

Surface conductivity of a metal: no- τ analysis

M. I. Kaganov, G. Ya. Lyubarskii, and J. Czerwonko

Kapitsa Institute for Physical Problems, Russian Academy of Sciences, Moscow; Physicotechnical Institute, Ukrainian Academy of Sciences, Khar'kov; and Instytut Fizyki, Politechnika Wroclawska, Poland
(Submitted 2 April 1992)

Zh. Eksp. Teor. Fiz. **102**, 1563–1570 (November 1992)

By including a p -scattering gain term in the collision integral, the near-surface conductivity $\sigma(z)$ is calculated for electrons reflected diffusely from the surface $z = 0$ of a bulk metallic sample. The mean-conductivity correction for a plate of thickness $D \gg l$, where l is the mean free path, and the asymptotic behavior of $\sigma(z)$ for $z \gg l$ are evaluated. It is shown that the average probability for scattering from the boundary is influenced by the nature of the collisions in the bulk of the metal.

Most of the studies on the electrical conductivity of thin metallic plates focus particular attention on the interaction between the electrons and the sample surface.¹⁻⁵ When the plate is very thin as compared to the loss-related electron mean free path l ($D \ll l$) then the scattering of electrons by surface roughnesses is the dominant dissipation mechanism. Even then, however, it has been shown⁶ that bulk collisions contribute to more than just the loss lifetime $\tau = l/v_F$, thus rendering the τ approximation suspect. The thin-plate ($D \ll l$) conductivity formulas of Ref. 6 were derived for extremely degenerate electrons with an isotropic dispersion law under the assumptions (i) that the (bulk) electron scattering probability $W(\theta)$ is an arbitrary function of the angle θ , and (ii) that the scattering of an electron from the (internal) surface of the sample is described by the Fuchs parameter¹ generally dependent on the electron incidence angle. One important result of Ref. 6 is that $\bar{\sigma}$, conductivity averaged over the thickness admits of no general formula of the type

$$\bar{\sigma} = \sigma f(l/D),$$

where σ is the "usual" bulk conductivity as calculated for $D/l \rightarrow \infty$, see below. Even for $D \ll l$, it is shown⁶ that the function f depends on quantities associated with the gain (or inscattering) collision term inherently neglected in the τ approximation.

Recently,⁷ our study of the surface impedance ζ of a metallic half-space revealed a very similar feature. It was found, namely that in contrast to normal skin-effect conditions, the mean-free-path dependence of ζ is not obtainable by simply replacing l by the transport mean free path l_{tr} . In the anomalous skin-effect regime, the surface impedance depends both on l and on some quantity related to the gain term of the collision integral. Notably, though, bulk collision characteristics are totally absent from the surface impedance expression in the *extreme anomalous limit* as $l/\delta \rightarrow \infty$, where $\delta = c/(2\pi\sigma\omega)^{1/2}$, c the speed of light and, ω the electromagnetic wave frequency.

The present analysis examines the propagation of near-surface direct current in a metal and follows Ref. 7 in assuming that the conduction-electron gas is degenerate and isotropic, and that, furthermore,

$$W(\theta) = W_0(1 + \alpha \cos \theta), \quad |\alpha| < 1. \quad (1)$$

Let the metal occupy a half-space $z > 0$. From the condition

$$\text{rot } \mathbf{E} = 0$$

it follows that

$$E_\alpha = \text{const},$$

where $\alpha = x, y$. To linear order in the electric field \mathbf{E} , it is shown in Ref. 7 that the nonequilibrium part f_1 of the electron distribution function may be written as

$$f_1 = -ev \frac{\partial f_F}{\partial \epsilon} \chi(z, \vartheta, \varphi), \quad \chi(z, \vartheta, \varphi) = \chi(z, \vartheta) \cos \varphi,$$

where f_F is the (equilibrium) Fermi function and $\chi(z, \vartheta)$ satisfies the equation

$$\cos \vartheta \frac{\partial \chi(z, \vartheta)}{\partial z} + \frac{1}{l} \chi(z, \vartheta) = \left(E + \frac{\alpha}{3\sigma} j(z) \right) \sin \vartheta. \quad (2)$$

