# Logarithmic corrections in the two-body problem in QED

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The logarithmic part of the Lamb shift, the contribution of relative order $\alpha^3 \log(1/\alpha)$ to the atomic state energy, is related to the usual infrared divergence. This fact allows one to easily calculate this correction in positronium. Logarithmic contributions of the next order, $\alpha^4 \log(1/\alpha)$, are of a different, relativisitic nature. Their calculation is reduced to ordinary perturbation theory for the nonrelativistic Schrödinger equation. The perturbation operators have the Breit-type structure and are found by evaluating free-particle diagrams. For positronium, the calculated logarithmic correction is nonzero only in $n^3S_1$ states and is equal to $5/24m\alpha^6 \log(1/\alpha)/n^3$.

## 1. INTRODUCTION

The increasing precision of the spectroscopic measurements in hydrogen, muonium and positronium demands higher accuracy of the theoretical calculations for the QED two-body bound states. Certainly, those problems are also of an independent theoretical interest.

The generally accepted theoretical approach to them goes back to Refs. 1–3. Its starting point is the introduction of a relativistic two-body wave equation, which can be solved exactly, and which in the nonrelativistic limit reduces to the Schrödinger equation. Then a perturbation series is developed about the exact solution.

Our approach is different. The corrections to the Lamb shift that are logarithmic in $\alpha$ originate from the effective operators which can be considered local with acceptable accuracy. These operators are found by evaluating free-particle diagrams. The corrections discussed are then computed in the standard perturbation theory for the nonrelativistic Schrödinger equation.

The logarithmic contribution to the Lamb shift, i.e., the energy correction of relative order $\alpha^3 \log(1/\alpha)$, is related to the usual infrared divergence. This allows one to easily calculate this contribution not only in hydrogen, but in positronium as well. Those considerations are presented in detail in Sec. 2.

The next logarithmic corrections to the energy, relative order $\alpha^4 \log(1/\alpha)$, are of different, relativistic origin. This fact is demonstrated in the next sections of the paper where those corrections are calculated explicitly. Meanwhile, in Introduction, we restrict ourselves to a somewhat formal argument in favor of the relativistic origin of the contributions discussed: these corrections of high order in $\alpha$ do not have any power of $\pi$ in the denominator, as distinct from the usual QED expansion. This is why we omit the usual adjective "radiative" with the noun "corrections' in our paper.

This approach in its simplest form was previously used by two of us to calculate the corrections of relative order $\alpha^2 \log(1/\alpha)$ to the para- and orthopositronium decay rates.[4] The calculation of the corrections of the relative order $\alpha^4 \log(1/\alpha)$ to the energy levels is a much more complicated problem. The main object of the present article is again positronium. However, to at least have an extra check on our results we consider the more general case of particles with different masses, $m$ and $M$.

The main result of the present work was presented in our recent note,[5] where for brevity we restricted ourselves to a formal scheme of calculations. Here we present in detail a somewhat different approach to the calculation of the logarithmic energy corrections. It naturally leads to the same result, but allows one to give physical interpretation of the different contributions.

## 2. THE LAMB SHIFT AND INFRARED DIVERGENCE: HYDGROGEN AND POSITRONIUM

The origin of the Lamb shift in hydrogen is closely related to the infrared divergence in electron scattering by a Coulomb center. Indeed, at the regularization via the introduction of the photon mass $\lambda$, the logarithmic dependence of the vertex part on it (Fig. 1a) is cancelled by the similar dependence of the bremsstrahlung (Fig. 1b). (We use the Comlomb gauge; the dashed line here and below refers to the Coulomb field, the wavy one to a transverse photon.)

If there is no acceleration, i.e., if the momentum transfer $q$ vanishes, the radiation vanishes also. Therefore, it is only natural that the infrared part of the vertex corection is proportional to $q^2$. Indeed, including this correction the potential of the electron interaction with a Coulomb center in the momentum representation is (see, e.g., Ref. 6, Sec. 117)

$$V(\mathbf{q}) = -\frac{4\pi\alpha}{q^2}\left(1 - \frac{\alpha q^2}{3\pi m^2}\log\frac{m}{\lambda}\right). \qquad (1)$$

Of course, in the bound state problem there is no infrared radiation. But the electron here is not on the mass shell, but deviates from it by an amount on the order of the binding energy, $\sim m\alpha^2$. On the other hand, the role of the photon mass in the bremsstrahlung is in fact to fix the minimum possible deviation of the final state invariant mass from that of the free electron. So, in the bound state problem one can put $\lambda \to m\alpha^2$ in Eq. (1) with logarithmic accuracy.

