated by Eq. (1.1) cannot be reduced to the expression

$$\int_{\mathbf{r}}\chi[\varphi(\mathbf{r})](\nabla\varphi)^2, \tag{F}$$

and essentially depends on the gradients of different components of the order parameter φ. This fact complicates analysis somewhat, since now it is necessary to seek the solution in the form

$$\Phi_1=\sum_{\alpha\beta}\int_{\mathbf{r}}\chi_{\alpha\beta}\nabla\varphi^{\alpha}\nabla\varphi^{\beta}, \tag{2.2}$$

where the $\chi_{\alpha\beta}$ are the components of the tensor $\hat{\chi}$ of rank n. The physical equivalence of index permutations implies that this tensor is symmetric about the principal diagonal. Substituting (2.2) into Eq. (1.2) and discarding higher-order gradients, we arrive at the following equation for the components of the tensor $\hat{\chi}$:

$$\dot{\chi}_{\alpha\beta}=-(\eta\delta_{\alpha\tau}+4f^{\tau\alpha})\chi_{\tau\beta}-(\Delta_{\varphi}\varphi^{\gamma}+2f^{\gamma})\chi_{\alpha\beta}^{\ \ \tau}$$
$$+{}^1/_2(f^{\tau\alpha}f^{\tau\beta}-\eta\delta_{\alpha\beta})+\chi_{\alpha\beta}^{\ \ \tau\tau}, \tag{2.3}$$

(here and in what follows summation over repeated indices is implied). It would seem that to find the coefficient η one must solve a system of $1+(n^2+n)/2$ coupled nonlinear second-order differential equations. Note, however, that the function f depends only on the absolute value of φ, that is, actually on φ^2 (see Ref. 17). If for the new independent variable we take $\varphi^2=x$, the number of equations can be reduced. In terms of x,

$$\int_{\mathbf{r}}f^{\tau\alpha}f^{\tau\beta}\nabla\varphi^{\alpha}\nabla\varphi^{\beta}=\int_{\mathbf{r}}[4f_x^2\delta_{\alpha\beta}+\varphi^{\alpha}\varphi^{\beta}16f_{xx}(xf_{xx}+f_x)]\nabla\varphi^{\alpha}\nabla\varphi^{\beta}. \tag{2.4}$$

Using this structure for the generation contribution, we look for the solution for the components of $\hat{\chi}$ in the form

$$\chi_{\alpha\beta}=\varkappa\delta_{\alpha\beta}+\varphi^{\alpha}\varphi^{\beta}\sigma, \tag{2.5}$$

where $\varkappa=\varkappa(x)$ and $\sigma=\sigma(x)$ $(x=\varphi^2)$ are functions only of the absolute value of φ. After substituting (2.5) into Eq. (2.3) and performing the necessary transformations we arrive at a system that consists of only two equations for the functions $\varkappa(x)$ and $\sigma(x)$:

$$\dot{\varkappa}=\varkappa_{\varphi\varphi}-\left(\Delta_{\varphi}\varphi+2f_{\varphi}-\frac{n-1}{\varphi}\right)\varkappa_{\varphi}$$
$$-\left(\eta+\frac{4}{\varphi}f_{\varphi}\right)\varkappa+\frac{1}{2}\left(\frac{1}{\varphi^2}f_{\varphi}^2-\eta\right)+2\sigma, \tag{2.6a}$$

$$\dot{\sigma}=\sigma_{\varphi\varphi}-\left(\Delta_{\varphi}\varphi+2f_{\varphi}-\frac{n+3}{\varphi}\right)\sigma_{\varphi}-\left(\eta+4f_{\varphi\varphi}+\frac{4}{\varphi}f_{\varphi}+2\Delta_{\varphi}\right)\sigma$$
$$+\frac{1}{2\varphi^2}\left(f_{\varphi\varphi}^2-\frac{1}{\varphi^2}f_{\varphi}^2\right)-\frac{4}{\varphi^2}\left(f_{\varphi\varphi}-\frac{1}{\varphi}f_{\varphi}\right)\varkappa. \tag{2.6b}$$

At $n=1$ the system of equations (2.6a) and (2.6b) transforms into a single equation, (1.4b). To clarify this step it has proved expedient to introduce a new function $\chi=\varkappa+x\sigma$ in place of $\varkappa$. In terms of the functions χ and σ, at a fixed point we have the following system of equations:

$$f_{\varphi'\varphi'}=-\frac{d_{eff}f}{d_{eff}-2}+\frac{1}{2}\varphi' f_{\varphi'}+f_{\varphi'}^2-\frac{(n-1)f_{\varphi'}}{\varphi'}, \tag{2.7a}$$

$$\chi_{\varphi'\varphi'}=\left[\left(1+\frac{4}{\varphi'}f_{\varphi'}\right)\frac{\varphi'}{2}-\frac{n-1}{\varphi'}\right]\chi_{\varphi'}$$
$$+\left(\frac{\eta}{2\Delta_{\varphi}}+4f_{\varphi'\varphi'}\right)\chi-\frac{1}{2}\left(2\Delta_{\varphi}f_{\varphi'\varphi'}^2-\frac{\eta}{2\Delta_{\varphi}}\right)-\frac{2(n-1)\sigma}{2\Delta_{\varphi}}, \tag{2.7b}$$

$$\sigma_{\varphi'\varphi'}=\left[\left(1+\frac{4}{\varphi'}f_{\varphi'}\right)\frac{\varphi'}{2}-\frac{n+3}{\varphi'}\right]\sigma_{\varphi'}+\left(\frac{\eta}{2\Delta_{\varphi}}+1+\frac{8}{\varphi'}f_{\varphi'}\right)\sigma$$
$$-\left(f_{\varphi'\varphi'}^2-\frac{1}{\varphi'^2}f_{\varphi'}^2\right)\frac{(2\Delta_{\varphi})^2}{2\varphi'^2}-\frac{8\Delta_{\varphi}}{\varphi'^2}\left(f_{\varphi'\varphi'}-\frac{1}{\varphi'}f_{\varphi'}\right)\chi. \tag{2.7c}$$

TABLE I.

d_{eff}	$f(\varphi=0)$	η	d
		$n=1$	
2,3	0,44(3)	0,279(6)	≈ 2
2,5	0,23(1)	0,15(8)	2,30
2,75	0,13(7)	0,07(5)	2,65
3	0,076(2)	0,03(4)	2,95
3,05	0,067(7)	0,03(1)	≈ 3
3,25	0,041(1)	0,013(4)	3,23
3,5	0,020(0)	0,004(1)	3,49

Here we have performed the change of variables, $\varphi = \varphi'(2\Delta_\varphi)^{1/2}$, and introduced the effective dimensionality $d_{\text{eff}} = 2d(2-\eta)$. The passage to the $n=1$ limit is straightforward.

