The influence of spatial dispersion and capillary effects on the propagation and attenuation of nonlinear surface waves

A. S. Kovalev and E. S. Syrkin

Low-Temperature Physico-Technical Institute, Ukrainian Academy of Sciences

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We investigate nonlinear surface shear waves in an anharmonic crystal and compare them to nonlinear bulk waves. We demonstrate that such waves can exist in the presence of spatial dispersion, and use an asymptotic method to construct explicit expressions for the wave properties in a narrow region of parameters near the dispersion law for linear bulk waves. When capillary effects are included, we find that nonlinear surface waves can exist for arbitrary values of the anharmonicity of the elastic medium. When both spatial dispersion and capillary effects are included, our analysis of the third harmonic of the nonlinear surface wave reveals that the latter radiates bulk waves from the surface into the depth of the crystal, leading to attenuation. We estimate the rate of this attenuation.

INTRODUCTION

For many decades, elastic surface waves have found a variety of applications, both as probes in the physical investigation of solids and as a basis for engineering devices. Especially interesting are high-frequency (hypersonic) elastic waves, whose proper description requires the inclusion of nonlinear effects (in particular, nonlinear interactions between elastic waves) and spatial dispersion (due to the decreasing wavelength of the hypersonic vibrations). In surface waves we expect the influence of anharmonicity to be more striking than in bulk waves, since the intensity of a surface wave considerably exceeds that of a bulk wave with the same total power due to the concentration of the surface wave energy in a thin layer near the surface. In addition, the fundamental characteristics of surface waves should be strongly influenced by surface distortion (capillary effects).[1–5] Surface waves are usually treated within the linear theory of elasticity; when these effects are included in the analysis, the surface waves will be modified in important ways, leading to a number of qualitatively new results, especially in the study of pure shear waves with horizontal polarization (SH-waves).

Mozhaev[6] showed that a new type of surface shear acoustic wave (SAW), whose localization at the crystal boundary is entirely due to inclusion of nonlinearity, can exist in a nonlinear elastic semi-infinite medium. In investigating a simple model of shear waves in a nonlinear medium, Gorentsveig *et al.*[7] found self-modulating nonlinear surface waves which generalize the SAW solutions found in Ref. 6. These nonlinear surface waves are described by soliton and multisoliton solutions to the nonlinear Schroedinger equation (NSE), which the dynamic equations of a nonlinear elastic medium reduce to in lowest approximation. We note here that a description of nonlinear SAW within the framework of the NSF has a number of peculiarities. First of all, the results obtained are single-frequency and do not contain higher harmonics. Second, due to the complete integrability of the NSE, there is no radiation of bulk elastic waves by the surface vibrations and hence no associated attenuation of the SAW, either for zero or for more general boundary conditions at the surface of the crystal. Finally, we note that the model investigated in Refs. 6 and 7 did not include spatial dispersion of the nonlinear acoustic waves.

The goal of this paper is to investigate nonlinear shear surface waves taking into account their spatial dispersion, and also attenuation of these waves due to radiation by higher harmonics when more general boundary conditions (capillary effects) are taken into account.

1. FORMULATION OF THE MODEL AND THE SIMPLEST NONLINEAR SURFACE WAVES

For a simple cubic elastic lattice with linear A and nonlinear C interactions between first (α), second (β), and third (γ) nearest neighbors described by the constants A_α, A_β, A_γ and C_α, C_β, C_γ respectively, atomic mass m, and interatomic distance a, the dynamic equations for the scalar displacement $u(x,y,z,t)$ in the medium have the following form in the long-wavelength approximation:[8]