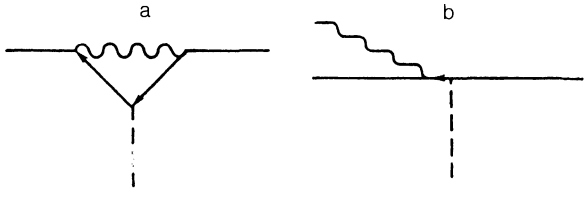


FIG. 1.

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Since the typical atomic momentum transfer is $q \sim m\alpha$, the relative magnitude of the correction to the potential, and that of the energy correction as well, $\alpha^3 \log(1/\alpha)$.

More accurately, in the momentum representation this radiative correction to the potential with logarithmic accuracy equals

$$\delta V = \frac{8}{3}\frac{\alpha^2}{m^2}\log\frac{1}{\alpha}. \tag{2}$$

In the coordinate representation it evidently equals

$$\delta V(\mathbf{r}) = \frac{8}{3}\frac{\alpha^2}{m^2}\log\frac{1}{\alpha}\delta(\mathbf{r}). \tag{3}$$

From it we find with logarithmic accuracy the known result for the Lamb shift in hydrogen:

$$\delta E_{nl} = \frac{8}{3}\frac{\alpha^2}{m^2}\log\frac{1}{\alpha}|\psi_{nl}(0)|^2 = \frac{8}{3}\frac{m\alpha^5}{\pi n^3}\log\frac{1}{\alpha}\delta_{l0}. \tag{4}$$

Here $n$ and $l$ are the principal and orbital quantum numbers of the atomic state.

It is useful to consider from the same point of view the Lamb shift in positronium. The infrared divergent radiative corrections to the electron-positron scattering amplitude are described by figures of the type 2, 3. In other words, the virtual transverse photon can be absorbed both by the same particle that has emitted it, and by the other one. It might be expected naively that the resulting perturbation operator will turn out to be four times as large as that in hydrogen, and the corresponding energy correction will be half as large (taking into account the decrease in the reduced mass by a factor of two and the resulting decrease in $|\psi(0)|^2$) by a factor of eight.

However, the situation is different for two reasons. The first of them allows for a quite simple physical explanation. The following arguments constitute a modification, as applied to positronium, of the intuitive description of the Lamb-shift in hydrogen, going back to Ref. 7. Due to the vacuum fluctuations of the electromagnetic field, the radius-vector $\mathbf{r}$ of a charged particle fluctuates according to $\mathbf{r} \to \mathbf{r} + \boldsymbol{\rho}$. Then the interaction potential of two such particles, averaged over the fluctuations, is equal to

$$\langle V(\mathbf{r}_1-\mathbf{r}_2+\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\rangle = V(\mathbf{r}_1-\mathbf{r}_2) + {}^1/_6\langle(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2\rangle\Delta V(\mathbf{r}_1-\mathbf{r}_2). \tag{5}$$

The mean square fluctuations of the electron and positron coordinates $\langle\boldsymbol{\rho}_{1,2}^2\rangle$, are evidently equal. With logarithmic accuracy each of them is proportional to (see Ref. 7)

$$\int_{m\alpha^2}^{m}\frac{d\omega}{\omega} = 2\log\frac{1}{\alpha}. \tag{6}$$

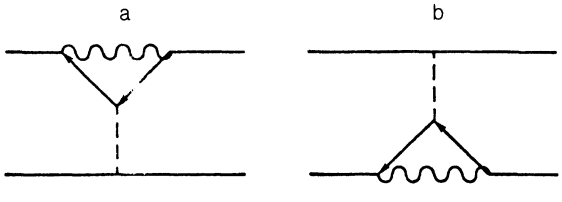


FIG. 2.

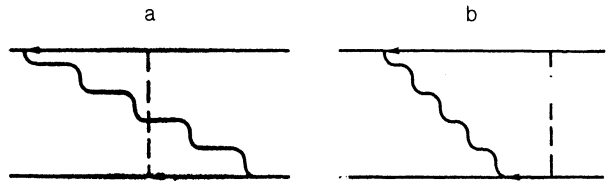


FIG. 3.

The corresponding contribution to the interaction opertor is in fact described by Figs. 2a,b.