Equation (2.7) must be solved numerically with boundary conditions that correspond to genuine critical behavior. The solution for f is unique (the appropriate boundary conditions are fixed by the symmetry $f_\varphi(\varphi=0)=0$ and the asymptotic behavior of f as $|\varphi|\to\infty$, that is, $f\to\varphi^2/2$. For the function χ in the limit $|\varphi|\to 0$ we have the constraints $\chi(0)=\chi_\varphi(0)=0$, which coincide with those at $n=1$. If we allow for parity, $\sigma_\varphi(0)=0$; however, the value of σ as $|\varphi|\to 0$ and that of η are fixed solely by the choice of the physical branch of the solution (i.e., by the asymptotic behavior). Thus, for $n>1$ we have the problem of selecting the physical branches of the solution for each function f, χ, and σ.

To select such branches we must analyze the system (2.7) in the limit of $\varphi^2\to\infty$. Bearing in mind that here $f\to\varphi^2/2$, we arrive at a Kummer equation for χ similar to the one for the case where $n=1$. For σ we also have a Kummer equation but, in contrast to the equation for χ, with a zero free term. It can be shown that each nontrivial solution of this equation is exponentially divergent for large φ. Hence, according to the asumption that nonlocalness is weak, such solutions must be discarded, with the result that the only way to satisfy the requirement that nonlocalness be weak is to select for σ a solution that tends to zero as $|\varphi|\to\infty$. This condition fixes unambiguously an additional free constant $\sigma(0)$, which suggests the following computational procedure.

For given n and d_{eff} we find the physical branch of the solution of the equation for f. The resulting numerical data on f are then substituted into Eq. (2.7b). Regarding the Fisher index η and $\sigma(0)$ as parameters, we find the branches of solutions for χ and σ that become constant. These solutions fix the desired values of η and $\sigma(0)$. In the last stage we restore the true dimensionality of space, $d=(2-\eta)d_{\text{eff}}/2$.

The results of numerical calculations for different values of n and d_{eff} are listed in Table II. To illustrate, Figs. 1–3 show the results of solving the system (2.7) numerically for the functions f, χ, and σ, respectively, in the most interesting case of $d=3$ and $n=3$. The solid curves indicate the physical branches, while the dashed curves represent the nonphysical branches closest to the physical.

The further expansion procedure appears to be fairly regular. In higher perturbation order we must retain terms with higher orders in $\nabla\varphi$ and the contribution $B\chi$ to the equation for f. The function $\eta(\mathbf{r}-\mathbf{r}')$ must also be expanded. The coefficient of the first nonvanishing correction is fixed by the requirement that $\Phi_2 \ll \Phi_{1,0}$ hold in the functional $H=\Phi_0+\Phi_1+\Phi_2+\dots$.

Note that even the first nontrivial order in this expansion yields very good results. On the other hand, this agrees well with the results of Refs. 4 and 5, in which the use of the "balance principle" revealed that the best results are obtained if many powers of φ^2 are retained in the fixed functional [specifically, about ten terms $(\varphi^2)^k$] and only one or

TABLE II.

d_{eff}	$f(0)$	$\sigma(0)$	η	d
		$n=2$		
2,3(5)	1,40(5)	0,47(3)	0,27(8)	≈2,0
2,5	0,80(0)	0,14(2)	0,25(7)	2,18
2,75	0,38(2)	0,05(3)	0,033(5)	2,62
3,0	0,19(3)	0,02(0)	0,043	2,94
3,06	0,16(8)	0,01(6)	0,031	≈3,0
3,25	0,10(3)	0,007	0,018	3,22
3,5	0,04(9)	0,001(8)	0,007	3,48
		$n=3$		
2,3(5)	2,90	0,87(1)	0,27(9)	≈2,0
2,5	1,587	0,27(5)	0,25(3)	2,17
2,75	0,714	0,036	0,038	2,62
3,0	0,358	0,023(4)	0,044	2,93
3,06	0,313	0,021(3)	0,043(5)	≈3,0
3,25	0,1812	0,0038	0,022	3,22
3,5	0,0849	0,0024	0,008	3,48
		$n=4$		
2,3(3)	4,87	1,38	0,27(2)	≈2,0
2,5	2,45	0,39	0,25(1)	2,18
2,75	1,09	0,12	0,038	2,65
3,0	0,539	0,037	0,034	2,95
3,06	0,470	0,030	0,025	≈3,0
3,25	0,259(6)	0,011	0,017	3,22
3,5	0,125(2)	0,002(9)	0,007(8)	3,49
		$n=5$		
2,3	7,61	2,13	0,23(5)	≈2,0
2,5	3,33	0,47	0,24(2)	2,19
2,75	1,49	0,14(2)	0,073	2,40
3,0	0,73(2)	0,045	0,020	2,96
3,06	0,64(0)	0,036(5)	0,018(5)	≈3,0
3,25	0,36(5)	0,013(5)	0,014(0)	3,22
3,5	0,16(9)	0,003(0)	0,007(5)	3,49

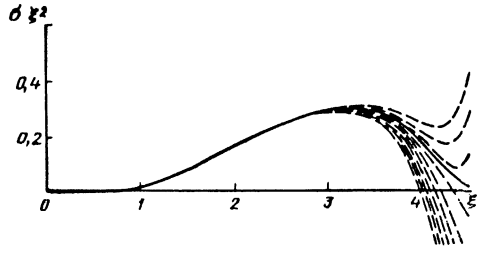

FIG. 3. The same as in Fig. 1 for function $\xi^2\sigma$.

two of the lowest gradients (i.e., powers of q^2). On the other hand, the expansion done here clearly demonstrates the reason for such a result since it is based on the use of a natural small parameter (see Appendix 2).

