Radiative transfer equation in a randomly inhomogeneous magnetized plasma

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We give a derivation of the radiative transfer equation in a randomly inhomogeneous unsteady cold magnetized plasma; this enables us to describe multiple scattering processes of electromagnetic waves when the geometric optics approximation is valid for the mean field.

INTRODUCTION

Multiple scattering effects may be of considerable importance for the propagation of electromagnetic waves in continuous media containing random inhomogeneities. A space or laboratory plasma in an external magnetic field is an important example of such a medium. According to multiple scattering theory[1,2] the average field satisfies the Dyson equation (DE) and the incoherent component the Bethe–Salpeter equation (BSE). It is well known that the solution of the Dyson equation is equivalent to determining the effective dielectric permittivity tensor. This problem was considered for a randomly inhomogeneous magnetized plasma in a number of papers (see, e.g., Ref. 3 where the Bourret approximation was used). When the vector nature of electromagnetic waves and the plasma gyrotropy is taken into account the Bethe–Salpeter equation encounters considerable mathematical difficulties. In our opinion a solution can be found in this case by transforming the BSE into a radiative transfer equation (RTE).

The RTE has been studied in detail in connection with its applications in astrophysics[4,5] and the physics of nuclear reactors[6] and also in the theory of wave propagation in randomly inhomogeneous media.[7,8] Various analytical and numerical methods have been developed for solving it.[9,10] Methods for changing from the BSE to a RTE have been considered in a large number of papers (see citations in Ref. 7), as a rule for scalar wave fields. The most general prescription for obtaining the RTE from the BSE for vector electromagnetic waves is expounded in Ref. 11 in which a RTE is obtained for an isotropic randomly inhomogeneous medium. A search of the literature shows that the transfer of radiation in a magnetized randomly inhomogeneous plasma has been considered only in the quasi-isotropic approximation,[12–14] which greatly restricts the possibility of using the results.

The problem of the present paper is to apply the method of Ref. 11 to derive a RTE for a magnetized plasma. The equation obtained is valid under the same conditions under which the geometric optics approximation is valid for the mean field. The potential range of its use is rather broad and includes the propagation of electromagnetic waves in space, near-earth, and laboratory plasmas.

We stick to the following plan in the present paper. We give the initial equations in Sec. 1 and list the basic assumptions. In Sec. 2 we collect some useful relations used in what follows. In Sec. 3 we develop a method of successive approximations to solve the Bethe–Salpeter equation. In Sec. 4 we analyze the conditions for compatibility of the equations of the zeroth and first approximations for the coherence function. We introduce in Sec. 5 the energy characteristics of the scattered waves. The RTE is brought into its canonical form in Sec. 6. In the Conclusion we analyze the applicability domain of the resulting equations and discuss possible generalizations.

1. INITIAL EQUATIONS AND BASIC APPROXIMATIONS

The electric field strength vector $\mathbf{E}(\mathbf{r},t)$ of an electromagnetic wave propagating in a randomly inhomogeneous nonstationary medium satisfies a wave equation which can be written as follows:

$$[\hat{L}(X)+\hat{V}(X)]\mathbf{E}(X)=0, \tag{1}$$

where we have introduced the notation $X=\{\mathbf{r},t\}$ while the deterministic operator $\hat{L}(X)$ and the random operator $\hat{V}(X)$ are defined by the equations

$$\hat{L}(X)\mathbf{E}(X)=\nabla^2\mathbf{E}(\mathbf{r},t)-\nabla(\nabla\mathbf{E}(\mathbf{r},t))$$
$$-c^{-2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{t}dt'\int d^3r'\langle\hat{\varepsilon}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')\rangle\mathbf{E}(\mathbf{r}',t'), \tag{2}$$

$$\hat{V}(X)\mathbf{E}(X)=-c^{-2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{t}dt'\int d^3r'\,\tilde{\hat{\varepsilon}}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')\mathbf{E}(\mathbf{r}',t'),$$

where $\hat{\varepsilon}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')$ is the kernel of the permittivity operator,

$$\tilde{\hat{\varepsilon}}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')=\hat{\varepsilon}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')-\langle\hat{\varepsilon}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t')\rangle.$$