Let us consider now the average $-2\langle\boldsymbol{\rho}_1\boldsymbol{\rho}_2\rangle$. Since the particles 1 and 2 have opposite charges and same masses, we get $\boldsymbol{\rho}_2 = -\boldsymbol{\rho}_1$, but only for fluctuations whose wavelength exceeds the size of the atomic system. For smaller wavelengths, or higher frequencies, $\omega > m\alpha$, the coordinate fluctuations are uncorrelated, i.e., $\langle\boldsymbol{\rho}_1\boldsymbol{\rho}_2\rangle = 0$. In other words, the upper limit for the integration over frequencies of the virtual quanta in the correlator $\langle\boldsymbol{\rho}_1\boldsymbol{\rho}_2\rangle$ is not $m$, as in Eq. (6), but $m\alpha$. Therefore, with logarithmic accuracy the contribution from this correlator,

$$-2\langle\boldsymbol{\rho}_1\boldsymbol{\rho}_2\rangle \propto 2\int_{m\alpha^2}^{m\alpha}\frac{d\omega}{\omega} = 2\log\frac{1}{\alpha}, \tag{7}$$

is also equal to Eq. (6). It can be easily seen that the correlator $-2\langle\boldsymbol{\rho}_1\boldsymbol{\rho}_2\rangle$ in question corresponds to Figs. 3a,b. Thus, the perturbation operator $\delta V_c(\mathbf{r})$, generated by Figs. 2, 3 with the Coulomb interaction,

$$\delta V_c(\mathbf{r}) = 8\frac{\alpha^2}{m^2}\log\frac{1}{\alpha}\delta(\mathbf{r}), \tag{8}$$

turns out to be three times as large as the perturbation (2) for the case of the external field.

Certainly, to this order in $\alpha$ we have considered all the figures with a true infrared divergence which is cut off at $m\alpha^2$. The above arguments, however, demonstrate that in Figs. 2, 3 with a double exchange there is a contribution cutting them off effectively at frequencies larger than the typical momentum transfer $q \sim m\alpha$. It is natural therefore to consider in the same region $m\alpha < \omega \leqslant m$ Fig. 4a with the double magnetic exchange (and the analogous diagram with the crossed wavy lines). With the acceptable accuracy one can neglect in them the three-dimensional external momenta of both particles. It is well-known that in this case, in the totally nonrelativistic limit, the scattering of a transverse photon on the electron or positron is described by the contact operator

$$V = \frac{e^2}{2m}\mathbf{A}^2. \tag{9}$$

Correspondingly, the double magnetic exchange is reduced

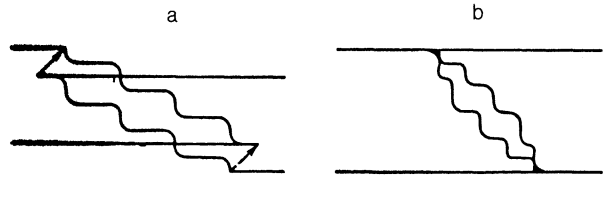


FIG. 4.

to the simple Fig. 4b with vertices generated by this operator. The calculation of the effective interaction arising in this way is of no difficulty. In the coordinate representation it is equal with logarithmic accuracy to

$$\delta V_M(\mathbf{r}) = -\frac{2\alpha^2}{m^2}\log\frac{1}{\alpha}\delta(\mathbf{r}). \tag{10}$$

The total Lamb-shift operator in positronium is

$$\delta V(\mathbf{r})=\delta V_C(\mathbf{r})+\delta V_M(\mathbf{r}) = \frac{6\alpha^2}{m^2}\log\frac{1}{\alpha}\delta(\mathbf{r}), \tag{11}$$

and the corresponding energy shift for a level with quantum numbers $n$, $l$ equals

$$\delta E_{nl} = \frac{6\alpha^2}{m^2}\log\frac{1}{\alpha}|\psi_{nl}(0)|^2 = \frac{3}{4}\frac{m\alpha^5}{\pi n^3}\log\frac{1}{\alpha}\delta_{l0}. \tag{12}$$

The answer obtained reproduces the part of the exact result for the Lamb-shift in positronium which is logarithmic in $\alpha$ found many years ago in Ref. 8 (see also Refs. 9, 10). The numerical difference between the logarithmic result (12) and the exact one is small for parapositronium (with total spin $S=0$): $\log 1/\alpha = 4.9$ is replaced by 4.7 in the exact result. The difference for orthopositronium ($S=1$) is larger: the coefficient analogous to $\log 1/\alpha = 4.9$, is equal to 3.0 in the exact result.

In this approach to the logarithmic contribution to the Lamb-shift in positronium, this lack of dependence on $S$ becomes quite obvious. For the correction $\delta V_C(\mathbf{r})$ related to the Coulomb interaction the spin of the infrared radiation does not enter. For the correction $\delta V_M(\mathbf{r})$ due to the double magnetic exchange the spin of the Thomson amplitude, i.e., of the nonrelativistic limit of the Compton scattering does not enter.

To conclude this section let us emphasize that just as in hydrogen, the logarithmic part of the Lamb-shift in positronium is completely described by the quantum electrodynamics of nonrelativistic particles. In other words, it is a true radiative correction to the nonrelativistic bound-state problem.