$$\begin{aligned} mu_{tt}={}&a^2(A_\alpha+4A_\beta+4A_\gamma)(u_{xx}+u_{yy}+u_{zz})+{}^1/_{12}a^4(A_\alpha+A_\beta+A_\gamma)\\ &\times(u_{xxxx}+u_{yyyy}+u_{zzzz})+a^4(A_\alpha+2A_\beta)(u_{xxyy}+u_{yyzz}+u_{xxzz})\\ &+3a^4(C_\alpha+4C_\beta+4C_\gamma)(u_x^2u_{xx}+u_y^2u_{yy}+u_z^2u_{zz})+6a^4(C_\beta+2C_\gamma)\\ &[u_x^2u_{yy}+u_x^2u_{zz}+u_y^2u_{xx}+u_y^2u_{zz}+u_z^2u_{xx}+u_z^2u_{yy}\\ &+4(u_xu_yu_{xy}+u_xu_zu_{xz}+u_yu_zu_{yz})], \end{aligned} \tag{1}$$

where $u_{tt}=\partial^2u/\partial t^2$, $u_{xx}=\partial^2u/\partial x^2$, etc.

Naturally, Rayleigh waves cannot propagate within such a model; however, a pure shear surface wave with horizontal polarization exists even in the scalar model.

In the limit $a^2\to0$, Eq. (1) becomes a wave equation without dispersion

$$u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz}), \tag{2}$$

where c is the velocity of sound, $c^2=(a^2/m)(A_\alpha+4A_\beta+4A_\gamma)$.

For a pure shear surface wave propagating along the x axis and uniform in the direction of the y axis, the proper choice of scales for time, the coordinates, and wave amplitudes ($[x]=a$, $[z]=a$, $[t]=a/c$, $[u]=\{|C_\alpha+4C_\beta+4C_\gamma|/(A_\alpha+4A_\beta+4A_\gamma)\}^{1/2}$ reduces Eq. (1) to dimensionless form (for definiteness we will assume $C_\alpha+4C_\beta+4C_\gamma<0$, which corresponds to a "focusing" elastic medium)

$$\begin{aligned} u_{tt}-u_{xx}-u_{zz}+\delta(3u_x^2u_{xx}+3u_z^2u_{zz}+\lambda(u_x^2u_z)_z+\lambda(u_z^2u_x)_x)\\ -\varkappa(u_{xxxx}+u_{zzzz}+\mu u_{xxzz})=0, \end{aligned} \tag{3}$$

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where $\lambda = 6(C_\beta + 2C_\gamma)/(C_\alpha + 4C_\beta + 4C_\gamma)$ and $\mu = (A_\beta + 2A_\gamma)/(A_\alpha + 4A_\beta + 4A_\gamma)$, and δ is the sign function, which equals 1 for focusing media and -1 for defocusing media with $C_\alpha + 4C_\beta + 4C_\gamma > 0$. In this model we have $\varkappa = 1/12$; however, we will introduce a formal multiplier $\varkappa$ into the "dispersion" term of Eq. (3) so that in what follows we can retain the option of treating the limiting case of a model without spatial dispersion. In the "dispersionless" limit $\varkappa \to 0$ (for $\lambda = 1$) we obtain from Eq. (3) the equation used in Ref. 6. We note that the choice of sign $\delta = 1$ for the anharmonic moduli [i.e., the choice of sign in front of the anharmonic terms in Eq. (3)] is not entirely orthodox: as a rule, investigators of soliton solutions in nonlinear elastic one-dimensional systems in terms of the nonlinear equations of an elastic string choose the opposite sign for the anharmonic terms.

Equation (3) must be supplemented by boundary conditions at the crystal surface (the plane $z = 0$). When stresses are absent at a free surface this condition takes the form

$$[u_z(1-u_z^2-\lambda u_x^2)+\varkappa u_{zzz}+\varkappa\mu u_{xxz}]_{z=0}=0. \tag{4}$$