Introducing the coherence matrix

$$\hat{\Gamma}(X_1,X_2)=\langle\mathbf{E}(X_1)\otimes\mathbf{E}^*(X_2)\rangle$$

and the correlation matrix of the scattered field,

$$\delta\hat{\Gamma}(X_1,X_2)=\hat{\Gamma}(X_1,X_2)-\hat{\Gamma}^0(X_1,X_2),$$

where

$$\hat{\Gamma}^0(X_1,X_2)=\langle\mathbf{E}(X_1)\rangle\otimes\langle\mathbf{E}^*(X_2)\rangle$$

and the $\otimes$ sign indicates the external product of vectors, we can write the Bethe–Salpeter equation in the "ladder" approximation:[2]

$$\hat{D}(X_1)\delta\hat{\Gamma}(X_1,X_2)$$
$$=\hat{G}^*(X_1)\langle\hat{V}(X_1)\hat{V}^*(X_2)\rangle\hat{\Gamma}(X_1,X_2), \tag{3}$$

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where $\hat{D}(X) = \hat{L}(X) - \hat{G}_0(X)\langle\hat{V}(X_1)\hat{V}(X_2)\rangle$ is the Dyson operator in the Bourret approximation, and we have $\hat{G}(X) = \hat{D}^{-1}(X)$, $\hat{G}^0(X) = \hat{L}^{-1}(X)$. Note that the Dyson operator can be written in the equivalent form

$$\hat{D}(X)=\nabla^2-\nabla\cdot\nabla-c^{-2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{t}dt'\int d^3r'\ \hat{\varepsilon}^{\text{eff}}(\mathbf{r},t,\mathbf{r}-\mathbf{r}',t-t'), \tag{4}$$

where $\hat{\varepsilon}^{\text{eff}}$ is the kernel of the effective dielectric permittivity operator.

To obtain the RTE from (3) we shall follow Ref. 11 and assume that the following conditions are satisfied.

1. The medium is quasistationary and quasiuniform, i.e., the inequality

$$\mu\ll 1, \tag{5}$$

is satisfied where

$$\mu=\max\left(\frac{\rho_0}{L},\frac{\lambda}{L},\frac{\tau_0}{T},\frac{\tau}{T}\right),$$

L and T are characteristic spatial and temporal scales on which the average properties of the medium change, λ and τ are the average wavelength and oscillation period of the radiation propagating in the medium, and ρ_0 and τ_0 are the spatial and temporal scales of the nonlocality (dispersion) of the medium. Condition (5) is the condition for the applicability of the geometric optics method.[15]

2. The correlation matrix of the scattered field,

$$\delta\hat{\Gamma}(X_1,X_2)=\delta\hat{\Gamma}\left(\frac{X_1+X_2}{2},\ X_1-X_2\right)=\delta\hat{\Gamma}(R,\rho)$$

depends more strongly on the difference variable ρ than on the "center of gravity" coordinate R (since the characteristic scale of changes in the quantity $\delta\hat{\Gamma}(R,\rho)$ with respect to R is as a rule much larger[2] than with respect to ρ):

$$\left|\frac{\partial\delta\hat{\Gamma}(R,\rho)}{\partial R}\right|\sim\mu\left|\frac{\partial\delta\hat{\Gamma}(R,\rho)}{\partial\rho}\right|\quad(\mu\ll 1). \tag{6}$$

3. The mean field is defined by the geometric optics method. It can then locally be written in the form of a superposition of plane waves satisfying the dispersion relation

$$\hat{D}(K)\langle\mathbf{E}(K)\rangle=\hat{D}(\mathbf{k},\omega)\langle\mathbf{E}(\mathbf{k},\omega)\rangle$$
$$=[k_ik_j-k^2\delta_{ij}+k_0^2\ \hat{\varepsilon}^{\text{eff}}_{ij}(\mathbf{k},\omega)]\langle E_j(\mathbf{k},\omega)\rangle=0, \tag{7}$$

where $\hat{D}(K)$ is the Fourier transform of the kernel of the Dyson operator ($K = (\mathbf{k}, -\omega)$) and the effective dielectric permittivity tensor $\hat{\varepsilon}^{\text{eff}}(\mathbf{k},\omega)$ is defined as the Fourier transform of the kernel of the effective dielectric permittivity operator).

