Stability of a uniform distribution of magnetization

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The stability of a uniform distribution of magnetization of a ferromagnetic plate is investigated. It is shown that when a certain condition is fulfilled a uniform distribution is stable for all thicknesses of the plate. The loss of stability against nonuniform deformations is investigated, and analytical expressions are obtained for the critical parameters of very thin and very thick plates. The results obtained are compared with known results. The phase diagram of a ferromagnet with cubic crystallographic anisotropy is investigated.

For investigation of the stability of a uniform distribution of magnetization it is convenient to use the dynamical criterion proposed in Ref. 1. The essence of the method consists in investigating the transient process after the system has undergone a small departure from its equilibrium position.

The Landau–Lifshitz equation of motion can be written conveniently in the Gilbert form:[2]

$$\frac{\partial \mathbf{M}}{\partial t}=-\gamma[\mathbf{MH}]+\alpha_1 M_0^{-1}\left[\mathbf{M}\frac{\partial \mathbf{M}}{\partial t}\right], \tag{1}$$

where α_1 is a dimensionless parameter, M_0 is the saturation magnetization, $\mathbf{H}$ is the effective magnetic field, $\mathbf{M}$ is the magnetization, and γ is the magnetogyric ratio. In the linear approximation,

$$\mathbf{M}=\mathbf{M}_0+\mathbf{m}_\sim,\quad \mathbf{H}=\mathbf{H}_0+\mathbf{h},$$

where $\mathbf{m}_\sim \mathbf{M}_0=0$, $\mathbf{M}_0\times\mathbf{H}_0=0$. The normal modes of the motion of the system have the form

$$\mathbf{m}_\sim=\mathbf{m}\exp(\gamma M_0\omega t-i\mathbf{kr}).$$

The position of the magnetization vector $\mathbf{M}$ is characterized by the polar angle θ and azimuthal angle φ (the z axis is perpendicular to the surface of the layer). Use of the equation of motion (1) and the well known Maxwell equations leads to coupled equations for φ and θ that determine small deviations of the system from the equilibrium values φ_0 and θ_0:

$$\begin{gathered}\left[\omega+4\pi\frac{k_\varphi k_\theta}{k^2}+\frac{W_{\varphi\theta}}{M_0^2\sin\theta_0}\right]\theta+\left[\mathbf{k}\hat{\alpha}\mathbf{k}+4\pi\frac{k_\varphi^2}{k^2}+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}+\alpha_1\omega\right]\varphi\sin\theta_0=0,\\ \left[\mathbf{k}\hat{\alpha}\mathbf{k}+4\pi\frac{k_\theta^2}{k^2}+\frac{W_{\theta\theta}}{M_0^2}-4\pi\cos^2\theta_0+\alpha_1\omega\right]\theta-\left[\omega-4\pi\frac{k_\varphi k_\theta}{k^2}-\frac{W_{\varphi\theta}}{M_0^2\sin\theta_0}\right]\varphi\sin\theta_0=0.\end{gathered} \tag{2}$$

Here $\hat{\alpha}$ is the exchange-interaction tensor, and the energy density W is the sum of the crystallographic-anisotropy, Zeeman, magnetoelastic, etc., energy densities; $W_{\alpha\alpha}$ are the second derivatives of W with respect to the corresponding angles, taken in the state of the uniform distribution (φ_0, θ_0) $\mathbf{k}_\varphi$ and $\mathbf{k}_\theta$ are the components of the vector $\mathbf{k}$ in the spherical coordinate system. The nontriviality condition leads to the dispersion relation

$$\begin{gathered}(1+\alpha_1^2)\omega^2+\alpha_1\omega\left[2\mathbf{k}\hat{\alpha}\mathbf{k}+4\pi\frac{k_\varphi^2+k_\theta^2}{k^2}+\frac{W_{\varphi\varphi}+(W_{\theta\theta}-4\pi M_0^2\cos^2\theta_0)\sin^2\theta_0}{M_0^2\sin^2\theta_0}\right]\\ +\left(\mathbf{k}\hat{\alpha}\mathbf{k}+4\pi\frac{k_\varphi^2}{k^2}+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}\right)\times\left(\mathbf{k}\hat{\alpha}\mathbf{k}+4\pi\frac{k_\theta^2}{k^2}+\frac{W_{\theta\theta}}{M_0^2}-4\pi\cos^2\theta_0\right)\\ -\left(4\pi\frac{k_\varphi k_\theta}{k^2}+\frac{W_{\varphi\theta}}{M_0^2\sin\theta_0}\right)^2=0.\end{gathered} \tag{3}$$