3. THE DEPENDENCE OF η ON n; SOLUTION OF THE RG EQUATION IN THE LIMIT OF $n \to \infty$

Figure 4 depicts the dependence of the Fisher index η on the number n of components in the order parameter at $d = 3$. The curve connecting the □ was obtained with the present approach, and the dashed curve corresponds to an extrapolation of the ε-expansion. The behavior of $\eta(n)$ for other values of d ranging from 2 to 4 is qualitatively similar to that illustrated by Fig. 4. At $n \approx 3$ there is a pronounced peak in this dependence [whose presence, generally, is not evident from Eqs. (2.7)], with a further increase in n leading to a rapid drop in η to zero. Actually, the value of η can be set equal to zero even for $n \sim 10$. For small values of n (1, 2, or 3) the values of η obtained numerically are found to practically coincide with those obtained from experiments,[24] while the ε-expansion strongly understates the value of η. Note that

$$\eta(n) = (n+2)\varepsilon^2/2(n+8)^2$$

(see, e.g., Ref. 19) in the ε-expansion also contains a peak (at $n = 4$) and demonstrates a slow decrease in η to zero as n increases ($\eta \sim 1/n$ for large n).

In the isotropic case the limit $n \to \infty$ leads to the exactly solvable spherical model.[20,21] Methodologically, an analysis of this limit by our approach is expedient since it is possible to verify the passage to the limit in an exactly solvable case.

This analysis has been carried out using the local approximation in Ref. 18. Following that paper, in Eqs. (2.7) we explicitly isolate the terms that vanish as $n \to \infty$ by replacing H with nH and φ^2 with $n\varphi^2$. To avoid any misunderstanding, we note that such normalization of the sum $\varphi^2 = \Sigma\varphi_\alpha^2$ is common.[21] Following the ideas developed in the papers published earlier on this subject, we retain this normalization condition everywhere in the present paper with the exception of the case $n \to \infty$. One can easily verify that with the normalization $H \to nH$ and $\varphi^2 \to n\varphi^2$ the quantity $f(0)$ remains small (at $d \gtrsim 3$) as n increases and tends to a constant as $n \to \infty$ instead of increasing in proportion to n (as is formally the case with the data in Table II).

Clearly, with the given normalization the term containing the second variation in H vanishes as $n \to \infty$. As a result the order of all the differential equations generated by the initial exact RG equation is reduced from the second to the first. In the final analysis it is owing to this effect that the equation becomes exactly solvable. For one thing, the equations for f and $\varkappa$ in the limit specified above in terms of the variables $x = \varphi^2/n$ and $y = f/n$ have the form

$$dy - (2\Delta_\varphi x - 2)y_x - 4xy_x^2 = 0, \tag{3.1a}$$

$$[2 - x(2\Delta_\varphi + 8y_x)]\varkappa_x - [\eta + 8y_x]\varkappa + [y_x^2 - \eta]/2 = 0. \tag{3.1b}$$

Let us show that the Fisher index η in this limit does indeed tend to zero. Equation (3.1a) implies that at the minimum point $x_{\min}$, at which $y_x = 0$, the function $y(x)$ vanishes. Differentiating Eq. (3.1b), we arrive at the following condition at the minimum point:

$$(\Delta_\varphi x_{min} - 1)y_{xx}(x_{min}) = 0.$$

It can easily be verified that since in a certain neighborhood of point $x_{\min}$ the function $y(x)$ is nonzero, the second derivative $y_{xx}(x_{\min})$ is nonzero, too, and hence $x_{\min} = 1/\Delta_\varphi$. Substituting this value of $x_{\min}$ into Eq. (3.1b), we find that at point $x_{\min}$

$$\eta(\varkappa + {}^1/_2) = 0. \tag{3.2}$$

This is possible for $\varkappa(x_{\min}) = -\frac{1}{2}$ or $\eta = 0$. However, the function $\varkappa$ is uniquely determined by the boundary condition as $x \to 0$ and the choice of asymptotic behavior. The requirement $\varkappa(x_{\min}) = -\frac{1}{2}$ is additional for this function and cannot be satisfied in the general case. The reader will also recall that Eqs. (3.1) are valid only if $\chi \ll f$, so that even if for a specific branch of the solution this condition accidentally holds true, the inequality $\chi \ll f$ breaks down and such a solution can be considered nonphysical. The only possibility of satisfying Eq. (3.2) is to put $\eta = 0$.

With allowance for the fact that $\eta = 0$ the equation for f reduces to the following form:[18]

$$\dot{y} = dy - [(d-2)x - 2]y_x - 4xy_x^2. \tag{3.3}$$

To establish the fixed point we have, respectively,

$$dy = [(d-2)x - 2]y_x + 4xy_x^2, \tag{3.4}$$

while to find the eigenvalues and eigenfunctions we must solve Eq. (3.4) linearized near the fixed point y^*:

$$\lambda\psi = d\psi - [(d-2)x - 2]\psi_x - 8xy_x^*\psi_x, \tag{3.5}$$

with $\psi = y - y^*$. In the case of a trivial fixed point, Eq. (3.5) leads to the eigenvalue $\lambda_1 = 2$ and, respectively, to Gaussian critical indices. For a nontrivial fixed point Eqs. (3.4) and (3.5) can be written in the form