## 3. CALCULATION OF THE CORRECTION $\sim\alpha^4\log(1/\alpha)$ TO THE BOUND-STATE ENERGY

In this section we will calculate the correction $\sim\mu\alpha^6\log(1/\alpha)$ to the energy of a bound state of two particles with masses $m$ and $M$ (here and below $\mu = mM/(m+M)$ is the reduced mass). Considering particles of different masses allows one to have an additional check on the calculations, in particular by comparing with the results for the hydrogen atom[11] obtained to the first order in $m/M$ for $m \ll M$.

The direct nonrelativistic approach to the problem is to solve the Bethe–Salpeter equation with the expansion in $v/c$ in the momenta region which contributes to the energy correction in the order of interest to us. This gives rise to an equation which resembles the Breit equation, but which in contrast to the latter includes not only the corrections to the Hamiltonian of the order $v^2/c^2$, but those $\sim v^4/c^4$ as well. We emphasize that such an expansion is possible since we are interested in the logarithmic energy corrections, which can be calculated in the nonrelativistic momenta region $p \ll \mu$. This equation is satisfied by the solution of the Bethe–Salpeter equation projected onto the positive-energy states and integrated over the relative energy of the two particles.

However, the simplest way to derive the equation discussed and the corresponding energy correction is to proceed along the lines used for the derivation of the Breit equation (Ref. 6, Sec. 83). In this method the correction to the effective interaction Hamiltonian is found by expanding the scattering amplitude on the mass shell in $v/c$. In the present paper we will use just this approach, which in our opinion, not only is simpler, but also allows transparent physical interpretation of different contributions to the correction discussed.

In the nonrelativistic region of interest to us it is convenient to use noncovariant perturbation theory and the Coulomb gauge. Since this technique is not too common, let us present the corresponding Feynman rules. We assume that the particles have opposite charges. Then the exchange by a Coulomb quantum of a momentum $\mathbf{q}$ is described by a factor $-4\pi\alpha/q^2$, and the exchange of a magnetic quantum the factor $-\alpha_i\otimes\alpha_j S_{ij}(\mathbf{q})4\pi a/2q$, where $S_{ij}(\mathbf{q}) = \delta_{ij} - q_i q_j/q^2$ is the photon polarization density matrix. We recall that in the noncovariant perturbation theory the frequency of an intermediate photon equals its momentum. The projectors onto the positive and negative energy states of a fermion with a momentum $\vec{p}$ are correspondingly

$$\Lambda_\pm(\mathbf{p}) = \frac{1}{2}\left(1 \pm \frac{\alpha\mathbf{p}+\beta m}{\omega_p}\right),$$

where $\omega_p = \sqrt{m^2+p^2}$. In the expression for the effective potential the projector $\Lambda_-$ contributes an extra minus sign. Any intermediate state introduces the factor $(E - E_n + i0)^{-1}$ where $E_n$ is the energy of the intermediate state and $E$ is the energy of the system.

### 3.1. Pure Coulomb exchange

Let us start with the correction due to Coulomb exchange. The calculations will be performed in the momentum representation in the center of mass frame where the particle of mass $m$ has momentum $\mathbf{p}$, and that of mass $M$ has momentum $-\mathbf{p}$, with $E \approx m + M$. Writing the Dirac spinors as

$$u = \left(\frac{\omega_p+m}{2\omega_p}\right)^{1/2}\left(1+\frac{\alpha\mathbf{p}}{\omega_p+m}\right)\begin{pmatrix}\varphi\\0\end{pmatrix},$$

we find easily that the correction to the Hamiltonian can be written as

$$V_c(\mathbf{p},\mathbf{p}') = V_c^{(2)}(\mathbf{p},\mathbf{p}') + V_c^{(4)}(\mathbf{p},\mathbf{p}').$$

The second-order Breit correction $V_C^{(2)}(\mathbf{p},\mathbf{p}')$, including in particular the relativistic correction to the dispersion law, is

$$V_c^{(2)}(\mathbf{p},\mathbf{p}') = -\frac{p^4}{8}\left(\frac{1}{m^3}+\frac{1}{M^3}\right)(2\pi)^3\delta(\mathbf{q}) + \frac{\pi\alpha}{2}\left(\frac{1}{m^2}+\frac{1}{M^2}\right) - \frac{i\pi\alpha}{q^2}[\mathbf{p}\mathbf{p}']\left(\frac{\boldsymbol{\sigma}}{m^2}+\frac{\boldsymbol{\sigma}'}{M^2}\right), \tag{13}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the Pauli matrices for the first and second particles, and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. The next relativistic correction to the Breit Hamiltonian in $v^2/c^2$ due to the Coulomb exchange is

$$V_C^{(4)}(\mathbf{p},\mathbf{p}')=-4\pi\alpha\left[\frac{q^2}{64m^2M^2}+\frac{3}{64}\left(\frac{1}{m^4}+\frac{1}{M^4}\right)(p^2+p'^2)\right]$$
$$-\frac{\pi\alpha}{32q^2}\left[5\left(\frac{1}{m^4}+\frac{1}{M^4}\right)(p^2-p'^2)^2-\frac{8}{m^2M^2}(\boldsymbol{\sigma}[\mathbf{pp}'])(\boldsymbol{\sigma}'[\mathbf{pp}'])\right]. \tag{14}$$