Stability of the system against uniform deformations leads to the well known inequalities

$$W_{\varphi\varphi}\geqslant 0,\quad W_{\varphi\varphi}(W_{\theta\theta}-4\pi M_0^2\cos 2\theta_0)-W_{\varphi\theta}^2\geqslant 0. \tag{4}$$

In a system that is stable against nonuniform deformations, only damping oscillations ($\operatorname{Re}\omega_{1,2}<0$) arise, and, consequently, all the coefficients in Eq. (3) are positive. The appearance of an excitation with $\operatorname{Re}\omega>0$ following a change in any particular external factor, the temperature of the layer, or the thickness of the layer is possible only if the sign of the free term changes. Thus, the vanishing of the free term of the dispersion relation ($\omega=0$) can be regarded as the criterion for the loss of stability of the uniform state against nonuniform deformations. Henceforth, it is assumed that $\alpha_1=0$, and the critical state is determined from the condition $\omega=0$.

To investigate specific critical states it is necessary to solve (3) with allowance for the boundary conditions. Before we do this, note that when

$$W_{\varphi\varphi}(W_{\theta\theta}-4\pi M_0^2\cos^2\theta_0)-W_{\varphi\theta}^2\geqslant 0 \tag{5}$$

all the coefficients of the dispersion relation (3) are positive, i.e., the uniform distribution is stable and in the region (5) only first-order phase transitions are possible.

Below, we investigate the region outside (5) but (naturally) inside (4). As can be seen from (3), to each value of $\varkappa=(k_x^2+k_y^2)^{1/2}$ there correspond six values of $k_z\equiv q_n$ ($n=1, 2, ..., 6$). The usual boundary conditions (see, e.g., Ref. 3) that must be obeyed by the solutions can be written at $z=\pm L$ ($2L$ is the film thickness) as follows:

$$\sum_{n=1}^{6} q_n\vartheta_n \sin(q_nL)=\sum_{n=1}^{6} q_n\theta_n \cos(q_nL)=\sum_{n=1}^{6} q_n\varphi_n \sin(q_nL)$$

$$=\sum_{n=1}^{6} q_n\varphi_n \cos(q_nL)=0,$$

$$\sum_{n=1}^{6} (k_\theta{}''\theta_n+k_\varphi\varphi_n \sin\theta_0)\,\frac{\varkappa\cos(q_nL)-q_n\sin(q_nL)}{k_n^2}$$

$$=\sin\theta_0\sum_{n=1}^{6}\theta_n\sin(q_nL), \tag{6}$$

$$\sum_{n=1}^{6} (k_\theta{}''\theta_n+k_\varphi\varphi_n \sin\theta_0)\,\frac{\varkappa\sin(q_nL)+q_n\cos(q_nL)}{k_n^2}$$

$$=-\sin\theta_0\sum_{n=1}^{6}\theta_n\cos(q_nL).$$

To illustrate the application of the relations (3)–(6) we consider the particular (but frequently encountered) cases $\mathbf{k}\alpha=\alpha k^2$, $W_{\varphi\theta}=0$. It is convenient to investigate two directions of wave propagation—in the plane containing **M** and the z axis ($k_\varphi=0$, $k_\theta^n=\varkappa\cos\theta_0-q_n\sin\theta_0$), and perpendicular to this plane ($k_\varphi=\varkappa$, $k_\theta^n=-q_n\sin\theta_0$). On the boundary of the region (4) the system is unstable against uniform deformation ($\varkappa=0$) for all thicknesses ($2L_{\rm cr}=0$). Since $W_{\theta\theta}M_0^{-2}-4\pi\cos 2\theta_0=\delta=0$ is not a singular point of Eq. (3), it should be expected that in the limit $\delta\to 0$ instability is possible against excitations with $\varkappa\to 0$ for $L_{\rm cr}\to 0$. In this case, in the region $\delta\ll 1$ for thin films ($q_nL\ll 1$) the system (6) is simplified:

$$\sum_{n=1}^{6} q_n\theta_n=\sum_{n=1}^{6} q_n^2\theta_n=\sum_{n=1}^{6} q_n\varphi_n=\sum_{n=1}^{6} q_n^2\varphi_n=0,$$

$$\sum_{n=1}^{6} (k_\theta{}''\theta_n+k_\varphi\varphi_n \sin\theta_0)\frac{\varkappa-q_n^2L}{k_n^2}=0, \tag{6a}$$

$$\sum_{n=1}^{6} (k_\theta{}''\theta_n+k_\varphi\varphi_n \sin\theta_0)\frac{q_n}{k_n^2}+\frac{\sin\theta_0}{1+\varkappa L}\sum_{n=1}^{6}\theta_n=0.$$