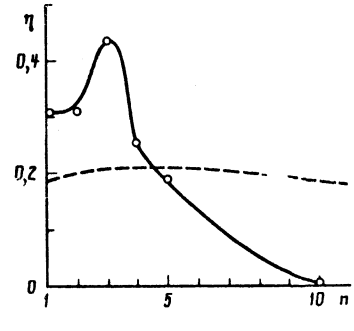

FIG. 4. Dependence of η on n. The dashed curve represents the extrapolation of the ε-expansion to $d = 3$.

$$2y_x^*(2y_x^*-1)\,dx/dy_x^*+(d-2)x-2+8xy_x^*=0, \tag{3.6}$$

$$(d-\lambda)\psi=2y_x^*(2y_x^*-1)\,d\psi/dy_x^*. \tag{3.7}$$

For $y_x^*\neq 0$ and $y_x^*\neq 1/2$ these equations are exactly integrable. For one thing, for ψ we have

$$\psi=C[2y_x^*/(1-2y_x^*)]^{(d-\lambda)/2}, \tag{3.8}$$

where C is the constant of integration. The solution of Eq. (3.6) determining $x(y^*)$ is elementary but somewhat more involved. As can be shown,[18] on its physical branch Eq. (3.8) leads to the spectrum $\lambda_k = d-2k$, which together with the condition that $\eta=0$ leads in turn to the known critical indices of the spherical model:

$$\nu=\frac{1}{\lambda_1}=\frac{1}{d-2}, \quad \gamma=\frac{2}{d-2},$$
$$\alpha=\frac{d-4}{d-2}, \quad \beta=\frac{1}{2}.$$

Similar results are yielded by the numerical solution of the initial system of equations for $n\gg 1$, which shows that our method is meaningful.

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APPENDIX 1. THE EXISTENCE OF A TRANSITION AT $d=2$ AND THE MERMIN–WAGNER–HOHENBERG THEOREM

As shown above, allowing for the generation of free-energy functional nonlocal behavior leads to an increase in the effective dimensionality both for $n=1$ and for $n\neq 1$. For one thing, in two-dimensional space, $d_{\text{eff}}\approx 2.3>2$ and, consequently, there exists a physical branch, so that a real second-order phase transition must exist, as it does for $d>2$. Formally this conclusion, however, violates the well-known rigorous Mermin–Wagner–Hohenberg (MWH) theorem concerning the absence of such a transition in degenerate two-dimensional systems.[25–27] Below we show that this contradiction is illusory.

We start by offering simple phenomenological arguments.[24] Let us suppose that a two-dimensional system is place in an external field h and that its order parameter deviates only slightly from the equilibrium value φ_0. We study the transverse fluctuations φ_a (for $n\neq 1$). Using the "principle of conservation of absolute value,"[24] we write the respective variation in the free-energy functional as

$$\Delta F=\frac{1}{2}\int d^dr\left[\frac{\varphi_\perp^2}{\chi_\perp}+c(\nabla\varphi_\perp)^2\right], \tag{A1.1}$$

where the transverse susceptibility $\chi_\perp$ in the field $\mathbf{h}$ can be represented in the form $\chi_\perp=\varphi_0/h$. The correlator of the fluctuation $\varphi_\perp$ is defined by the following integral:

$$\langle\varphi_\perp\rangle=\int_{\mathbf{q}} G_\perp(\mathbf{q})=\frac{T}{(2\pi)^d}\int\frac{d^dq}{h/\varphi_0+cq^2}, \tag{A1.2}$$

where T is the temperature. The quantity $\langle\varphi_\perp\rangle$ is divergent if we have $d\leqslant 2$ for $\varphi_0\neq 0$ and $h\to 0$. This means that ordering is absent at $d=2$ and $h=0$. Qualitatively the initial arguments of Mermin and Wagner[25,27] and Hohenberg[26] were the same but more rigorous. In their proof they used the exact Bogolyubov inequality

$$\tfrac{1}{2}\langle[A,A^+]\rangle\langle[[C,H],C^+]\rangle\geqslant T|\langle[C,A]\rangle|^2, \tag{A1.3}$$

where H is the Hamiltonian

$$\langle X\rangle\equiv \mathrm{Sp}[X\exp(-H/T)]/\mathrm{Sp}(X),\ [X,Y],$$

and $[X,Y]$ denotes a commutator.

To be definite, we follow Ref. 25 and give its main result. Let us consider the Hamiltonian of the Heisenberg model:

$$H=-\frac{1}{2}\sum_{ll'}J_{ll'}s_ls_{l'}-h\sum_l s_l^z e^{ikl}. \tag{A1.4}$$

Here $k=0$ in a ferromagnetic system and $e^{ikl}=\pm 1$ (for different sublattices) in an antiferromagnet. Setting $C=s_+(q)$ and $A=s_-(-q-k)$ and using the Bogolyubov inequality, we obtain

$$s(s+1)>\frac{2T(s^z)^2}{N}\sum_l\left[s(s+1)\left(\sum_l l^2J(l)\right)\frac{q^2}{2}+|hs^z|\right]^{-1},$$

or, in the thermodynamic limit $V\to\infty$,

$$s(s+1)>\frac{2T(s^z)^2}{(2\pi)^d\rho}\times\int\frac{d^dq}{s(s+1)}\left[\int d^dl\,l^2J(l)\frac{q^2}{2}+|hs^z|\right]^{-1}, \tag{A1.5}$$

where $1/\rho$ is the volume per spin. If the exchange interaction $J(l)$ is fairly short-range, that is, if $\int d^dl\,l^2J(l)$ is finite, the inequality (A1.5) leads to the same result as Eq. (A1.2).