The energy correction $\delta E_C'$ due to the operator $V_C^{(4)}$ equals its value averaged over nonrelativistic wave functions. Evaluating the integral with logarithmic accuracy in the region $\mu\alpha \ll p, p' \ll \mu$, we obtain

$$\delta E_C'=\varepsilon\mu^2\left[\frac{5}{4}\left(\frac{m}{M^3}+\frac{M}{m^3}\right)+\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}'}{6mM}\right], \tag{15}$$

where $\varepsilon\equiv\alpha^6(\mu^3/mM)\log(1/\alpha)\delta_{l0}/n^3$. The logarithmic contribution arises only from the last two terms from Eq. (14). Note that in the expansion in $p/m$ we get operators which do not lead to contributions logarithmic in $\alpha$, but lead to momentum integrals that diverge linearly, not logarithmically at the upper limit (e.g., the term $\sim p^6$ from the expansion of $\omega_p$). Those operators give rise to corrections $\sim\alpha^5$ and hence can be omitted.

The contribution to the energy $\sim\mu\alpha^6\log(1/\alpha)$ from the Breit Hamiltonian $V_C^{(2)}$ arises in the second order perturbation theory, i.e., from iterating $V_C^{(2)}$:

$$\delta E_C''=\int\frac{d\mathbf{p}\,d\mathbf{p}'\,d\mathbf{P}\,d\mathbf{P}'}{(2\pi)^{12}}\psi^*(\mathbf{p})V_C^{(2)}(\mathbf{p},\mathbf{P})G(\mathbf{P},\mathbf{P}'|E)V_C^{(2)}$$
$$(\mathbf{P}',\mathbf{p}')\psi(\mathbf{p}'). \tag{16}$$

Here $G(\mathbf{P},\mathbf{P}'|E)$ is the nonrelativistic Coulomb Green's function. Actually, we only need the zeroth and the first terms of its expansion in $\alpha$:

$$G^{(0)}(\mathbf{P},\mathbf{P}'|E)=\frac{(2\pi)^3\delta(\mathbf{P}-\mathbf{P}')}{E-P^2/2\mu}, \tag{17}$$

$$G^{(1)}(\mathbf{P},\mathbf{P}'|E)=\frac{1}{E-P^2/2\mu}\frac{-4\pi\alpha}{(\mathbf{P}-\mathbf{P}')^2}\frac{1}{E-P'^2/2\mu}. \tag{18}$$

Simply counting the powers of the momenta in the integrand of (16) demonstrates that only the function $G^{(1)}$ contributes to the logarithm. So we find with logarithmic accuracy

$$\delta E_C''=-\varepsilon\frac{\mu^2}{4m^3M^3}(M-m)^4. \tag{19}$$

This contribution is evidently spin-independent, since the only spin-dependent term in expression (13) is the spin-orbit interaction, which is absent in the $S$-state.

Then, there is a contribution due to the negative-energy intermediate states. It is described by diagrams like that of Fig. 5. Since some intermediate states in these diagrams are "heavy," i.e., satisfy $|E-E_n|\sim 2m$, $2M\gg p$, the line going in the opposite direction contracts into a point. Diagrams like that on Fig. 5 lead to the effective operator

$$\bar{V}_C(\mathbf{p},\mathbf{p}')=-\frac{(\pi\alpha)^2}{8}\left(\frac{1}{m^3}+\frac{1}{M^3}\right)q. \tag{20}$$

while all the other diagrams with heavy intermediate states can be shown to be nonlogarithmic. The perturbation (20) gives the energy shift

$$\delta\bar{E}_C=-\varepsilon\left(\frac{m}{M}+\frac{M}{m}-4\right). \tag{21}$$

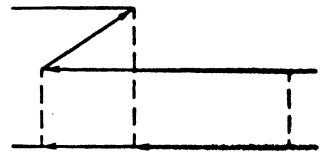

FIG. 5.

Adding up (15), (19), and (21), we get the total energy correction due to Coulomb exchange:

$$\delta E_C=\varepsilon\frac{\mu^2}{mM}\left(-\frac{3}{2}+\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}'}{6}\right). \tag{22}$$

In complete accord with the exact solution of the Dirac equation in the Coulomb field, this correction, which is logarithmic in $\alpha$, vanishes when one of the particles becomes infinitely heavy. The spin-dependent part of expression (22) has been found previously in Refs. 2, 12, and 13.