For a wave propagating at a right angle to the **Mz** plane the free term of the dispersion relation is equal to zero:

$$\left(\alpha k_n^2+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}\right)\left(\alpha k_n^2+\delta-4\pi\frac{\varkappa^2}{k_n^2}\sin^2\theta_0\right)$$
$$+4\pi\frac{\varkappa^2}{k_n^2}(\alpha k_n^2+\delta-4\pi\sin^2\theta_0)=0 \tag{3a}$$

($k_n^2=\varkappa^2+q_n^2$). From (6a) and (3a), with (2) taken into account, it is easy to show that the uniform structure is unstable against a wave with $\theta\sim\cos qz$, with $\varkappa$ satisfying the equation

$$\alpha\varkappa^3L+\alpha\varkappa^2+\varkappa L(\delta-4\pi\sin^2\theta_0)+\delta=0.$$

This equation has a real positive solution for

$$L\geqslant L_{\rm cr}=(\alpha\delta)^{1/2}/2\pi\sin^2\theta_0$$

or

$$\alpha\delta_{\rm cr}\leqslant 4\pi^2L^2\sin^4\theta_0. \tag{7}$$

The critical thickness corresponds to

$$\varkappa_{\rm cr}=(\delta/\alpha)^{1/2}. \tag{8}$$

Analysis of the roots of Eq. (3a) shows that the approximation $q_nL\ll 1$ is valid for thicknesses

$$2L\ll 2(\alpha/W_{\varphi\varphi})^{1/2}M_0\sin\theta_0.$$

For a wave in the **M z** plane we have

$$\varkappa_{\rm cr}=(\delta/\alpha)^{1/2},\quad L_{\rm cr}=-2(\alpha\delta)^{1/2}M_0^2W_{\theta\theta}^{-1}.$$

Thus, this excitation destroys the structure for $W_{\theta\theta}<0$ (in the first case, for $W_{\theta\theta}-4\pi M_0^2\cos^2\theta_0<0$) and for large thicknesses, if $\cos\theta_0\neq 0$.

In the region $W_{\theta\theta}M_0^{-2}-4\pi\cos^2\theta_0=-\delta\,(0\leqslant\delta\leqslant 1)$ we must expect loss of stability at large thicknesses, since for $\delta=0$ a uniform distribution is stable for all L [this can be seen from (5)]. It follows from the dispersion relation that in this case too we have $\alpha\varkappa^2\leqslant\delta$, i.e., the system is unstable against long-wavelength excitations.

For a wave propagating at a right angle to the **Mz**, plane the dispersion relation has the form

$$\omega^2+\left(\alpha k_n^2+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}+4\pi\frac{\varkappa^2}{k_n^2}\right)(\alpha k_n^2-\delta)$$
$$+4\pi\frac{q_n^2}{k_n^2}\sin^2\theta_0\left(\alpha k_n^2+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}\right)=0. \tag{3b}$$

Close to the loss of stability ($\omega\approx 0$) the solutions of Eq. (3b) have the form

$$\alpha k_1^2\approx\alpha\varkappa^2\sim\delta,\quad \alpha k_2^2\approx-4\pi\sin^2\theta_0,$$

$$\alpha k_3^2\approx-\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}<0,\quad q_1^2\approx\varkappa^2(\delta-\alpha k_1^2).$$

The solutions k_2 and k_3 ($q_2^2, q_3^2<0$) correspond to surface waves. From the boundary conditions it can be seen that the amplitudes of the surface waves are much smaller than the amplitude of the bulk wave, and the influence of the surface waves on the magnetization distribution inside the layer can be neglected. For a wave with $\theta\sim\cos q_1z$ it follows from (6) that

$$\operatorname{ctg}(q_1L)\approx-(\delta-\alpha\varkappa^2)^{1/2}/4\pi\sin\theta_0\ll 1,$$

$$q_{1m}L\approx\frac{\pi}{2}(2m+1)+\frac{\varkappa}{q_{1m}}\frac{\alpha\varkappa^2-\delta}{4\pi\sin^2\theta_0}\qquad(m=0,1,2\ldots).$$