4. The local damping of the mean field due to collisional losses and scattering is small:

$$|D_{ij}^{a}(K)|\sim\mu|D_{ij}^{\text{H}}(K)|\qquad(\mu\ll 1), \tag{8}$$

where $D^{\text{H}}_{ij}(K)$ and $D^{a}_{ij}(K)$ are the Hermitean and anti-Hermitean components of the tensor $D_{ij}(K)$.

The Bethe–Salpeter equation (3), which is the starting point for obtaining the RTE, is written in the "ladder" approximation for the intensity operator. It is then natural to find the mean field from the Dyson equation in the Bourret approximation for the mass operator.[1,2] It is permissible in the Bourret approximation to neglect the change in the real part of the refractive index, assuming that the action of the inhomogeneity reduces solely to a change in its imaginary part. We can thus put $\hat{D}^{\text{H}}(R,K) = \hat{\Lambda}(R,K)$ where the dispersion tensor $\hat{\Lambda}(R,K)$ of the regular plasma has a structure determined by Eq. (7) with the substitution $\hat{\varepsilon}^{\text{eff}}(\mathbf{k},\omega)\rightarrow\langle\hat{\varepsilon}(\mathbf{k},\omega)\rangle$.

2. RAY EQUATION AND SOME USEFUL RELATIONS

The requirements we have formulated are, in fact, the conditions for the applicability of the geometric optics method in space-time form, taking into account the spatial and temporal dispersion. The equations for the ray trajectories in a magnetized plasma have the form[15]

$$\frac{dR}{ds}=\frac{\partial H}{\partial K}\left|\frac{\partial H}{\partial\mathbf{k}}\right|^{-1},\quad\frac{dK}{ds}=-\frac{\partial H}{\partial R}\left|\frac{\partial H}{\partial\mathbf{k}}\right|^{-1}, \tag{9}$$

where we have written $R = \{\mathbf{r},t\}$ and $K = \{\mathbf{k}, -\omega\}$, $H = \frac{1}{2}[k^2 - k_0^2n^2(R,\omega)]$ is the ray Hamiltonian, $n(R,\omega)$ is the refractive index for the ordinary or the extraordinary wave, and d/ds is the derivative along the ray trajectory. For a compact representation of the subsequent calculations it is convenient to introduce Poisson brackets of two functions of arguments R and K:

$$[f,g]=\frac{\partial f}{\partial K}\frac{\partial g}{\partial R}-\frac{\partial f}{\partial R}\frac{\partial g}{\partial K}. \tag{10}$$

Using (9) and (10) we can obtain the following relation:

$$[H,f]=\left|\frac{\partial H}{\partial\mathbf{k}}\right|\frac{df}{ds}. \tag{11}$$

We introduce into our considerations the polarization vector $\mathbf{e}$ which satisfies the relations

$$\Lambda_{ij}e_j=0,\qquad e_ie_i^*=1.$$

The identity

$$e_i^*\Lambda_{ij}e_j=-2QH, \tag{12}$$

with $Q = |\mathbf{e}|^2 - |\mathbf{ek}|^2/k^2$ is then valid. Using (12) and the fact that $H = 0$ holds in the transparency region of the plasma we can obtain the relations

$$e_i^*\frac{\partial\Lambda_{ij}}{\partial K}e_j=-2Q\frac{\partial H}{\partial K},\qquad e_i^*\frac{\partial\Lambda_{ij}}{\partial R}e_j=-2Q\frac{\partial H}{\partial R}. \tag{13}$$