### 3.2. Exchange of a single magnetic photon

Let us consider now the contribution to the energy due to the exchange of one magnetic photon. Before we take the expectation value over the Dirac spinors the interaction operator looks as follows

$$V_M=-\frac{4\pi\alpha}{2q}S_{ij}(\mathbf{q})\alpha_i\otimes\alpha_j$$
$$\left[\frac{1}{E-\Omega_{p'}-\omega_p-q}+\frac{1}{E-\Omega_p-\omega_{p'}-q}\right], \tag{23}$$
$$\Omega_p=(M^2+p^2)^{1/2}.$$

We are interested in the region of the variables $q\gg|\Delta E|=|E-\Omega_{p'}-\omega_p|$. The energy corrections arise both from expanding in $v^2/c^2$ of the expectation value of the operator $\alpha_i\otimes\alpha_j$, and from expanding the denominators in the ratio $\Delta E/q$. Taking the leading terms in the numerator and denominator, we obtain the well-known magnetic contribution to the Breit Hamiltonian:

$$V_M=\frac{\pi\alpha}{mMq^2}\left[4\frac{[\mathbf{pp}']^2}{q^2}-q^2\boldsymbol{\sigma}\boldsymbol{\sigma}'+(\mathbf{q}\boldsymbol{\sigma})(\mathbf{q}\boldsymbol{\sigma}')\right.$$
$$\left.+2i[\mathbf{pp}'](\boldsymbol{\sigma}+\boldsymbol{\sigma}')\right]. \tag{24}$$

To find the contribution of this operator to the energy shift, one has to calculate the second order of the perturbation theory, taking $V_M$ as one of the operators and $V_C^{(2)}$ as the other. Now the logarithmic contributions arise due to the functions $G^{(0)}$ and $G^{(1)}$. These energy corrections are

$$\delta E_{1M}^{(1)}=-2\varepsilon\left(1-\frac{4\mu^2}{mM}\right), \tag{25}$$

$$\delta E_{1M}^{(2)}=\frac{2}{3}\varepsilon\boldsymbol{\sigma}\boldsymbol{\sigma}'\left(1-\frac{4\mu^2}{mM}\right). \tag{26}$$

Retaining the next term of the expansion in $v^2/c^2$ in the matrix element of the operator $\alpha_i\otimes\alpha_j$, we easily get for this relativistic correction

$$\delta E_{1M}^{(3)}=4\varepsilon\left(1+\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}'}{6}\right)\left(1-\frac{2\mu^2}{mM}\right). \tag{27}$$

Let us turn to the expansion of the denominator in Eq.

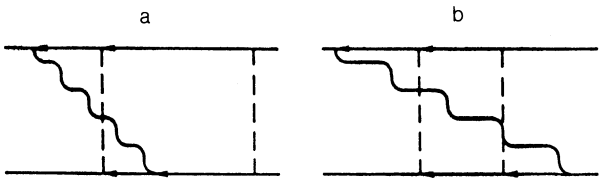


FIG. 6.

(23) in powers of $\Delta E/q$. Its first term gives the contribution $\sim\mu\alpha^5\log(1/\alpha)$ to the Lamb shift. This correction originates from the region $p,p'\sim\mu\alpha$, $\mu\alpha^2\ll q\ll\mu\alpha$. The next term of the expansion gives the necessary contribution due to the region $\mu\alpha\ll p,p'\ll\mu$:

$$\delta E_{1M}^{(4)}=2\varepsilon\frac{\mu^2}{mM}\left(1+\frac{2}{3}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right). \tag{28}$$

Now we have to consider the possibility of Coulomb exchange between the emission and absorption of a magnetic photon, where both particles stay in the positive-energy states. Counting the powers of the momentum in the integrand demonstrates that the logarithmic contribution originates from Figs. 6a,b with the exchange of one and two Coulomb quanta in the intermediate state. The energy denominators here are of the form $E-E'-k$ where $E'$ is the energy of the particles in the intermediate state and $k$ is the photon energy. The logarithmic contributuion arises from the energy region $k\gg|E-E'|$. Therefore the denominators can be expanded in the ratio $\Delta E/k$. In the case of the single Coulomb exchange the leading term of the expansion gives the Lamb-shift correction $\sim\mu\alpha^5\log(1/\alpha)$. One could expect the energy correction of the necessary order of magnitude, $\mu\alpha^6\log(1/\alpha)$, to appear if the next term of the expansion in $\Delta E/k$ is included. It would evidently correspond to the correction to the Lamb-shift of first order in $v/c$. However, there is one more correction of the same order of magnitude originating from Fig. 6b. Its meaning is the expansion of the Green's function not in $\Delta E/k$, but in the ratio of the Coulomb potential to $k$. Meanwhile a relativistic correction should start from $v^2/c^2\sim\alpha^2$. Therefore the total contribution of Figs. 6a and 6b to the energy correction we are interested in, is equal to zero. The vanishing of this contribution is also confirmed by the direct calculation.[5] (Unfortunately, in our paper[5] Figs. 2d and 2c were interchanged).