In the limit $\varkappa = 0$ (i.e., in the lowest approximation with respect to a^2) expression (4) reduces to the Maradudin boundary condition,[9] and, as was shown in Ref. 7, the existence of small-amplitude weakly localized nonlinear surface waves requires only the condition

$$u_z|_{z=0}=0. \tag{5}$$

Along with condition (5), in what follows we will make use of a more general boundary condition, which takes into account near-boundary distortion of the lattice (capillary effects). Within the framework of Eq. (3), even if we neglect nonlinear and dispersive ($\sim\varkappa$) terms, to accuracy of terms of order $\sim a$ our boundary condition has the form $|u_z|_{z=0} = \Delta z(u_{tt} - u_{xx})|_{z=0}$ (in the variables we have chosen the thickness of the near-surface layer is $\Delta z \approx a = 1$), and consequently for a solution periodic in t and x [$\sim\sin(kx - \omega t)$] it reduces to

$$(u_z+\gamma u)|_{z=0}=0, \tag{6}$$

where $\gamma = \Delta z(\omega^2 - k^2)$. The parameter γ has a more complicated form when its dependence on the crystal plane and elastic modulus in the near-surface layer are taken into account;[5] however, γ always contains terms $\sim\omega^2$ and $\sim k^2$, and therefore we will always assume it is small ($\gamma \ll 1$).

For the case of an unbounded crystal linear wave of the form $u(x,z,t) = \sin(kx + qz - \omega t)$ have the following dispersion law

$$\omega^2=(k^2-\varkappa k^4)+q^2-\varkappa q^4-\varkappa\mu k^2q^2, \tag{7a}$$

while for a wave independent of

$$\omega^2=k^2-\varkappa k^4\ . \tag{7b}$$

When we neglect spatial dispersion ($\varkappa = 0$) we obtain the linear spectrum $\omega = k$, which is illustrated by the dashed line in Fig. 1. In this same figure the solid curve shows the dispersion law (7b). Of course, the expressions we have obtained are meaningful for small k ($k \ll 1$), but the small-k behavior is not of interest here.

In the expression for the simplest type of surface waves, the coordinate x along the surface and the time t appear in the combination $\zeta = x - Vt$, and the dependence on the phase $\vartheta = kx - \omega t = k\zeta$ is assumed to be periodic (here, k is the wave vector of the surface wave and $\omega = kV$ is its frequency). For such a wave Eq. (3) reduces to the following:

$$(1-V^2)u_{\zeta\zeta}+u_{zz}-\delta[3u_\zeta^2u_{\zeta\zeta}+3u_z^2u_{zz}+\lambda(u_\zeta^2u_z)_z+\lambda(u_z^2u_\zeta)_\zeta] +\varkappa(u_{\zeta\zeta\zeta\zeta}+\mu u_{\zeta\zeta zz}+u_{zzzz})=0. \tag{8}$$

The authors of Refs. 6 and 7 studied the fundamental harmonic of a solution to this equation localized in z of the form $u \approx f(z)\sin(kx - \omega t)$ for $\varkappa = 0$, neglecting the z-dependence in the nonlinear terms (8). This solution has the standard soliton form

$$f(z)=\frac{2^{3/2}}{3^{1/2}}\frac{(k^2-\omega^2)^{1/2}}{k^2}\,\mathrm{ch}^{-1}[(k^2-\omega^2)^{1/2}z]. \tag{9}$$

A somewhat unusual feature is the presence of the wave vector k in the denominator of the expression for the soliton amplitude, which is connected with the structure of the nonlinear term in the original equation. It is clear from (9) that this solution exists for $\omega < k$, i.e., in the "below-spectrum" region, for linear waves with $V = 1$ (the single dashes in Fig. 1). In Refs. 6 and 7 it was noted that the solutions so obtained are small in amplitude only when the inequality $\omega - k \ll k$ is satisfied.