In the above, ϑ and φ are the spherical angles in the electron \mathbf{p} space, the angle ϑ is measured from the z axis (alias the surface normal), and the x axis is directed along \mathbf{E} . Using the δ -function approximation for $-(\partial f_F / \partial \epsilon)$, the electric current density is found to be¹⁾

$$j(z) = \frac{3\sigma}{4l} \int_0^\pi \chi(z, \vartheta) \sin^2 \vartheta d\vartheta, \quad \sigma = \frac{ne^2 l}{P_F}. \quad (3)$$

Now since the influence of the boundary is not felt in the sample interior ($z \gg l$), we may write

$$\chi|_{z \rightarrow \infty} \equiv \chi_\infty(\vartheta) = \left(E + \frac{\alpha}{3\sigma} j_\infty \right) l \sin \vartheta. \quad (4)$$

which when combined with (3) and (4) yields

$$j_\infty = \sigma_r E, \quad \sigma_r = \sigma / (1 - \alpha/3). \quad (5)$$

If we let

$$\begin{aligned} \chi(z, \vartheta) &= \chi_\infty(\vartheta) + \psi(z, \vartheta), \\ \chi_\infty(\vartheta) &= E l_{tr} \sin \vartheta, \quad l_{tr} = l / (1 - \alpha/3). \end{aligned} \quad (6)$$

then $\psi(z, \vartheta)$ satisfies the equation

$$\cos \vartheta \frac{\partial \psi}{\partial z} + \frac{1}{l} \psi(z, \vartheta) = \frac{\alpha}{3\sigma} [j(z) - j_\infty] \sin \vartheta, \quad (7)$$

with the boundary-induced current-density perturbation in the right-hand side. Note that

$$\lim_{z \rightarrow \infty} j(z) = j_\infty, \quad \lim_{z \rightarrow \infty} \psi(z) = 0. \quad (8)$$

Thus far, the boundary conditions to be satisfied by the electron distribution function at $z = 0$ have not yet been discussed. While in the specular-reflection limit the near-surface conductivity does not differ from its bulk value, diffusive reflection naturally maximizes this difference.¹ We accordingly assume that the electrons are reflected diffusely, so that

$$\chi(0, \vartheta) = 0 \quad \text{for} \quad \cos \vartheta \leq 0. \quad (9)$$

From Eqs. (6) through (9),

$$\psi(z, \vartheta) = \begin{cases} \frac{\alpha}{3\sigma} \int_0^z [j(z) - j_\infty] \frac{\sin \vartheta}{\cos \vartheta} \exp\left(-\frac{z-z'}{l \cos \vartheta}\right) dz' \\ -El_r \exp\left(-\frac{z}{l \cos \vartheta}\right) \sin \vartheta, & \cos \vartheta > 0. \\ \frac{\alpha}{3\sigma} \int_z^\infty [j(z) - j_\infty] \frac{\sin \vartheta}{|\cos \vartheta|} \exp\left(-\frac{z'-z}{l |\cos \vartheta|}\right) dz' \\ \cos \vartheta < 0. \end{cases} \quad (10)$$

Using the definition (3) [see also (6)], an integral equation for the current density $j(z)$ may now be derived. Noting that the electric field is independent of z , it is helpful to introduce the *specific conductivity*

$$\sigma(z) = j(z)/E.$$

which satisfies the integral equation

$$\begin{aligned} \sigma(z) - \sigma_{ir} - \frac{\alpha}{3l} \int_0^\infty [\sigma(z') - \sigma_{ir}] K(z-z') dz' \\ = -\frac{3}{4} \sigma_{ir} \int_0^{\pi/2} \sin^3 \vartheta \exp\left(-\frac{z}{l \cos \vartheta}\right) d\vartheta. \end{aligned} \quad (11)$$

where we have defined

$$K(z) = \frac{3}{4} \int_0^{\pi/2} \frac{\sin^3 \vartheta}{\cos \vartheta} \exp\left(-\frac{|z|}{l \cos \vartheta}\right) d\vartheta. \quad (12)$$