At last, let us consider the energy shift due to the transitions to the negative-energy states. The corresponding diagrams of the noncovariant perturbation theory are shown in Fig. 7. As it was mentioned above, the line corresponding to a particle of negative energy can be contracted into a point. In this case an effective vertex arises corresponding to the emission of Coulomb and magnetic quanta. Such an interaction is evidently of a spin nature. After simple calculations we find the effective operator corresponding to the diagrams in Fig. 7,

$$V_M^{(5)}=-\frac{(\pi\alpha)^2q}{6\mu mM}\boldsymbol{\sigma}\boldsymbol{\sigma}', \tag{29}$$

and the energy correction induced by it:

$$\delta E_{1M}^{(5)}=-\frac{4}{3}\varepsilon\boldsymbol{\sigma}\boldsymbol{\sigma}'. \tag{30}$$

Adding up the corection $\delta E_{1M}^{(1-5)}$ we get the total contribution to the energy from the single magnetic exchange:

$$\delta E_{1M}=\varepsilon\left(2+2\frac{\mu^2}{mM}-\frac{8}{3}\frac{\mu^2}{mM}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right). \tag{31}$$

Its spin-dependent part was calculated previously.[2] The first term in the brackets agrees with the corresponding correction for hydrogen found recently.[11]

### 3.3. Double magnetic exchange

Our consideration of the double-magnetic-exchange contribution to the energy shift will start from the second-order perturbation theory in $V_M$ [cf. Eq. (16), where $V_M$ should now replace $V_C^{(2)}$]. Again we have to take into account in the Green's function $G$ the terms $G^{(0)}$ and $G^{(1)}$ only. In the first case the energy correction is

$$\delta E_{2M}^{(1)}=\varepsilon\frac{\mu^2}{mM}(1+3\boldsymbol{\sigma}\boldsymbol{\sigma}'), \tag{32}$$

in the second one

$$\delta E_{2M}^{(2)}=-\varepsilon\frac{\mu^2}{mM}\left(\frac{3}{2}-\frac{5}{6}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right). \tag{33}$$

Let us consider now the contributions of the negative-energy states, starting from the case presented in Fig. 8 when only one particle goes over into the negative-energy state. In our approximation such a zigzag in a diagram contracts into the vertex described by the two-photon operator $e^2\mathbf{A}^2/2m$ of the nonrelativistic electrodynamics. Since this vertex is spin-independent, the perturbation operator originating from the diagrams presented in Fig. 8 is also spin-independent:

$$V_{2M}^{(3)}=-\frac{(\pi\alpha)^2q}{8\mu mM}. \tag{34}$$

The corresponding energy correction equals

$$\delta E_{2M}^{(3)}=-\varepsilon. \tag{35}$$

Let us consider now the case when both particles go over into the negative-energy states (see Fig. 9). By the same reasoning as above, these contributions are spin-independent. If one retains the photon energy $k+q$ only in the ener-

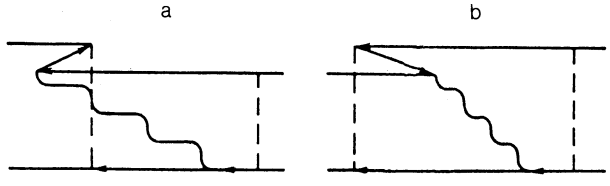


FIG. 7.

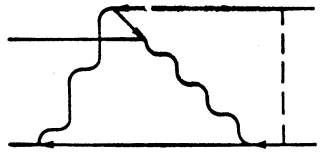

FIG. 8.

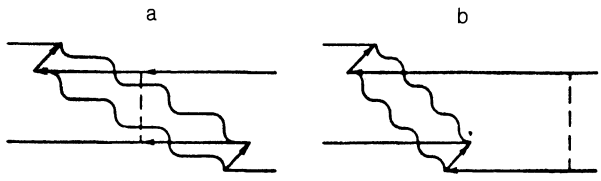


FIG. 9.

gy denominators of Fig. 9a corresponding to two particles with positive energies, a Lamb-shift contribution (10) of order $\mu\alpha^5 \log(1/\alpha)$ arises. As in the case of single magnetic exchange, the corrections to the Lamb-shift which are first order in $v/c$ arising due to the next term of the expansion of Fig. 9a in $\Delta E/(k+q)$, exactly cancel with the next correction of the same order of magnitude originating from Fig. 9b, its meaning being the expansion of the Green function in the ratio of the Coulomb potential to $k+q$. Direct calculation (see Ref. 5) also confirms that the total contribution of Fig. 9 to the energy correction $\sim\mu\alpha^6 \log(1/\alpha)$ vanishes.