The direct link between these two approaches can easily be established. Bearing in mind that the quantum nature of the system is unimportant in the fluctuation range,[28] we demonstrate this link for the classical analog of Hamiltonian (A1.4). This derivation can easily be generalized to the case of quantum systems. We linearize in s_l the exponent in the partition function

$$Z\propto\int Ds\exp[-H[s]/T]$$

by using the Stratonovich–Hubbard transformation:

$$Z\propto\int\mathcal{D}s\int D\varphi\exp\left[-\frac{1}{2}\sum\varphi_lJ^{-1}_{ll'}\varphi_{l'}+\frac{1}{T}\sum_l(\varphi_l+h_l)s_l\right], \tag{A1.6}$$

where $\Sigma J_{lm}J^{-1}_{ml'}=\delta_{ll'}$, and the $s(l)$ are classical vectors. Integrating with respect to s_l yields

$$Z\propto\int D\varphi\exp\left\{-\frac{1}{2}\sum_{ll'}\varphi_lJ^{-1}_{ll'}\varphi_{l'}+\sum\ln\left[\Gamma\left(\frac{n}{2}\right)\left(\frac{2T}{\varphi_l+h_l}\right)^{n/2-1}\times I_{n/2-1}\left[\left(\frac{T}{\varphi_l+h_l}\right)^{-1}\right]\right]\right\}, \tag{A1.7}$$

where $\Gamma(x)$ and $I_\nu(x)$ are the gamma function and the modified Bessel function, respectively. Formally Eq. (A1.7)

gives the exact representation of the initial free-energy functional. Using the gradient expansion, in the continuum limit we obtain[29]

$$H(\varphi)=\frac{1}{2}\int d^dl[c(\nabla\varphi_l)^2+f(\varphi_l)], \qquad \text{(A1.8)}$$

where $c=\int d^d l\, l^2 J(l)/\int d^d l J(l)$. This formula for c makes the connection between the two approaches quite obvious. If the interaction $J(l)$ decreases fairly rapidly, c is finite and no phase transition occurs.

We believe the assumption of short-range interaction to be realistic. Strictly speaking, however, $J(l)$ usually contains long-range contributions that are small in amplitude. Often these contributions are unimportant and can be ignored, but not in the case of the critical region, where nonlocal corrections to $H[\varphi]$ in the form of (A1.8) are sure to be generated. Formally this causes the propagator $G_0(q)\propto 1/q^2$ to be replaced by $G(q)\propto 1/q^{2-\eta}$, which corresponds to the divergence of $c\propto g^{-\eta}\to\infty$ and $q\to 0$ and leads in turn to the substitution of d_{eff} for d in Eq. (A1.2). In other words, long-range effects in the system are generated in the critical region even if they are negligible far from it, that is, although the MWH theorem remains valid within the framework of its hypotheses, one of these [the one concerning the finiteness of $\int d^d l\, l^2 J(l)$] becomes invalid. The same situation occurs, naturally, in the case where the initial interaction falls off fairly slowly.[30,31] To conclude our discussion of the relation between the obtained results and the MWH theorem, three remarks are in order.

1. The above arguments are valid for various two-particle systems. A Heisenberg ferromagnet was taken only as an illustration directly related to the original work of Mermin and Wagner.[25]

2. All the arguments are applicable only to the critical region. If we assume that after the transition the two-dimensional system becomes ordered and that in the process the temperature "leaves" the critical region, we arrive at a contradiction. Indeed, outside this region the theorem's hypotheses are satisfied and ordering is impossible. This means that the "ordered phase" must lie completely inside the critical region. Of course, this cannot be an ordered phase in the ordinary mean-field sense since the fluctuations against the background of the phase are large (in this sense the MWH theorem is still valid). However, a second-order phase transition does exist in the two-dimensional system.

3. If d_{eff} is formally set equal to two, η diverges together with $f(0;d_{\text{eff}})$ and in any case reaches $\eta=2$. Here, however, $d=(2-\eta)d_{\text{eff}}/2$ vanishes. In other words, when the real dimensionality d decreases to zero, the expected generation of nonlocal interactions is so high that d_{eff} always remains greater than two. The behavior of d as a function of d_{eff} is depicted schematically in Fig. 5. It is unclear to what extent this dependence can be reconstructed experimentally, but theoretically the behavior is of considerable interest because it reveals that the critical behavior of an idealized system (infinite and pure) for any value of d which is nonnegative but less than four must be the same as at $d=3$. The presence of a boundary is, apparently, an important factor in forming a specifically critical behavior of real quasi-low-dimensional systems, while the idea that such behavior can be modeled by an infinite system with $d\leqslant 2$ is to a certain extent illusory.

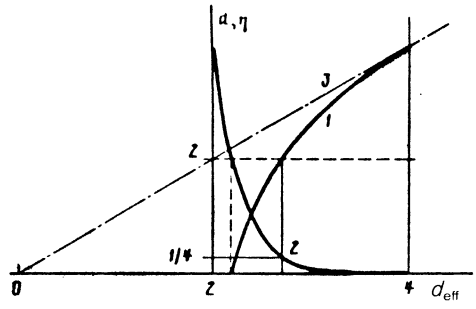

FIG. 5. The behavior of d and η as functions of d_{eff}; the dot-dash line corresponds to $d=d_{\text{eff}}$.

APPENDIX 2. THE NATURE OF THE SMALL PARAMETER

The idea of using the local approximation as the zeroth approximation in perturbation theory was based on the experimentally established smallness of the Fisher index, which reflects the onset of nonlocal behavior. In itself, however, the quantity η requires calculation and does not enter into the initial concepts of the theory. More than that, in the actual expansion we used the smallness of the amplitude $f(0)$ of the physical branch of the solution of the local equation, while η was determined indirectly. Note that the same small parameter was used when the local equation was studied analytically (although this is not needed in a numerical solution). The quantity $f(0)$ in turn was also calculated in the solution process, and its smallness appeared to be even more accidental than that of η. To grasp the nature of the corresponding perturbation theory one would want a "genuine small parameter" that would be a combination of the parameters d and n and would enter into the equation in such a manner that the desired problem would be clear-cut.