The equation

$$e_i^*[\Lambda_{ij},f]e_j=-2Q\left|\frac{\partial H}{\partial\mathbf{k}}\right|\frac{df}{ds} \tag{14}$$

also follows from (11) and (13). In what follows we need the following relations, which hold for waves in a magnetized plasma:[16]

$$e_i^*e_j=\lambda_{ij}\,\mathrm{Sp}^{-1}\hat{\lambda}, \tag{15}$$

where $\hat{\lambda}$ is the matrix of the cofactors of the matrix elements

Λ_{ij} ($\varepsilon_{i\alpha\beta}$ is the Levi–Civita tensor),

$$\lambda_{ij}={}^1/_2\varepsilon_{i\alpha\beta}\varepsilon_{j\gamma\delta}\Lambda_{\gamma\alpha}\Lambda_{\delta\beta}, \tag{16}$$

$$\mathrm{Sp}\,\hat{\lambda}={}^1/_2(\Lambda_{ii}\Lambda_{jj}-\Lambda_{ij}\Lambda_{ji}), \tag{17}$$

$$\lambda_{ij}\lambda_{kl}=\lambda_{il}\lambda_{kj}+\Lambda\varepsilon_{ikm}\varepsilon_{jnl}\Lambda_{mn}, \tag{18}$$

and also the relations

$$\frac{1}{\Lambda^*}-\frac{1}{\Lambda}=2i\pi\delta(\Lambda), \tag{19}$$

$$\delta(\Lambda)=|\mathrm{Sp}^{-1}\hat{\lambda}|\{Q^{-1}[\delta(k^2-k_1^2)+\delta(k^2-k_2^2)]+\delta(k_i\langle\varepsilon_{ij}\rangle k_j)\}. \tag{20}$$

3. EQUATIONS OF THE SUCCESSIVE APPROXIMATIONS FOR THE QUANTITY $\delta\hat{\Gamma}$

We consider separately the left-hand and the right-hand sides of Eq. (3). Using the integral form of the operator representation we can write the left-hand side of (3) in the form

$$\int \hat{D}(X_1,X_1-X')\delta\hat{\Gamma}(X',X_2)dX'. \tag{21}$$

In (21) we change to the variables R and ρ and expand the integrand in a Taylor expansion in the variable R, retaining terms through first order in μ. As a result we find

$$\int\Big[\hat{D}(R,\rho-X')\delta\hat{\Gamma}(R,X')+\frac{\rho}{2}\frac{\partial\hat{D}^{\mathrm{H}}(R,\rho-X')}{\partial R}\delta\hat{\Gamma}(R,X') +\frac{X'-\rho}{2}\hat{D}^{\mathrm{H}}(R,\rho-X')\frac{\partial}{\partial R}\delta\hat{\Gamma}(R,X')\Big]dX'. \tag{22}$$

By expanding all quantities occurring in the integrand in Fourier integrals in the difference variable ρ we change (22) to the form

$$(2\pi)^4\int\Big\{\hat{D}(R,K)\delta\hat{\Gamma}(R,K)+\frac{1}{2i}\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial K}\frac{\partial\delta\hat{\Gamma}(R,K)}{\partial R} -\frac{1}{2i}\frac{\partial}{\partial K}\Big[\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial R}\delta\hat{\Gamma}(R,K)\Big]\Big\}\exp(iK\rho)dK. \tag{23}$$

Similarly we get for the right-hand side of (3)

$$(2\pi)^4\int\hat{G}^*(R,K)\hat{B}(R,K-K',K',-K') \times[\hat{\Gamma}^0(R,K')+\delta\hat{\Gamma}(R,K')] \times\exp(iK\rho)\,dK\,dK', \tag{24}$$

where $\hat{B}(R,K-K',K',-K')$ is the Fourier transform with respect to the three difference coordinates of the quantity

$$\hat{B}(R,X_1-X_2,X_1-X',X_2-X'')=\langle\hat{\hat{\varepsilon}}(X_1,X_1-X') \times\hat{\hat{\varepsilon}}^*(X_2,X_2-X'')\rangle.$$

Comparing (23) and (24) we transform Eq. (3) to the following form:

$$\hat{D}(R,K)\delta\hat{\Gamma}(R,K)+\frac{1}{2i}\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial K}\frac{\partial\delta\hat{\Gamma}(R,K)}{\partial R} -\frac{1}{2i}\frac{\partial}{\partial K}\Big[\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial R}\delta\hat{\Gamma}(R,K)\Big]$$
$$=\int\hat{G}^*(R,K)\hat{B}(R,K-K',K',-K') \times[\hat{\Gamma}^0(R,K')+\delta\hat{\Gamma}(R,K')]dK'. \tag{25}$$