2. NONLINEAR SURFACE AND INTERNAL WAVES IN A DISPERSIVE MEDIUM

We now include spatial dispersion in the discussion, and find the true surface wave solution, which includes higher harmonics. Since the function u is periodic in one of its variables, we can use an asymptotic procedure to find the solution of Eq. (8),[10] according to which the function $u(z,\zeta)$ is written as a Fourier series in the variable ζ with expansion coefficients that depend on z (in this case, the parameter ω which is contained in ζ is arbitrary for a given k), while the amplitudes of the harmonics are expanded in a power series with respect to a small parameter which characterizes the deviation of the solution from a linear wave:

$$u(z,\zeta)=\sum_{n=0}^{\infty}f_{2n+1}(z)\sin[(2n+1)\vartheta]. \tag{10}$$

A natural requirement for the convergence of this expansion is fulfillment of a series of inequalities $f_{s+2} \ll f_s$ (in what follows we will verify that this expansion can be used to find the small-amplitude solutions with $f_1 \ll 1$). Substituting (10) into Eq. (8) leads to an infinite system of ordinary differential equations for the functions f_s. These equations are too

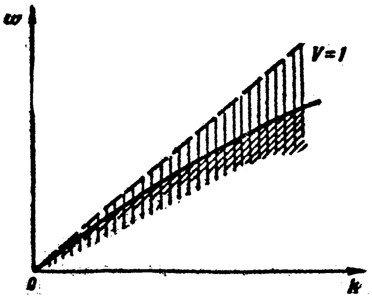

FIG. 1. Region in which nonlinear surface waves exist in the dispersionless limit (vertical hatching), and when spatial dispersion is taken into account (crosshatching).

complicated to write in explicit form. However, their overall structure is as follows:

$$-s^2(k^2-\varkappa s^2k^2-\omega^2)f_s-(1-\varkappa\mu s^2k^2)\partial^2 f_s/\partial z^2+\varkappa\partial^4 f_s/\partial z^4 + k^4\Phi_s+\lambda k^2 F_s+U_s=0. \quad (11)$$

Here the Φ_s are infinite sums of products $f_i f_j f_k$, F_s contains triple products of the functions f_i and their first and second derivatives with respect to z (the total number of derivatives in each term equals 2), and U_s consists of triple products of the form f_i', f_j', f_k'', where the primes denote derivatives with respect to z.

In using the asymptotic procedure the natural small parameter for expanding the functions f_s is the deviation of the frequency ω of the nonlinear surface wave from the value of the frequency of the linear wave from the same value of wave number. It is easy to set up a hierarchy ordering the functions f_s:

$$f_s=\sum_{n=0}^{\infty} f_{s,2n+s}\varepsilon^{2n+s}. \quad (12)$$

$$\varepsilon=(1-\varkappa k^2-V^2)^{1/2}\ll 1. \quad (13)$$

Since there is a connection between the smallness of the amplitude of the nonlinear surface wave and its weak localization near the surface, it is convenient to introduce a new coordinate scale along the z axis: $\eta=\varepsilon z$. Let us write down the equations for the coefficients f_{ij} for the first three harmonics of the solution through order ε^5 (for $\delta=1$):

$$(1-\varkappa\mu k^2)f_{11}''-k^2 f_{11}+{}^3/_4 k^4 f_{11}^3=0, \quad (14)$$

$$(1-\varkappa\mu k^2)f_{13}''-(k^2-{}^9/_4 k^4 f_{11}^2)f_{13} =-\varkappa f_{11}''''-{}^9/_4 k^4 f_{11}^2 f_{33}+{}^1/_{12}\lambda k^2(f_{11}^3)'', \quad (15)$$

$$72\varkappa k^4 f_{33}+{}^3/_4 k^4 f_{11}^3=0. \quad (16)$$

$$72\varkappa k^4 f_{35}-9k^2 f_{33}(1-9\varkappa\mu k^2)f_{33}''+{}^9/_4 k^4 f_{11}^2 f_{13} -{}^5/_4\lambda k^2 f_{11}(f_{11}')^2-{}^1/_4\lambda k^2 f_{11}^2 f_{11}''=0, \quad (17)$$

$$600\varkappa k^4 f_{55}+9k^4 f_{11}^2 f_{33}=0, \quad (18)$$

where the primes denote derivatives with respect to η.