To establish the relation between $f(0)$ and d and n, we reproduce the estimate of $f(0)$ done in Ref. 17. The local equation is taken in the form

$$f_{\xi\xi}+\frac{n-1}{\xi}f_\xi+\frac{4d}{d-2}f-2\xi f_\xi=f(0)f_\xi^2, \qquad \text{(A2.1)}$$

where $\xi=z^{1/2}$, $z=(d-2)\varphi^2/4$, and all solutions are normalized at $\xi=0$ to unity. The linear equation

$$zf_{zz}^{(0)}+\left(\frac{n}{2}-z\right)f_z^{(0)}+\frac{d}{d-2}f^{(0)}\approx 0 \qquad \text{(A2.2)}$$

has a solution in the form of the Kummer function[23]

$$f^{(0)}=M\left(-\frac{d}{d-2};\frac{n}{2};z\right),$$

such that

$$M\left(-\frac{d}{d-2};\frac{n}{2};0\right)=1.$$

The physical branch of the solution departs from this function near the turning point of its derivative (i.e., at $f_{\xi\xi}^{(0)}\approx 0$). Using the recursion formulas for the Kummer function, we can easily obtain the equation defining the turning point:

$$M\left(-\frac{d}{d-2};\frac{n}{2};z\right)=\frac{n-1}{n}M\left(-\frac{d}{d-2};\frac{n}{2};z\right) \qquad \text{(A2.3)}$$

(here we are interested in the second positive "zero" of this equation, $z \equiv z_0$). The value of the derivative at the point can be estimated by the Sonin–Polia theorem.[22]

For $d \leqslant 3$ the parameter $a = d/(d-2)$ is fairly large, so that the corresponding estimate can be simplified by employing the approximate representation of the Kummer function in the form

$$M(a;b;z) \propto \Gamma(b) \exp\left(\frac{z}{2}\right) \times \left(\frac{bz}{2} - az\right)^{1/4 - b/2} \pi^{1/2} \cos\left[(2b-4a)^{1/2} - \frac{b\pi}{2} + \frac{\pi}{4}\right]. \quad \text{(A2.4)}$$

If at the minimum point $f_\xi^{(0)}$ is sufficiently large,

$$f(0)\,(f_\xi^{(0)})^2 \propto [4d/(d-2)]\, f^{(0)},$$

it is the first point where the solution of the nonlinear equation (A2.1) departs from the Kummer function as $f(0)$ increases. Therefore, the condition

$$f(0)\,(f_\xi^{(0)})^2 = c_0 [4d/(d-2)]\, f^{(0)},$$

provides an estimate for $f(0)$ on the physical branch of the solution. The numerical factor c_0 can be chosen in such a way that the above relation provides good estimates of $f(0)$ for different d and n [$c_0 \approx 2$; see Fig. 6(a)].

Note that in this case we are interested not so much in good numerical estimates for $f(0,d,n)$ as in clarifying the meaning of the smallness of $f(0,d,n)$. For this reason we restrict our discussion to $n = 1$. Equation (A2.3) and (A2.4) then simplify considerably. Combining them with the condition imposed on $f(0)$, we arrive at the following estimate:

$$f(0) \approx \frac{d-2}{d} z_0^{1/2} \exp\left(-\frac{z_0}{2}\right) \frac{\cos[(z_0(5d-2)/(d-2))^{1/2}]}{[\cos[(z_0(3d+2)/(d-2))^{1/2}]]^2}. \quad \text{(A2.5)}$$

The presence of the formal singularity $(d-2)^{-1}$ as $d \to 2$ is unimportant since in the local approximation instead of d we must use the effective dimensionality d_{eff}, which does not become smaller than $d_{\text{eff}} \approx 2.3$. More important is the presence of the factor $\exp(-z_0/2)$, owing to which $f(0)$ rapidly decreases as $z_0 \propto d-2$ increases. Figure 6(b) illustrates this exponential decrease. This exponential factor is actually the small parameter sought. But although it depends on d, its presence in the equation is not obvious and the smallness of $f(0)$ is an accidental corollary of the properties of the Kummer function.

Yet, since the answer is known, it is easy to clarify the nature of the parameter obtained. The equation

$$df - (d-2)\varphi \nabla_\varphi f/2 + \nabla_\varphi^2 f - (\nabla_\varphi f)^2 = 0 \quad \text{(A2.6)}$$

can be interpreted as an Euler–Lagrange equation for the functional

$$\Phi = \frac{1}{2}\int d^d\varphi \exp\left[-\frac{(d-2)\varphi^2}{4} - 2f\right]\left[(\nabla_\varphi f)^2 + d\left(f + \frac{1}{2}\right)\right] \sim \frac{1}{2}\int d\varphi\, \varphi^{n-1} \exp\left[-\frac{(d-2)\varphi^2}{4}\right] F\{f(\varphi)\}. \quad \text{(A2.7)}$$

Let us examine the behavior of the integrand on the physical branch of $f(\varphi)$. At

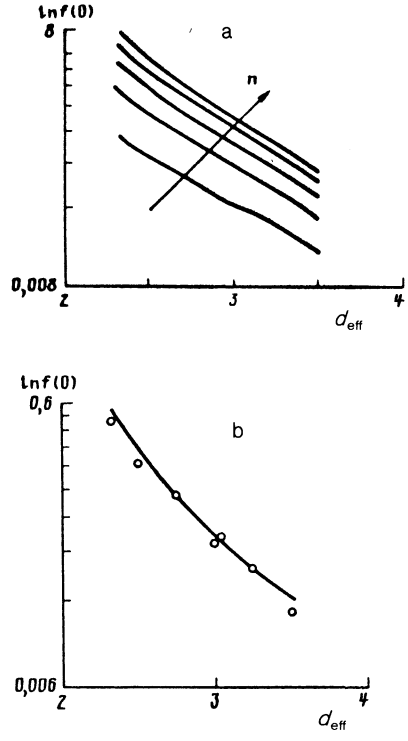

FIG. 6. (a) $\ln f(0)$ vs d_{eff}. The variation of η is depicted by an arrow. (b) The same dependence at $n = 1$ obtained through an analytical estimate. The □ represent the results of numerical calculations.

$$\varphi \approx \varphi_0 = [4z_0/(d-2)]^{1/2}$$

the function $F\{f(\varphi)\}$ follows the universal asymptotic behavior. Its contribution to the integral is obviously small, proportional to $\exp(-z_0)$. For $\varphi \ll \varphi_0$ the smallness is absent and must be balanced by the smallness of $f(0)$, whence

$$f(0) \propto \exp[-(d-2)\varphi_0^2/4].$$

The presence of the factor φ^{n-1} leads to an increase in $f(0)$ with n and in turn is balanced by the obvious substitutions of $n\varphi^2$ for φ^2 and of nf for f used above to study the limit of $n \to \infty$.