In addition, we have checked that the diagrams where magnetic quanta are emitted and absorbed by the same particle, do not contribute to the correction of interest to us.

Therefore, the total contribution of the double magnetic exchange equals

$$\delta E_{2M}=\varepsilon\left(-1-\frac{1}{2}\frac{\mu^2}{Mm}+\frac{23}{6}\frac{\mu^2}{Mm}\sigma\sigma'\right). \tag{36}$$

The spin-dependent part of this expression was found previously in Ref. 2. The first term in the brackets, $-1$, agrees with the result for hydrogen obtained in Refs 11, 14.

## 4. DISCUSSION

Thus, our total result for the contribution $\sim\mu\alpha^6 \log(1/\alpha)$ to the energy of the two-body bound state in QED, the sum of (22), (31), and (36), equals

$$\delta E(n,l)=\frac{\mu^2}{Mm}\alpha^6 \log\frac{1}{\alpha}\left(1+\frac{4}{3}\frac{\mu^2}{Mm}\sigma\sigma'\right)\frac{\delta_{l,0}}{n^3}. \tag{37}$$

Note that this energy shift is in fact a relativistic correction. The relativistic origin of the contributions arising to the second order in the terms $\sim v^2/c^2$ in the Breit equation and to first order in the corrections $\sim v^4/c^4$, is self-evident. For other contributions, due to negative-energy states, this assertion is somewhat more a matter of convention.

As for positronium, one should add to (37) the annihilation contribution,[15]

$$\delta E_a(n,l)=\frac{1}{96}m\alpha^6 \log\frac{1}{\alpha}(3+\sigma\sigma')\frac{\delta_{l,0}}{n^3}. \tag{38}$$

Since the annihilation operator reduces with the required accuracy to the same form as the contact magnetic spin-spin interaction in the Briet equation (see Ref. 6, Sec. 83), the calculation of this correction can be also easily performed within the approach used here.

The final result for positronium is

$$\delta E(n,l)=\frac{5}{96}m\alpha^6 \log\frac{1}{\alpha}(3+\sigma\sigma')\frac{\delta_{l,0}}{n^3}, \tag{39}$$

which can be rewritten also in a more compact form:

$$\delta E(n,l,s)=\frac{5}{24}m\alpha^6 \log\frac{1}{\alpha}\frac{\delta_{l,0}\delta_{s,1}}{n^3}. \tag{40}$$

In other words, this correction in positronium is nonvanishing only for triplet $S$-states.

Our result (40) differs from the recent one,[16]

$$\delta E(n,l)=\frac{1}{96}m\alpha^6 \log\frac{1}{\alpha}(3+5\sigma\sigma')\frac{\delta_{l,0}}{n^3}, \tag{41}$$

obtained via a relativistic two-particle equation for positronium. The absence of the results for separate contributions in Ref. 16, as well as the difference in the calculation technique, makes it difficult to investigate the reasons of the disagreement. However, based on private communications with R. Fell, our results agree for the pure Coulomb and single-magnetic exchanges.

Adding (40) to the known contributions of orders lower in $\alpha$, we obtain the following theoretical values for transition frequencies in positronium (in MHz):

$$E(2^3S_1)-E(1^3S_1)=1\,233\,607\,211{,}7; \tag{42}$$

$$E(2^3S_1)-E(2^3P_2)=8\,627{,}7; \tag{43}$$

$$E(2^3S_1)-E(2^3P_1)=13\,013{,}3; \tag{44}$$

$$E(2^3S_1)-E(2^3P_0)=18\,498{,}5. \tag{45}$$

The correction (40) contributes $-16.7$ MHz to the first of these frequencies and 2.4 MHz to all others.

A comparison with the experimental values

$$E(2^3S_1)-E(1^3S_1)=1\,233\,607\,218{,}9\,(10{,}9)\ \ [17]; \tag{46}$$

$$E(2^3S_1)-E(2^3P_2)=8\,628{,}4\,(2{,}8)\,[18];\ 8\,619{,}6\,(2{,}7)\ (0{,}9)\,[19]; \tag{47}$$

$$E(2^3S_1)-E(2^3P_1)=13\,001{,}3\,(3{,}9)\ (0{,}9)\ \ [19]; \tag{48}$$

$$E(2^3S_1)-E(2^3P_0)=18\,504{,}1\,(10{,}0)\ (1{,}7)\ \ [19] \tag{49}$$

shows that the calculated correction will be essential for the next generation of experiments.

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