There are several points which must now be made:

a) it is only because of the gain term present in the collision integral that the above equation for $\sigma(z)$ is of integral form; for $\alpha = 0$,

$$\sigma(z) = \sigma \left[1 - \frac{3}{4} \int_0^{\pi/2} \sin^3 \vartheta \exp\left(-\frac{z}{l \cos \vartheta}\right) d\vartheta \right]. \quad (13)$$

b) the correction to the thickness-averaged (or mean) conductivity

$$\bar{\sigma} = \frac{1}{D} \int_0^D \sigma(z) dz$$

is determined for $D \gg l$ by the integral

$$\int_0^\infty u(z) dz,$$

where

$$u(z) = \sigma_{ir} - \sigma(z).$$

In fact,

$$\begin{aligned} \bar{\sigma} = D^{-1} \int_0^D \sigma(z) dz = D^{-1} \int_0^D \sigma_{ir} dz \\ - \frac{2}{D} \int_0^D u(z) dz \approx \sigma_{ir} - \frac{2}{D} \int_0^\infty u(z) dz, \end{aligned} \quad (14)$$

the last approximation being valid if $D \gg l$. The correction is exponentially small [i.e., $\propto e^{-D/l}$]. In writing (14), diffusive reflection *from both sides of the plate* have been assumed.

c) For α values lying in the range $|\alpha| < 1$ but otherwise arbitrary, Eq. (11) does not contain any small quantities capable of providing an approximate solution to the problem.²⁾ This is most readily shown by changing to the dimensionless variable

$$\xi = z/l$$

and introducing a dimensionless function $v(\xi)$ such that

$$u(z) = \frac{3}{4} \sigma_{ir} v(\xi). \quad (15)$$

The function $v(\xi)$ satisfies the equation

$$v(\xi) - \frac{\alpha}{3} \int_0^\infty K(\xi - \xi') v(\xi') d\xi' = \int_0^1 (1-s^2) e^{-\xi/s} ds, \quad \xi > 0, \quad (16)$$

in which

$$K(\xi) = \frac{3}{4} \int_0^1 (s^{-1} - s) e^{-\xi/s} ds, \quad (16')$$

and which does not contain any parameters other than α . Once (16) is solved, the mean conductivity follows as

$$\bar{\sigma} = \sigma_{ir} - \frac{3}{4} \sigma_{ir} \frac{l}{D} \int_0^\infty v(\xi) d\xi, \quad (17)$$

according to (14) and (15), and the asymptotic behavior of the current density $j(z)$ for $z \gg l$ is found from

$$j(z) = \sigma_{ir} \left[1 - \frac{3}{4} v\left(\frac{z}{l}\right) \right] E. \quad (18)$$

The steps of the derivation are outlined in the Appendix, and here we only quote the results.

Let us introduce the notation

$$A(\alpha) = \frac{16}{3\pi\alpha} \int_1^\infty \frac{dx}{x^2} \operatorname{arctg} \Phi(x, \alpha) \quad (19)$$

for the integral in (17). The asymptotic form of $v(\xi)$ for $\xi \gg 1$ is

$$v(\xi) \sim \frac{B(\alpha)}{2\pi} \frac{e^{-\xi}}{\xi^2}, \quad \xi \gg 1,$$

$$B(\alpha) = -\frac{16\pi}{3} \frac{1-\alpha/3}{(1-\alpha/2)^2} \exp\left\{\frac{1}{\pi} \int_1^\infty \frac{dx}{x+1} \operatorname{arctg} \Phi(x, \alpha)\right\}. \quad (20)$$

The function $\Phi(x, \alpha)$ in (19) and (20) is given by

$$\Phi(x, \alpha) = \frac{\alpha\pi}{4x^3} \frac{x^2-1}{1-(\alpha/3)[(1-1/x^2)\ln[(x+1)/(x-1)]+2/x]}. \quad (21)$$

If required, $A(\alpha)$ and $B(\alpha)$ may be obtained numerically. For $\alpha=0$ they are both nonzero: $A(0)=1/3$, $B(0)=-16\pi/3$.

DISCUSSION

Because of the absence of any limiting cases to consider, the present formulation not only necessitates, not unexpectedly, that some amount of numerical work be done [to calculate $A(\alpha)$ and $B(\alpha)$, to be specific], but also leads to results which, on the face of it, do not differ qualitatively from their τ approximation counterparts: For $\alpha=0$, for example, the ratio

$$(\bar{\sigma}-\sigma_{tr})/\sigma_{tr}$$

is again inversely proportional to the plate thickness D , and the asymptotic behavior of $\sigma(z)$ for $z \gg l$ is the same for all in α ; as implied by (20), the α dependence only enters through the factor $B(\alpha)$.