For the simplest boundary condition (5), to lowest order we obtain from (14) the following solution which describes nonlinear surface waves:

$$\varepsilon f_{11}=\frac{2^{3/2}}{3^{1/2}}\frac{(k^2-\varkappa k^4-\omega^2)^{1/2}}{k^2}\mathrm{ch}^{-1}\left[\left(\frac{k^2-\varkappa k^4-\omega^2}{1-\varkappa\mu k^2}\right)^{1/2} z\right]. \quad (19)$$

This expression is closest to the solution (9) obtained in Refs. 6 and 7, and reduces to it as $\varkappa\to 0$. The region in which surface waves exist in a dispersive medium is shown in Fig. 1 by oblique crosshatching and lies under the dispersion curves for linear waves (7b). The lower boundary of this region is found from the condition of convergence of the asymptotic procedure: $\varepsilon^2 f_{i,j+2}\ll f_{ij}$, $\varepsilon^2 f_{i+2,i+2}\ll f_{ii}$. Note an unusual feature of the location of the region of parameters ω and k in which the surface waves exist. Usually in one-dimensional systems spatially localized "bion" solutions exist for values of the parameter adjacent to the dispersion curve and in the direction in which this curve is convex (i.e., solitons localized along the x axis having parameters that lie above the solid curve in Fig. 1, but they occur when the sign of the nonlinearity is reversed).

The solution of the system of equations (14)–(18) is found rather simply and has the following form (due to its complexity, we will not write out the solution for f_{35}):

$$f_{11}=\frac{2^{3/2}}{3^{1/2}k}\mathrm{ch}^{-1}pz,\qquad f_{33}=-\frac{2^{3/2}}{3^{1/2}k}\frac{1}{18\varkappa k^2}\mathrm{ch}^{-3}pz,$$

$$f_{55}=\frac{2^{3/2}}{3^{1/2}k}\frac{1}{360\varkappa^2k^4}\mathrm{ch}^{-5}pz,$$

$$f_{13}=\frac{2^{3/2}}{3^{1/2}k}\frac{1}{36\varkappa k^2}\left[36g^2\frac{pz\,\mathrm{sh}\,pz}{\mathrm{ch}^2 pz}+(2+25\lambda g^2-972g^2)\mathrm{ch}^{-1}pz -(1-8\lambda g^2+288g^2)\mathrm{ch}^{-3}pz\right], \quad (20)$$

where $g=\varkappa k^2/(1-\varkappa\mu k^2)$ and $p=\varepsilon(g/\varkappa)^{1/2}$.

From the expressions given here there follow the following estimates: $\varepsilon^2 f_{33}/f_{11}\propto\varepsilon^2 f_{55}/f_{33}\propto\varepsilon^2 f_{13}/f_{11}\propto\varepsilon^2/\varkappa k^2$. Thus, the true small parameter for the expansion we have made is the quantity $\varepsilon^2/\varkappa k^2$. This is characteristic of asymptotic methods, in which the expansion is in powers of the ratio of the deviation of the frequency of the nonlinear wave from the linear frequency to the magnitude of the dispersion D (where $D=\partial^2\omega/\partial k^2\sim\varkappa$). The region of applicability of the asymptotic method (the crosshatching in Fig. 1 mentioned above) is determined by the inequality $(k^2-\varkappa k^4)-\omega^2\ll\varkappa k^4$; i.e., there is considerably less space between the dispersive and nondispersive spectra (the dashed and solid curves in Fig. 1). We find that as the dispersion of the medium decreases (i.e., as $\varkappa$ decreases) the range of applicability of the method contracts, until it disappears in the nondispersive limit ($\varkappa=0$). Therefore, the question of the existence of nonlinear surface waves of the form (9) remains open for a nondispersive medium.