For a wave with $\theta\sim\sin q_1z$ we have

$$\operatorname{tg}(q_1L)\approx-\frac{q_1}{\varkappa}\frac{W_{\varphi\varphi}}{W_{\varphi\varphi}+4\pi M_0^2\sin^2\theta_0}\ll 1,$$

$$q_{1m}L\approx\pi m-\frac{q_{1m}}{\varkappa}\frac{W_{\varphi\varphi}}{W_{\varphi\varphi}+4\pi M_0^2\sin^2\theta_0}\qquad(m=1,2,3\ldots).$$

Thus, in a film of thickness $2L$ there arises a spectrum of waves with $q_{1m}\approx\pi m/2L$ ($m=1,2,3\ldots$), satisfying the dispersion relation

$$\omega_m^2+\left(4\pi+\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}\right)(\alpha\varkappa^2-\delta)+\frac{\pi^2m^2}{\varkappa^2L^2}\frac{W_{\varphi\varphi}}{M_0^2\sin^2\theta_0}=0.$$

Each of the branches of the spectrum has a maximum at

$$\alpha\varkappa^2=\left(\alpha\frac{4\pi W_{\varphi\varphi}}{W_{\varphi\varphi}+4\pi M_0^2\sin^2\theta_0}\right)^{1/2}\frac{\pi m}{2L}=\frac{mL_0}{2L}$$

and touches the axis $\omega = 0$ at

$$\delta_m = mL_0/L$$

or for thicknesses

$$2L_m = 2L_0\delta^{-1}m.$$

The uniform distribution loses stability when

$$2L_{cr} = 2L_0\delta^{-1}, \tag{9}$$

$$2\alpha\varkappa_{cr}^2 = \delta. \tag{10}$$

As in the case of thin films, a wave in the **Mz** plane destroys the structure for thicknesses greater than (9).

The analogous problem was first solved for the particular case of a film with perpendicular anisotropy, placed in a magnetic field **H** parallel to the surface.[3,4] For this system,

$$W = -0{,}5\beta M_0^2\cos^2\theta - HM_0\sin\theta\sin\varphi,$$

where β is the anisotropy constant. In all the cases considered, $\sin\varphi_0 = 1$ and $W_{\varphi\theta} = 0$. From the conditions (4) and (5) the region of existence of the critical fields for the phase with $\sin\theta_0 = 1$ is

$$\beta M_0 > H \geqslant (\beta - 4\pi)M_0.$$

In the region of large thicknesses,

$$-\delta = HM_0^{-1} - \beta, \quad W_{\varphi\varphi} = HM_0 \approx \beta M_0^2.$$

From (9),

$$\delta = \frac{2\pi}{L}\left(\frac{\pi\alpha\beta}{\beta+4\pi}\right)^{1/2}$$

and

$$2\alpha\varkappa_{cr}^2 = \delta, \quad H_{cr} = \beta - \delta M_0.$$

In the region of small thicknesses,

$$\delta = HM_0 - \beta + 4\pi \equiv HM_0 - \beta_\bullet.$$

From (7),

$$2L_{cr} = \frac{1}{\pi}\left[\frac{\alpha(H-\beta_\bullet M_0)}{M_0}\right]^{1/2},$$

$$H_{cr} = \beta_\bullet M_0 + 4\pi^2L^2\alpha^{-1}M_0,$$

$$2\alpha\varkappa_{cr}^2 = 4\alpha^{-1}\pi^2L^2.$$

In the uniform tilted structure, $\sin\theta_0 = \beta_\bullet^{-1}HM_0^{-1}$. The region of fields that destroy the structure is

$$\beta_\bullet M_0 \geqslant H \geqslant \beta_\bullet^{3/2}\beta^{-1/2}M_0.$$

In the region of large thicknesses,

$$\delta = \frac{\beta H^2}{\beta_\bullet^2M_0^2} - \beta_\bullet, \quad H_{cr}^2 = \left[\frac{\beta_\bullet^3}{\beta} + \frac{2\pi\beta_\bullet^4}{\beta^2L}(\alpha\pi)^{1/2}\right]M_0^2.$$

For the case considered in Ref. 3, $\beta \gg 4\pi$ holds and, taking into account that $\alpha^{1/2} \ll L$, we have

$$H_{cr} = M_0\left[\frac{\beta_\bullet^{3/2}}{\beta^{1/2}} + \frac{\alpha^{1/2}}{L}\left(\frac{\pi\beta_\bullet}{\beta}\right)^{3/2}\right].$$

In the region of small thicknesses,

$$\delta = \beta_\bullet - \beta_\bullet^{-1}(HM_0^{-1})^2, \quad W_{\theta\theta} = -4\pi,$$

$$2L_{cr} = \frac{1}{\pi}\left[\alpha\beta_\bullet^{-1}M_0^{-2}(\beta_\bullet^2M_0^2 - H^2)\right]^{1/2},$$

$$H_{cr} = \beta_\bullet M_0 - 2\frac{\pi^2L^2}{\alpha}M_0.$$

All the above results repeat exactly the results of Refs. 3 and 4.