The presence of the factor $\exp[-(d-2)\varphi^2/4]$ in Φ is the result of the transformation properties of field φ under scaling. The perturbation theory based on this factor operates least effectively for small values of d. On the other hand, as $d \to 4$, the value of z_0 increases, the solution degenerates into the φ^4-model, and estimates of the form (A2.5) lose all meaning. In the final analysis this is due to the scaling properties of φ and technically manifests itself, within the framework developed here, in the loss of accuracy of numerical calculations as $d \to 4$. In this limit the quantity η is of the order of $f^2(0)$, so that the expansion in $f(0)$ is actually done in terms of $\eta^{1/2}$, as in the traditional approach.[24] In the opposite limit, $2 \leqslant d \leqslant 3$, the dependence of η on $f(0)$ becomes linear (Fig. 7) and the expansion within this range is done in η, as expected in the first *a priori* ideas concerning the possibility of an expansion in the Fisher index.

Summarizing, we can say that the perturbation theory based on the exotic small parameter $f(0) \propto \exp[-(d-2)\varphi_0^2/4]$ is the most effective at $d \approx 3$.

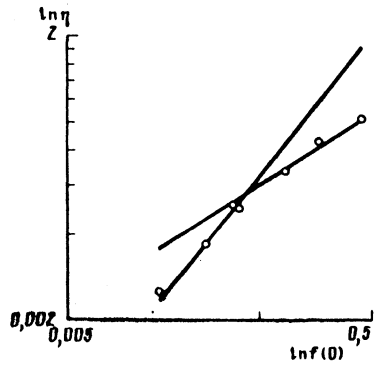

FIG. 7. Plots of η vs $f(0)$ on the log-log scale. Crossover from $\eta \propto f^2(0)$ as $d \to 4$ to $\eta \propto f(0)$.

APPENDIX 3. ALLOWING FOR ANISOTROPY

Our discussion up to now has been restricted to the isotropic case. Real physical systems usually require anisotropic models for their investigation (for reviews see Refs. 9 and 32). In such cases there is need to consider free-energy functionals with a large number of fluctuating fields and/or various invariants in each order in powers of the components φ_i^α. It is well known that phase trajectories leave the region within which the fourth-order anisotropic form discovered within the framework of the ε-expansion is positive definite. Customarily this is said to signal a possible sudden transformation of a continuous phase transition to a discontinuous one, caused by fluctuations.[29,33,34] Confirmation of this can be obtained by theoretical means via approximate free-energy calculations, by the ring approximation,[35] and in exactly solvable models.[36] But there is always the suspicion that, apart from the use of the ε-expansion or the limit $n \to \infty$, the results are corollaries of the φ^4-model, of which, as the above shows, the fluctuation region is in itself a crude approximation.

A rigorous answer to the question of the possibility of a fluctuation transformation of a phase transition can be obtained only by solving the exact RG equation in variational derivatives. As shown above, however, even the local version serves is a very good approximation. Bearing in mind that allowing for anisotropy greatly complicates the problem, we start with the local RG equation. A formal generalization to the anisotropic multicomponent case can be done very simply:

$$f = \hat{R}f = df + \sum_i \left[-\frac{d-2}{d} \varphi_i \nabla_{\varphi_i} f + \Delta_{\varphi_i} f - (\nabla_{\varphi_i} f)^2 \right]. \quad \text{(A3.1)}$$

The simplicity of Eq. (A3.1) is illusory because f is a function of various invariants (allowed by specific symmetry) built on the basis of the vectors $\{\varphi_i^\alpha\}$ and the partial derivatives mix the components φ_i^α. Even a numerical solution is highly problematical.

Nevertheless, several general statements can be made concerning the fixed points of Eq. (A3.1). First, the $n = \Sigma n_i$-component $O(n)$-symmetric free-energy-functional density $f = f(\varphi^2)$, where $\varphi^2 = \Sigma \varphi_i^2$, always has a straightforward isotropic solution. One can easily show that the sum of noninteracting solutions, $f = \Sigma f(\varphi_i)$, is a solution, too. Next we allow for the fact that asymptotically $f \approx \varphi^2/2$. This means that the isotropic solution $f(\varphi^2) = \Sigma \varphi_i^2/2$ has the same asymptotic behavior as the sum $\Sigma f(\varphi_i)$. On the other hand, any physical solution of the equation $\hat{R}f = 0$ must also have the asymptotic forms $f \approx \varphi^2/2$, which coincides with the asymptotic behavior of the sum of noninteracting solutions:

$$f = \sum f(\varphi_i) \approx \sum \varphi_i^2/2 = \varphi^2/2.$$

The symmetry group transformations do not change this isotropic asymptotic form but formally map $f = \Sigma f(\varphi_i)$ into the remaining solutions of the equation $\hat{R}f = 0$, compatible simultaneously with the symmetry of the system and the requirement $f \to \varphi^2/2$ as $\varphi \to \infty$. As a result we can name all the fixed points without solving Eq. (A3.1).