Nonlinear surface waves possess a number of features that distinguish them from ordinary linear surface waves. First of all, in linear surface waves (Rayleigh waves) the frequency is fixed for a given value of k and differs from that of internal waves with the same k. In contrast, for a given value of k the frequency of a nonlinear surface wave, though it must lie below the frequency of the linear internal wave, can have a continuous spectrum of values. The frequency of a nonlinear surface wave is connected with its amplitude. As is clear from (19),

$$\omega^2=(k^2-\varkappa k^4)-{}^3/_8 k^4 G^2, \quad (21)$$

where G is the amplitude of the surface wave. For fixed frequency and wave amplitude (given by the source of radiation) the excited nonlinear surface wave has a smaller wavelength than the linear one.

Furthermore, in the nonlinear case delocalized nonlinear internal waves also exist for a given k at the frequency of the nonlinear surface wave. Actually, it is easy to verify that Eq. (8) admits solutions that are uniform in z, in which the strain is described by the formula

$$u_\zeta^{w}=[2^{3/2}rK(r)\varkappa^{1/2}/\pi]\,k\,\mathrm{sn}[2K(r)(kx-\omega t)/\pi,\ r]; \quad (22)$$

here $K(r)$ is the complete elliptic integral of the first kind, $\mathrm{sn}[\varphi,r]$ is the Jacobi elliptic sine, r is the modulus of the elliptic function and, as before, $k=2\pi/\Lambda$ is the wave number (where Λ is the wavelength). For fixed values of k and ω the modulus r (which gives the wave amplitude) is determined implicitly by the expression

$$(k^2-\omega^2-\varkappa k^4)^{1/2}/k^2=\varkappa^{1/2}[4K^2(r)(1+r^2)/\pi^2-1]^{1/2}. \qquad (23)$$

For comparison we present the expression for the strain of a surface wave for $z=0$:

$$u_\zeta^s=[2^{1/2}(k^2-\varkappa k^4-\omega^2)^{1/2}/3^{1/2}k^2]k\sin(kx-\omega t). \qquad (24)$$

Note that Eqs. (22) and (24) differ even for small amplitudes: for $r\ll 1$ we have $u_\zeta^w = u_\zeta^s/2^{1/2}$; i.e, the amplitude of the internal wave is smaller than that of the surface wave.

Let us rewrite relation (23) in a somewhat different way, so that it takes the form of a nonlinear dispersion law: if we introduce the strain amplitude $A=2^{3/2}rK(r)\varkappa^{1/2}/\pi$ of the nonlinear wave, and use the asymptotic form $K(r)\approx\pi(1+r^2/4)/2$ for small amplitudes (and therefore for small $r\ll 1$), we obtain the nonlinear dispersion law for internal waves:

$$\omega^2=k^2-\varkappa k^4-3k^2A^2/4 \qquad (25)$$

(as it turns out, $A\approx Gk/2^{1/2}$).

Knowledge of the dispersion laws (21) and (25) allows us to draw conclusions about the longitudinal stability of internal and surface nonlinear waves against their decay into solitons localized along the direction of propagation of the wave (the z axis). In fact, from Eqs. (21) and (25) it follows that for small k the quantity $(\partial^2\omega/\partial k^2)/(\partial\omega/\partial A^2)$ is greater than zero, and equals $16\varkappa$ and $8\varkappa$ for surface and bulk waves, respectively. According to the well-known Lighthill criterion,[11] for this sign of the inequality nonlinear waves of constant amplitude are modulationally stable. Consequently, the surface waves investigated here are also stable against decay into a train of solitons localized in the plane of the crystal surface.