To illustrate the application of the results obtained, we shall investigate a plate of magnetic material with cubic crystallographic anisotropoy. If the x, y, and z coordinate axes coincide with the [100], [010], and [001] axes, the energy density is given by the expression

$$W = 0.25K_1M_0^2(\sin^4\theta\sin^2 2\varphi + \sin^2 2\theta) + 0.25K_2M_0^2\sin^4\theta\cos^2\theta\sin^2 2\varphi,$$

where K_1 and K_2 are the anisotropy constants. If we disregard the metastable states, the principal states are three uniform phases,[5] for all of which $W_{\varphi\theta} = 0$ holds.

1. $\varphi_0 = 0, \theta_0 = 0.5\pi$ [the (100) phase]

This state is stable against uniform deformations for $K_1 > 0$. From (5) we have

$$W_{\theta\theta} - 4\pi M_0^2\cos^2\theta_0 = 2K_1M_0^2 \geqslant 0,$$

whence it follows that the uniform (100) phase is stable for $K_1 > 0$ for all thicknesses, and in this region only first-order phase transitions are possible. The unbounded crystal also has an analogous phase in the same region.[6]

2. $\varphi_0 = 0.25\pi, \theta_0 = 0.5\pi$ [the (110) phase]

The phase is stable against uniform deformations in the region

$$K_1 < 0, \quad 2K_1 + K_2 + 8\pi > 0.$$

The region of stability for all thicknesses is, from (5),

$$2K_1 + K_2 \geqslant 0,$$

and coincides with the region of stability of the analogous phase of the unbounded crystal. In the region $2K_1 + K_2 < 0$ the uniform state is stable for layer thicknesses smaller than a certain critical value that decreases to zero on the straight line $2K_1 + K_2 + 8\pi = 0$.

3. $\varphi_0 = 0.25\pi, 3K_2\sin^2\theta_0 = K_2 - 3K_1 - [(K_2+3K_1)^2 - 24\pi K_2]^{1/2}$

This phase (the tilted $(110)_<$ phase) is stable against uniform deformation in the region

$$2K_1 + K_2 + 8\pi < 0,$$

$$-8\pi K_2 < (K_2 + 3K_1)^2 > 24\pi K_2.$$

The region of stability (5) against nonuniform deformations is well approximated by the expression

$$2K_1 + K_2 + 12\pi \leqslant 0.$$

The phase diagram of the layer is given in Fig. 1. For K_1, $K_2 \gg \pi$ the diagram goes over into the well known phase diagram of the unbounded crystal.[6] The transitions between the phases are first-order phase transitions (they are not shown

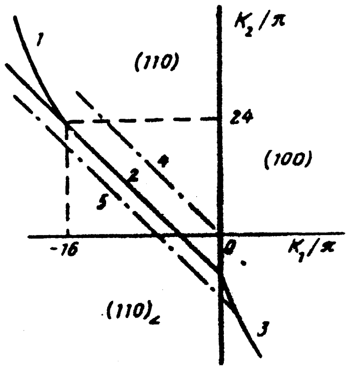

FIG. 1. Phase diagram of a layer: *1)* $(K_2+3K_1)^2=24\pi K_2$; *2)* $2K_1+K_2=-8\pi$; *3)* $(K_2+3K_1)^2=-8\pi K_2$; *4)* $2K_1+K_2=0$; *5)* $2K_1+K_2=-12\pi$.

in Fig. 1). The segment of the transition between the (110) and $(110)_<$ phases for $0>K_1>-16\pi$ may be an exception. It is necessary, however, to take into account that on the line $2K_1+K_2+8\pi=0$ the critical thickness of the uniform phase is equal to zero, i.e., for any layer of finite thickness the transition between phases 2 and 3 occurs via a nonuniform state. The type of phase transition on this segment requires additional investigation.

Thus, the problem of the stability of a uniformly magnetized magnetic plate has been solved in general form in this paper. The expression (5), which determines the region of stability of uniform magnetization in a plate of any thickness, has been obtained for the first time. The results of the paper have been tested on well investigated[3,4] magnetic layers with perpendicular anisotropy. The phase diagram of a layer (perpendicular to the [001] axis) of a magnet with cubic crystallographic anisotropy has been investigated.

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