A similar effect should occur in the ε-expansion. To confirm this statement, we take the simple example of a tetragonally symmetric system:

$$H = \frac{1}{2} \int d^d r \left[\tau \varphi^2 + c(\nabla \varphi)^2 + \frac{1}{4} u(\varphi_1^4 + \varphi_2^4) + \frac{1}{2} v \varphi_1^2 \varphi_2^2 \right], \quad \text{(A3.2)}$$

where $\varphi^2 = \varphi_1^2 + \varphi_2^2$. For this system the RG equations in the first ε-approximation are

$$\dot{u} = \varepsilon u - {}^9/_2 u^2 - {}^1/_2 v^2, \quad \dot{v} = v[\varepsilon - 3u - 2v]. \quad \text{(A3.3)}$$

The respective fixed points are

$$\begin{aligned} &\mu_1^*: \ u^* = 2\varepsilon/9, \ v^* = 0, \\ &\mu_2^*: \ u^* = \varepsilon/5, \ v^* = \varepsilon/5, \\ &\mu_3^*: \ u^* = \varepsilon/9, \ v^* = \varepsilon/3. \end{aligned} \quad \text{(A3.4)}$$

Let us now perform the only rotation allowed by the symmetry group of the system. We have

$$\begin{aligned} u(\varphi_1^4 + \varphi_2^4) + 2v\varphi_1^2\varphi_2^2 = (u+v)(\mu^2+v^2)^2/2 \\ - (3u-v)\mu^2 v^2, \end{aligned}$$

which corresponds to the following substitution:

$$u \to (u+v)/2, \ v \to (3u-v)/2. \quad \text{(A3.5)}$$

We can easily see that (A3.5) maps μ_1^* and μ_3^* into each other, but does not change the isotropic point μ_2^*. A similar procedure can be applied to the equation $\hat{R}f = 0$. Bearing in mind that in the given case a rational basis of invariants consists of $x = \varphi^2$ and $y = \varphi_1^2 \varphi_2^2$, we have $f = f(x,y)$. It is convenient to introduce a new variable, $s = 2y^{1/2}$, and write the equation as

$$\begin{aligned} \hat{R}f = df - (d-2)(f_x x + f_s s) \\ + 4(x f_{xx} + 2s f_{xs} + x f_{ss}) \\ + 4 f_x - 4(x f_x^2 + 2s f_{xs} + x f_s^2) = 0. \end{aligned} \quad \text{(A3.6)}$$

The transformation $\varphi_{1,2} = (\mu \pm v)/2^{1/2}$ takes on the form $\xi_\pm = x \pm s$. We assume that $f(x,s)$ is a solution to Eq. (A3.6). Let us now show that $f = f(\xi_+) + f(\xi_-)$ is a solution, too, that is,

$$\begin{aligned} \hat{R}(f(\xi_+) + f(\xi_-)) = \hat{R}f(\xi_+) \\ + \hat{R}f(\xi_-) = 0. \end{aligned}$$

Directly substituting $\xi_\pm = x \pm s$ into Eq. (A3.6) yields

$$\begin{aligned} \hat{R}f = \hat{R}(f(\xi_+) + f(\xi_-)) \\ - 4[2x f_{\xi_+ \xi_-} + 2s(f_{\xi_+ \xi_+} - f_{\xi_- \xi_-}) \\ - 2x f_{\xi_- \xi_+}] = \hat{R}f(\xi_+) + \hat{R}f(\xi_-) = 0. \end{aligned} \quad \text{(A3.7)}$$

TABLE III.

k	n = 1	n = 2	n = 3	n = 4	n = 5
0	0,076	0,195	0,355	0,540	0,765
1	−0,456	−0,585	−0,710	−0,810	−0,918
2	0,373	0,378	0,385	0,379	0,383
3	−0,141	−0,140	−0,141	−0,134	−0,132
4	0,067	0,067	0,066	0,061	0,060
5	−0,036	−0,035	−0,035	−0,032	−0,030
6	0,020	0,020	0,020	0,017	0,016

The inverse transformation maps $f=f(\xi_+)+f(\xi_-)$ into the solution $f(x,s)$.

Unfortunately, knowing only the fixed points does not answer the question of the global structure of the phase portrait. Even a numerical solution of Eq. (A3.1), however, with the asymptotic boundary condition $f\to\varphi^2/2$ as $\varphi\to\infty$ is highly problematical as noted earlier. At the same time, this portrait can easily be obtained by truncating the hierarchy of equations for the coefficients ("vertices") of the expansion $f=\Sigma g_k O_k(\varphi_i)$ in k th-order invariants. For Eq. (A3.1) this procedure is done in a simple way in the ε-expansion.[37] Truncation of the series for f becomes nonphysical for values of d much smaller than $d=4$ ($d\leqslant 3$). Below we show that this operation can be corrected with allowance for the physical asymptotic behavior $f\approx\varphi^2/2$.

Let us write the system of equations for the g_k in the isotropic case:

$$\dot{g}_k=[d-(d-2)k]g_k(n)+(2k-n)(k+1)g_{k+1}(n)/2 - \sum_{m=1}^{k+1} m(k-m+1)g_m(n)g_{k+1+m}(n)=0. \qquad \text{(A3.8)}$$

This system can be solved as a recursion formula determining all leading vertices in terms of $f(0)=g_0$:

$$g_{k+1}(n)=\hat{G}_{k+1}[g_k(n);\ g_{k-1}(n);\ \ldots;\ g_0(n)]. \qquad \text{(A3.9)}$$

The vertex $g_0(n)$ in turn is determined uniquely on the physical branch, so that the sequence of $g_{k+1}(n)$ can be calculated in a simple manner. Table III list several such first vertices $g_k(n)$ for $n=1,2,\ldots,5$. Clearly, despite the large discrepancy between the $g_0(n)$ for different n, the sequence of $g_k(n)$ becomes universal as k increases, that is, $g_k(n)\to g_k(1)$. Indeed, since the asymptotic behavior $f\approx\varphi^2/2$ is controlled by the sequences of $g_k(n)$ for large k, the sequences of $g_k(n)$ must become indistinguishable for the scalar and multicomponent cases. This property can be used to truncate the series naturally for an anisotropic system. Instead of the formal condition $g_{k>k\max}=0$ (see Ref. 9) used in an ε-expansion, we can use the condition $g_{k\max+1}/g_{k\max}=A(g_{k\max})$, where $A(g_{k\max})$ can be found from the data of Table III.

The procedure described has been implemented for a tetragonal system and has led to a phase portrait that is qualitatively similar to the one obtained in the φ^4-model.

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