The specific features of nonlinear surface waves we have discussed here have the following physical interpretation. In the linear theory, elastic waves (phonons) in an unbounded volume do not interact; Rayleigh waves arise from the interaction of longitudinal and transverse phonons at the surface. In a nonlinear medium with the sign of the anharmonicity we have chosen (a focusing medium) an effective attraction in the perpendicular direction arises between phonons propagating along the x axis [note that when we substitute the wave solution $\sin(kx-\omega t)$ into the anharmonic term in Eq. (3) of the form $(\partial u/\partial x)^2(\partial^2u/\partial x^2)$, the sign of this nonlinear term changes and the wave interaction acquires the characteristics of an attraction]. Because of this attraction the phonons form bound multiphonon states (dynamic solitons) even in a nondispersive medium, which is equivalent to focusing of phonons (localization of the phonon current in a plane perpendicular to the direction of its propagation). If we view such states as solitons, we may also consider surface waves to be "half a soliton." What is unusual about this situation is the fact that in the majority of nonlinear evolution systems the attraction of elementary excitations leads to the formation of bound states localized in all directions (multidimensional solitons), whereas in the problems under discussion here the phonons repel in the direction of propagation and attract in the transverse direction. This results in a nonlinear wave that is localized near the surface and uniform along it.

3. THE EFFECT OF CAPILLARY PHENOMENA ON NONLINEAR SURFACE WAVES

Up to now we have treated the simplest boundary condition (5). We will now discuss the influence of capillary effects on the propagation of nonlinear surface waves (this problem was investigated for the first time in Ref. 12). For this we use boundary condition (6) at the crystal surface. In order to satisfy this condition, it is sufficient to use the solution (20) we have found for the fundamental harmonic, with a shift in its argument. To accuracy up to ε^3 inclusive, the solution for the fundamental harmonic has the form

$$u_1=\sin(kx-\omega t)\frac{2^{1/2}}{3^{1/2}k}\Big\{2\varepsilon\,\mathrm{ch}^{-1}[p(z+z_0)] + \frac{\varepsilon^3}{36\varkappa k^2}\Big[36g^2\frac{p(z+z_0)}{\mathrm{ch}^2[p(z+z_0)]}\mathrm{sh}[p(z+z_0)] + (2+25\lambda g^2-972g^2)\mathrm{ch}^{-1}[p(z+z_0)] - (1-8\lambda g^2+288g^2)\mathrm{ch}^{-3}[p(z+z_0)]\Big]\Big\}. \qquad (26)$$

The boundary condition (6) is easily satisfied by choosing the constant z_0. It is also in the form of a series in even powers of the small parameter ε. Using Eq. (26), we can write z_0 to accuracy ε^2. However, in what follows we will require only the lowest approximation for the quantity z_0:

$$z_0=\frac{1}{p}\,\mathrm{Arcth}\,\frac{\gamma}{p}. \qquad (27)$$

First of all, it is clear from Eq. (27) that when we take capillary effects into account the nonlinear surface wave exists only when the inequality $|\gamma|<p$ holds. We recall that $p=\varepsilon k^2/(1-\varkappa\mu k^2)^{1/2}\propto\varepsilon\ll 1$; i.e., the amplitude of the surface wave ($u_{\max}\propto\varepsilon$) should exceed the capillary parameter γ (which in this case remains small in order to ensure convergence of the asymptotic expansion). This condition is easily fulfilled since we have assumed $\gamma\ll 1$.

Second, it is clear from (27) that the character of the nonlinear surface wave depends significantly on the sign of the capillary parameter γ. For $\gamma>0$ the dependence of the field amplitude on position z becomes closer to the corresponding dependence in a linear surface wave (curve *1* in Fig. 2). On the same figure, in the form of curve *2* we show the profile of the nonlinear surface wave for $\gamma<0$. In this case a competition between the nonlinear and capillary effects occurs. Because of the capillary effects, the wave is "repelled" from the boundary; this is balanced by the nonlinear "attraction." This nonlinear surface wave is apparently unstable against "breaking away" from the boundary to form a bulk soliton.