In our view, however, the results are important in that they predict the surface scattering intensity to depend substantially on the nature of bulk collisions. Rewriting (17) as

$$\bar{\sigma} = \frac{1}{\bar{\sigma}} = \frac{p_F}{ne^2} \frac{1}{l}, \quad \frac{1}{l} = \frac{1}{l_{tr}} + \frac{3}{4} \left(1 - \frac{\alpha}{3}\right) \frac{A(\alpha)}{D}, \quad D \gg l$$

we see that the second term in $1/\bar{l}$ may naturally be thought of as the surface scattering probability per unit length and that, importantly, it is quite α dependent. If the p -scattering contribution happens to depend on temperature T , this will add—through α —to the temperature dependence of $1/\bar{l}$.

It should be emphasized that the α dependence of surface impedance parameters is virtually unpredictable and may hardly be “guesstimated” (see Ref. 7). It must be admitted therefore that when posed for a specific metal, the problem can only be treated with approximate models and, most important, depends on numerical work for its solution.

M. I. K. is grateful to the members of the International Laboratory for Low Temperatures and Large Magnetic Fields (Wrocław, Poland) for the hospitality given to him during the course of this work. J. C. would like to acknowledge the financial support of The Polish Scientific Research Committee (Project No. 209439101).

APPENDIX

1. Both of the problems in the text are simply solved if the Fourier transform

$$\bar{v}(\kappa) = \int_0^\infty v(\xi) e^{i\kappa\xi} d\xi \quad (22)$$

of the required function $v(\xi)$ is known: as implied by (17), the mean conductivity is determined by $\bar{v}(0)$. To obtain the asymptotic form of $v(\xi)$ it is convenient to employ the inversion formula

$$v(\xi) = (2\pi)^{-1} \int_{-\infty}^\infty \bar{v}(\kappa) e^{-i\kappa\xi} d\kappa. \quad (23)$$

As we will see, $\bar{v}(\kappa)$ may be analytically continued to the lower half-plane with a branch cut $(-i\infty, -i)$, the point $\kappa = -i$ appearing as a branch point for $\bar{v}(\kappa)$. Let $\bar{v}(\kappa+0)$ and $\bar{v}(\kappa-0)$ be the values of $\bar{v}(\kappa)$ on the right and left rims of the cut, respectively. By deforming the integration contour in (23) we then obtain

$$v(\xi) = (2\pi)^{-1} \int_{-i}^{-i\infty} [\bar{v}(\kappa+0) - \bar{v}(\kappa-0)] e^{-i\kappa\xi} d\kappa.$$

The asymptotic properties of this type of integral are known to be determined by the behavior of the pre-exponential near $\kappa = -i$.

2. The Fourier transform $\bar{v}(\kappa)$ of $v(\xi)$ can be obtained by applying the standard Wiener-Hopf method with the result that

$$\bar{v}(\kappa) = H(\kappa) \exp(-G_+(\kappa)). \quad (24)$$

where

$$H(\kappa) = (2\pi i)^{-1} \int_{-i\epsilon-\infty}^{-i\epsilon+\infty} \frac{d\xi}{\xi-\kappa} \exp(G_-(\kappa)) Q(\kappa), \quad \operatorname{Im} \kappa > -\epsilon, \quad (25)$$

$$Q(\kappa) = \int_{-i\infty}^{-i} \frac{dt}{t-\kappa} (t^{-2} + t^{-4}) \quad (26)$$

$$G_+(\kappa) = (2\pi i)^{-1} \int_{-i\epsilon-\infty}^{-i\epsilon+\infty} \frac{d\xi}{\xi-\kappa} \ln F(\xi), \quad \operatorname{Im} \xi > -\epsilon,$$

$$G_-(\kappa) = (2\pi i)^{-1} \int_{i\epsilon-\infty}^{i\epsilon+\infty} \frac{d\xi}{\xi-\kappa} \ln F(\xi), \quad \operatorname{Im} \xi < \epsilon. \quad (27)$$