We write the quantity $\delta\hat{\Gamma}(R,K)$ as a power series in the small parameter μ:

$$\delta\hat{\Gamma}(R,K)=\sum_{n=0}^{\infty}\delta\hat{\Gamma}^{(n)}(R,K), \tag{26}$$

where we have $\delta\hat{\Gamma}^{(n)}\propto\mu^n$. Using (26) we can reduce Eq. (25) to an infinite chain of equations, the first two of which have the form

$$\hat{D}^{\mathrm{H}}(R,K)\delta\hat{\Gamma}^{(0)}(R,K)=0, \tag{27}$$

$$\hat{D}^{\mathrm{H}}(R,K)\delta\hat{\Gamma}^{(1)}(R,K)=\hat{Z}(R,K), \tag{28}$$

where

$$\hat{Z}(R,K)=-\hat{D}^{a}(R,K)\delta\hat{\Gamma}^{(0)}(R,K) -\frac{1}{2}\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial iK}\frac{\partial\delta\hat{\Gamma}^{(0)}(R,K)}{\partial R} +\frac{1}{2}\frac{\partial}{\partial iK}\Big[\frac{\partial\hat{D}^{\mathrm{H}}(R,K)}{\partial R}\delta\hat{\Gamma}^{(0)}(R,K)\Big]$$
$$+\int\hat{G}^*(R,K)\hat{B}(R,K-K',K',-K')[\hat{\Gamma}^0(R,K') +\delta\hat{\Gamma}^{(0)}(R,K')]dK'.$$

The compatibility condition of the equations of the zeroth (27) and the first (28) approximation give us the radiative transfer equation[11]

4. ANALYSIS OF THE ZEROTH AND FIRST APPROXIMATION EQUATIONS

Using (8) and the equation $\hat{D}^{\mathrm{H}}(R,K)=\hat{\Lambda}(R,K)$ we can write the solution of Eq. (27) in the form

$$\delta\Gamma_{ij}^{(0)}(R,K)=e_ie_j^*C\delta(\Lambda), \tag{29}$$

where we have $\Lambda(R,K)=\det\|\Lambda_{ij}(R,K)\|$ and where we shall show below that C characterizes the radiation intensity.

For the compatibility of the zeroth and first approximation equations it is necessary to satisfy the condition that the matrix $\hat{Z}(R,K)$ be orthogonal to the polarization vectors:

$$\mathbf{e}^*\hat{Z}(R,K)\mathbf{e}=0. \tag{30}$$

This condition leads to the equation

$$e_i^*\Big\{e_je_k^*[\Lambda_{ij},C]+C[\Lambda_{ij},e_je_k^*]+2e_i^*\Big(iD_{ij}^ae_je_k^*-\frac{1}{2}\frac{\partial^2\Lambda_{ij}}{\partial K\,\partial R}$$
$$\times e_je_k^*\Big)C\Big\}e_k\delta(\Lambda)=2ie_i^*e_kG_{kp}^*(R,K)$$
$$\times\int B_{ilpm}(R,K-K',K',-K')[\Gamma_{lm}^0(R,K')$$
$$+\delta\Gamma_{lm}^{(0)}(R,K')]dK'. \quad (31)$$

For further transformations it is convenient to add Eq. (31) term by term to the equation which is its complex conjugate. After this we have for the left-hand side of Eq. (31)

$$\Big\{2e_i^*[\Lambda_{ij},C]e_j+(e_i^*[\Lambda_{ij},e_je_k^*]e_k+e_i[\Lambda_{ij}^*,e_j^*e_k]e_k^*)C$$
$$+2\Big(2iD_{ij}^ae_je_i^*-\frac{\partial^2\Lambda_{ij}}{\partial R\,\partial K}e_je_i^*\Big)C\Big\}\delta(\Lambda). \quad (32)$$

Using (15) we can reduce the first term in the braces in (32) to the form

$$-4Q\left|\frac{\partial H}{\partial \mathbf{k}}\right|\frac{dC}{ds}. \quad (33)$$