The (positive) quantity ϵ is sufficiently small that the strip $|\operatorname{Im} \kappa| < \epsilon$ contains not even one zero of $F(\xi)$. The function $F(\xi)$ is defined by

$$F(\kappa) = 1 - \frac{\alpha}{3} \bar{K}(\kappa), \quad (28)$$

and $\bar{K}(\kappa)$ is the Fourier transform of the integral kernel $K(\xi)$,

$$\bar{K}(\kappa) = \frac{3}{2} \int_0^1 \frac{1-s^2}{1+\kappa^2 s^2} ds. \quad (29)$$

3. The expression (24) for $\bar{v}(\kappa)$ is conveniently simplified by performing the analytic continuation of $\bar{K}(\kappa)$, $Q(\kappa)$, $F(\kappa)$, and $G_{\pm}(\kappa)$. The function $\bar{K}(\kappa)$ as defined by (29) is analytic in the complex κ plane with cuts $(-i\infty, -i)$ and $(i, i\infty)$. For $\kappa > 1$ we have the Laurent expansion

$$\bar{K}(\kappa) = \frac{3\pi}{4\kappa} - \frac{3}{\kappa^2} + \frac{3\pi}{4\kappa^3} - \frac{4}{\pi\kappa^4} \psi(\kappa),$$

$$\psi(\kappa) = \sum_{k=0}^{\infty} \left(-\frac{1}{\kappa^2}\right)^k \frac{1}{(2k+1)(2k+3)}, \quad (30)$$

which converges in the ring $|\kappa| > 1$ and defines in it an analytic function $K_+(\kappa)$ identical to $\bar{K}(\kappa)$ for

$$\text{Re } \kappa > 0, \quad |\kappa| > 1.$$

If $\kappa < -1$, $\bar{K}(\kappa)$ is described by another Laurent expansion,

$$K_-(\kappa) = -\frac{3\pi}{4\kappa} - \frac{3}{\kappa^2} - \frac{3\pi}{4\kappa^3} - \frac{3}{2\kappa^4} \psi(\kappa). \quad (31)$$

The functions $K_+(\kappa)$ [$K_-(\kappa)$] may be analytically continued to the entire right [left] half-planes by setting

$$K_-(\kappa) = \bar{K}(\kappa), \quad \text{Re } \kappa < 0,$$

$$K_+(\kappa) = \bar{K}(\kappa), \quad \text{Re } \kappa > 0. \quad (32)$$

From the definition of the functions $K_{\pm}(\kappa)$ we see that

$$K_+(-\kappa) = K_-(\kappa), \quad K_+(\kappa) - K_-(\kappa) = \frac{3\pi}{2} \left(\frac{1}{\kappa} + \frac{1}{\kappa^3} \right). \quad (33)$$

The last relationship enables $K_+(\kappa)$ and $K_-(\kappa)$ to be analytically continued to the complex κ plane with a cut $(-i, i)$.

The functions

$$F_{\pm}(\kappa) = 1 - \frac{\alpha}{3} K_{\pm}(\kappa) \quad (34)$$

are analytic in the same region as $K_{\pm}(\kappa)$.

The functions $G_{\pm}(\kappa)$ are analytic in the respective half-planes $\text{Im}\kappa > -\varepsilon$ and $\text{Im}\kappa < \varepsilon$ and may be analytically continued since

$$G_+(\kappa) - G_-(\kappa) = \ln F(\kappa). \quad (35)$$

This analytically continues $G_+(\kappa)$ [$G_-(\kappa)$] to the lower [upper] half κ plane with a cut $(-i\infty, -i)$ [$i, i\infty$], and we note that

$$G_+(\kappa-0) - G_+(\kappa+0) = \ln \frac{F_-(\kappa)}{F_+(\kappa)}. \quad (36)$$

The way it is defined in (26), the function $Q(\kappa)$ is analytic in the κ plane with a cut $(-i\infty, -i)$, and the difference of the values of $Q(\kappa)$ on both rims of the cut is

$$Q(\kappa-0) - Q(\kappa+0) = 2\pi i (\kappa^{-2} + \kappa^{-4}). \quad (37)$$

With the results above, the integral (25) can be written

$$H(\kappa) = \frac{16}{3\alpha\kappa} [\exp(G_+(\kappa)) - \exp(G_-(0))], \quad (38)$$

following the same argument as in Ref. 7.