Using Eqs. (16) to (18) we can write the second term as

$$\{2\,\mathrm{Sp}^{-1}\hat{\lambda}e_i^*e_j[\Lambda_{ij},\mathrm{Sp}\,\hat{\lambda}]+\mathrm{Sp}^{-2}\hat{\lambda}[\Lambda_{ij},\Lambda](\delta_{ij}\Lambda_{nn}-\Lambda_{ji})$$
$$+\Lambda\,\mathrm{Sp}^{-2}\hat{\lambda}[\Lambda_{ij},\delta_{ij}\Lambda_{nn}-\Lambda_{ji}]+\mathrm{Sp}^{-1}\hat{\lambda}[\Lambda_{ij},\lambda_{ij}]\}C. \quad (34)$$

By virtue of (15) the first term in (34) is equal to

$$8Q\left|\frac{\partial H}{\partial \mathbf{k}}\right|\mathrm{Sp}^{-1}\hat{\lambda}\frac{d}{ds}\mathrm{Sp}\,\hat{\lambda},$$

the second can be reduced to the form

$$-4Q\left|\frac{\partial H}{\partial \mathbf{k}}\right|\mathrm{Sp}^{-1}\hat{\lambda}\frac{d}{ds}\mathrm{Sp}\,\hat{\lambda},$$

and the third and fourth terms vanish because in the transparency region of the plasma we have $\Lambda=0$ and $[\Lambda_{ij},\lambda_{ij}]=0$. As a result the whole expression can be written in the form

$$-4Q\left|\frac{\partial H}{\partial \mathbf{k}}\right|\Big\{\frac{dC}{ds}-\mathrm{Sp}^{-1}\hat{\lambda}\frac{d}{ds}\mathrm{Sp}\,\hat{\lambda}+\frac{1}{L}\Big\}C\delta(\Lambda), \quad (35)$$

where we have introduced the notation

$$\frac{1}{L}=-Q^{-1}\left|\frac{\partial H}{\partial \mathbf{k}}\right|^{-1}\Big(iD_{ij}^ae_je_i^*-\frac{1}{2}\frac{\partial^2\Lambda_{ij}}{\partial R\,\partial K}e_je_i^*\Big). \quad (36)$$

After we add Eq. (31) to its complex conjugate the right-hand side has the following form:

$$-4\pi\,\mathrm{Sp}\,\hat{\lambda}\delta(\Lambda)\int e_i^*e_l'B_{jlpm}(R,K-K',K',-K')e_pe_m'^*$$
$$\times\Big[\sum_{\alpha=1}^{2}C_\alpha^0(R,K')\delta(\mathbf{k}_\alpha-\mathbf{k}')\delta(\omega_\alpha-\omega')+C(R,K')\delta(\Lambda)\Big]dK', \quad (37)$$

where the quantity $C_\alpha^0(R,K')$ characterizes the intensity of the coherent component of the wave field.

5. ENERGY OF THE SCATTERED WAVE AND RAY INTENSITY

The energy density of the scattered field can be determined by means of a well known formula (see, e.g., Refs. 16 and 17) which in the present case takes the form

$$w(R,\omega,\mathbf{k})=\frac{1}{16\pi}\frac{\partial\omega^2\langle\varepsilon_{ij}(R,\omega)\rangle}{\omega\,\partial\omega}\delta\Gamma_{ij}^*(R,K). \quad (38)$$

Restricting ourselves in the expression for the coherence matrix of the scattered field $\delta\hat{\Gamma}$ to zeroth-order terms in the small quantity μ and using (29) we note that the expression for the total energy of the scattered waves,

$$W(R)=\int d\omega\int d^3k\,w(R,\omega,\mathbf{k}),$$

can be written as follows:

$$W(R)=\sum_{\alpha=1}^{2}\int d\omega\int d\Omega_\alpha\frac{J_\alpha}{|\mathbf{v}_\alpha|}. \quad (39)$$

where

$$J_\alpha=\frac{\omega c}{16\pi}n_\alpha^2\frac{C_\alpha\,\mathrm{Sp}\,\hat{\lambda}_\alpha}{|\cos\vartheta_\alpha|}$$

the index α labels the type of the normal wave, n_α, $\mathbf{v}_\alpha$, ϑ_α are the refractive index, the group velocity vector, and the angle between the group velocity vector and the wavevector, and $d\Omega_\alpha$ is an element of solid angle in the wavevector space of waves of type α. The Poynting vector of the scattered waves is defined by the relation

$$\mathbf{S}(R)=\sum_{\alpha=1}^{2}\int d\omega\int d\Omega_\alpha\frac{\mathbf{v}_\alpha}{|\mathbf{v}_\alpha|}J_\alpha. \quad (40)$$