From (32) it follows that

$$\bar{v}(\kappa) = \frac{16}{3\alpha\kappa} \{1 - \exp[G_+(0) - G_+(\kappa)]\} \quad (39)$$

which when combined with (32) automatically provides an analytical continuation of $H(\kappa)$ and $\bar{v}(\kappa)$ to the κ plane with a cut $(-i\infty, -i)$.

4. We now evaluate $\bar{v}(0)$. From (24) and (38),

$$\bar{v}(0) = \frac{16}{3\alpha} G_+'(0). \quad (40)$$

Now from (27) we have

$$G_+'(0) = (2\pi i)^{-1} \int_{-i\infty}^{-i\infty+\infty} \frac{d\xi}{\xi^2} \ln F(\xi) = (2\pi i)^{-1} \int_{-i\infty}^{-i} \frac{d\xi}{\xi^2} \ln \frac{F_-(\xi)}{F_+(\xi)}.$$

The last integral can be written in the form [see (21)]

$$G_+'(0) = \frac{1}{\pi} \int_1^{\infty} \frac{dt}{t^2} \arctan \Phi(t, \alpha),$$

which when substituted in (40) yields (19).

The asymptotic behavior of $v(\xi)$ is obtained by first deforming the integration contour in (23) to give

$$v(\xi) = (2\pi)^{-1} \int_{-i\infty}^{-i} [\bar{v}(\kappa-0) - \bar{v}(\kappa+0)] e^{-i\kappa\xi} d\kappa.$$

From (33), (35), and (36) we get

$$\bar{v}(\kappa-0) - \bar{v}(\kappa+0) = \frac{8\pi}{3} \exp[G_+(0) - G_-(\kappa)] \frac{\kappa^{-1} + \kappa^{-3}}{F_-(\kappa)F_+(\kappa)}.$$

Near $\kappa = -i$ we have

$$\bar{v}(\kappa-0) - \bar{v}(\kappa+0) = -\frac{16\pi}{3} \frac{(\kappa+i) \exp[G_+(0) - G_-(\kappa)]}{F_-(\kappa)F_+(\kappa)}.$$

Now

$$K_-(\kappa) = K_+(\kappa) = \frac{3}{2}, \quad F_-(\kappa) = F_+(\kappa) = 1 - \frac{\alpha}{2},$$

$$G_+(0) = \ln \left(1 - \frac{\alpha}{3} \right).$$

Hence

$$\bar{v}(\kappa-0) - \bar{v}(\kappa+0) = (\kappa+i)B(\alpha),$$

where $B(\alpha)$ is given by (20) and (21) in accordance with the asymptotic equation in the text.

¹In the conventional notation, n is the electron density, p_F the Fermi momentum, e the electron charge.

²See Ref. 7 where the presence of the skin depth δ in the problem enables two limiting cases, $l \ll \delta$ and $l \gg \delta$, to be considered.

¹K. Fuchs, Proc. Cambr. Phil. Soc. **34**, 100 (1938).

²M. Ya. Azbel' and M. I. Kaganov, Zh. Eksp. Teor. Fiz. **27**, 762 (1954).

³R. B. Dingle, Proc. Roy. Soc. A **201**, 545 (1950).

⁴M. Ya. Azbel', S. D. Pavlov, I. A. Gamanya, and A. I. Vereshchagin, Pis'ma Zh. Eksp. Teor. Fiz. **16**, 295 (1972) [JETP Lett. **16**, 207 (1972)].

⁵V. I. Okulov and V. V. Ustinov, Fiz. Nizk. Temp. **5**, 213 (1979) [Sov. J. Low Temp. Phys. **5**, 101 (1979)].

⁶J. Czerwonko, Z. Phys. B **80**, 225 (1990); Physica A **174**, 438 (1991).

⁷M. I. Kaganov, G. Ya. Lyubarskiĭ, and J. Czerwonko, Zh. Eksp. Teor. Fiz. **102**, 1351 (1992) [Sov. Phys. JETP **75**, 733 (1992)].

Translated by E. Strelchenko