6. RADIATIVE TRANSFER EQUATION

Using (35), (37), and the definition (39) of the ray intensity we get from Eq. (31) by equating the coefficients of $\delta(k^2-k_\alpha^2)$

$$\frac{n_\alpha^2}{|\cos\vartheta_\alpha|}\frac{d}{ds}\Big[\frac{J_\alpha|\cos\vartheta_\alpha|}{n_\alpha^2}\Big]$$
$$=-\frac{1}{L_\alpha}J_\alpha+\sum_{\beta=1}^{2}\sigma_{\alpha\beta}J_\beta^0+\sum_{\beta=1}^{2}\int d\omega\,d\Omega_\beta\sigma_{\alpha\beta}J_\beta, \quad (41)$$

where

$$\sigma_{\alpha\beta}=e_{\alpha i}^*e_{\beta l}'B_{jlpm}(R,K-K',K',-K')e_{\alpha p}e_{\beta m}'^*$$
$$\times\left|\frac{\partial H_\alpha}{\partial\mathbf{k}}\right|^{-1}\left|\frac{\partial H_\beta}{\partial\mathbf{k}}\right|^{-1}Q_\alpha^{-1}Q_\beta^{-1}$$

is the cross-section for the scattering of a wave of type β into a wave of type α, while the ray intensity of the coherent component is determined by the relation

$$J_\beta^0=\frac{cn_\beta}{8\pi}\frac{|\langle\mathbf{E}_\beta(R)\rangle|^2}{|\cos\vartheta_\beta|}\Big[|\mathbf{e}_\beta|^2-\frac{|\mathbf{e}_\beta\mathbf{k}|^2}{k^2}\Big]. \quad (42)$$

The set of Eqs. (41) is the analog of the radiative transfer equation for a randomly inhomogeneous magnetized plasma. When one can neglect conversion of the normal waves in the scattering, the set (41) splits into two independent equations describing the transfer of radiation in the ordinary and the extraordinary components. We note that the structure of the left-hand side of Eq. (41) is analogous to the corresponding structure of the RTE in a regular magnetized plasma.[18]

In the case of a stationary cold collisionless magnetized plasma and when there is no spatial dispersion the second term in (36) vanishes. The first term in (36) can be written in the form

$$iD_{ij}^{a}e_i^{*}e_j={}^{1}/{}_{2}k_0[\ \varepsilon_{ij}^{\text{eff}}(R,\mathbf{k},\omega)-\varepsilon_{ji}^{\text{eff}*}(R,\mathbf{k},\omega)\,]. \tag{43}$$

For a statistically uniform and stationary magnetized plasma the effective dielectric tensor in the Bourret approximation is determined by the expression[3]

$$\varepsilon_{ij}^{\text{eff}}(\mathbf{k},\omega)=\langle\hat{\varepsilon}_{ij}(\omega)\rangle-k_0^2[\langle\hat{\varepsilon}_{il}(\omega)\rangle-\delta_{il}]$$
$$\times[\langle\hat{\varepsilon}_{jn}(\omega)\rangle-\delta_{jn}]\int G_{ln}{}^{0}(\omega,\mathbf{p})\Phi_N(\mathbf{k}-\mathbf{p})\,d^3p, \tag{44}$$

where $\Phi_N(\varkappa)$ is the spatial spectrum of the random inhomogeneities. Bearing in mind that in a uniform magnetized plasma the relation[16] $G_{ln}^{0}(\omega,\mathbf{p})=\lambda_{ln}(\omega,\mathbf{p})/\Lambda(\omega,\mathbf{p})$ holds, and substituting (44) into (43), we obtain

$$iD_{ij}^{a}e_i^{*}e_j=i\frac{k_0}{2}\int \operatorname{Sp}\hat{\lambda}(\omega,\mathbf{p})\,|e^{*}(\mathbf{p})\,(\langle\hat{\varepsilon}(\omega)\rangle-\hat{1})\,e(\mathbf{k})\,|^2$$
$$\times\left[\frac{1}{\Lambda^{*}(\mathbf{p},\omega)}-\frac{1}{\Lambda(\mathbf{p},\omega)}\right]\Phi_N(\mathbf{k}-\mathbf{p})\,d^3p.$$

Using (18) and (19) after simple transformations we get the following relation:

$$iD_{ij}^{a}\,e_{\alpha i}^{*}e_{\alpha j}=k_0^2\sigma_{0\alpha}n_\alpha(\omega)Q_\alpha|\cos\vartheta_\alpha|^{-1}, \tag{45}$$

where

$$\sigma_{0\alpha}=\sum_{\beta=1}^{2}\int d\omega\, d\Omega_\beta\sigma_{\beta\alpha}$$

is the total (integral) scattering cross-section, while the partial scattering cross-section $\sigma_{\alpha\beta}$ has the well known form[16]

$$\sigma_{\alpha\beta}=\frac{\pi k_0^2}{4}\frac{|\mathbf{e}_\alpha^{*}(\langle\hat{\varepsilon}\rangle-\hat{1})\mathbf{e}_\beta|^2}{Q_\alpha Q_\beta}\frac{n_\alpha}{n_\beta}|\cos\vartheta_\beta|\,\Phi_N(\mathbf{r},\mathbf{k}_\alpha-\mathbf{k}_\beta).$$

Substitution of (45) into (36) gives the following relation:

$$L_\alpha^{-1}=\sigma_{0\alpha}. \tag{46}$$

If there are dissipative losses in the medium we must add the appropriate absorption coefficient in the right-hand side of Eq. (46).[8] We have obtained Eq. (46) using Eq. (44), which is valid in a statistically uniform medium. It is clear that Eq. (44) and hence Eq. (46) retain their validity for a statistically inhomogeneous and nonstationary medium in the geometric optics approximation [provided conditions (5) to (8) are satisfied].

In that case Eqs. (41) can thus be written in the form

$$\frac{n_\alpha^2}{|\cos\vartheta_\alpha|}\frac{d}{ds}\left[\frac{J_\alpha|\cos\vartheta_\alpha|}{n_\alpha^2}\right]=-\sigma_{0\alpha}J_\alpha+\sum_{\beta=1}^{2}\sigma_{\alpha\beta}J_\beta^{0}$$
$$+\sum_{\beta=1}^{2}\int d\omega\, d\Omega_\beta\sigma_{\alpha\beta}J_\beta. \tag{47}$$

The last transformation gives a completely clear physical meaning to all terms occurring in the equation. The RTE describes the change in the intensity of the scattered radiation along a ray trajectory passing through a point in space in a given direction. The first term on the right-hand side of (47) describes the decrease in the intensity of the incoherent component due to secondary scattering. The second term corresponds to the influx of energy into the incoherent component when the coherent component is scattered. The third term gives the influx of energy of the incoherent radiation in the given direction due to secondary scattering of waves of both types, which had already been scattered, propagating in all possible directions. We note that the total energy flux can be found by using (40) to sum over all ray trajectories passing through a given point. In the case of a magnetized plasma the use of the RTE thus gives the result which could have been obtained by means of a phenomenological approach using the geometric-optics energy balance of the scattered components.

CONCLUSION

Equations (41) can be used to analyze multiple scattering processes in a randomly inhomogeneous magnetized plasma. The range of applicability is restricted only by the condition that the geometric-optics approximation be valid. This means in particular that these equations cannot be used near the region where the normal waves are reflected, where $n_\alpha=0$ and there where $n_1\approx n_2$. In the latter case one can apply the RTE in the quasi-isotropic geometric-optics approximation.[12–14] The equations do not describe inverse scattering processes, since in that case it is necessary to take into account wave corrections which requires leaving the framework of the linear radiative transfer theory.[7] In conclusion we note that the method used in the present paper to obtain the radiative transfer equation can apparently easily be generalized to the case of any randomly inhomogeneous anistropic medium.

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