Let us turn to an investigation of higher harmonics, i.e., the parts of the solution proportional to $\sin 3(kx-\omega t)$. In this case it turns out that the shift in the argument of solution (20) for f_{33} we found earlier does not result in an expression that satisfies the boundary condition (6). The physical rea-

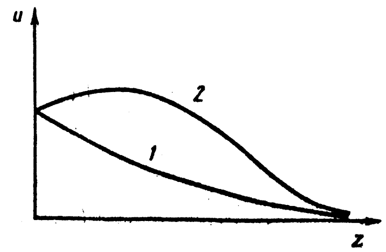

FIG. 2. Profile of the envelope of a nonlinear surface wave for different signs of the capillary parameter. Curve *1* corresponds to the case $\gamma > 0$, curve *2* to the case $\gamma < 0$.

son for this is the fact that internal sound waves at the third harmonic are radiated from the surface into the depth of the crystal. Expression (20) for f_{33} follows from Eq. (16), which was obtained by means of the asymptotic method. In deriving the hierarchy of Eqs. (14)–(18) we assumed that the derivative $\partial/\partial z$ is small, and terms containing $\partial^n f_{33}/\partial z^n$ were neglected in Eq. (16) by virtue of their smallness in comparison to the lowest-order term, which is proportional to f_{33}. The investigation of internal waves in which $u \propto \sin(qz)$ requires that we include these terms in Eq. (16), which now has the following form:

$$72\varkappa k^4 f_{33} + (1-9\varkappa\mu k^2)\frac{\partial^2 f_{33}}{\partial z^2} + \varkappa\frac{\partial^4 f_{33}}{\partial z^4} = -\frac{3}{4}k^4 f_{11}^3. \qquad (28)$$

(Previously the second and third terms on the left-hand side were discarded.) Now we can choose an approximate solution to Eq. (28) in the form

$$f_{33} = f_{33}^{(0)} + P\cos qz, \qquad (29)$$

where $f_{33}^{(0)}$ is described by an expressional analogous to (20) with argument $p(z+z_0)$. When we substitute the general solution for the uniform equation into the left side of (28), we obtain for the z component of the wave vector q the equation

$$\varkappa q^4 - (1-9\varkappa\mu k^2)q^2 + 72\varkappa k^4 = 0. \qquad (30)$$

It is easy to verify that this equation coincides to lowest order in ε with expression (7a) for the dispersion law of bulk elastic waves, if in the latter we make the substitution $k \to 3k$, $\omega \to 3\omega$ (which corresponds to treating the third harmonic) and make use of the definition (13) in the form $\omega^2 = k^2 - \varkappa k^4 - k^2\varepsilon^2$. Thus, for a surface wave with fixed wave vector k linear internal waves are radiated at the third harmonic into the interior of the crystal. This phenomenon is entirely a consequence of including spatial dispersion. In the dispersionless limit $\varkappa = 0$ there is no wave emerging from the surface ($q = 0$). For small dispersion ($\varkappa \ll 1$) we have

$$q \approx 6(2\varkappa)^{1/2}k^2 \ll 1. \qquad (31)$$

Thus, these waves propagate into the interior of the crystal at a very small angle τ to the surface: $\tan\tau = q/3k = 2(2\varkappa)^{1/2}k \ll 1$. [Actually, for our model $\varkappa = 1/12$, and for small surface wave vectors $k \ll 1$ we have $q \approx 6^{1/2}k^2$ and $\tau = (2/3)^{1/2}k \ll 1$.]

The final expression for the third-harmonic solution, taking into account the direction of propagation of the wave from the boundary, has the following form:

$$u_3 = -\frac{2^{1/2}}{3^{1/2}k}\frac{\varepsilon^3}{18\varkappa k^2}\frac{\sin(3kx-3\omega t)}{\mathrm{ch}^3[p(z+z_0)]} + P\sin(3kx-3\omega t+qz-\alpha). \qquad (32)$$

For the components that do not contain localized corrections [$\propto\cos(3kx-3\omega t)$] the boundary conditions give the following relation for the constant α:

$$\mathrm{tg}\,\alpha = q/\gamma, \qquad (33)$$

requiring that the boundary condition for the components that are $\propto\sin(3kx-3\omega t)$ be satisfied let us find the amplitude of the outgoing wave from the surface P:

$$P = \frac{2^{3/2}}{3^{1/2}}\frac{\gamma\varepsilon^3}{18\varkappa k^3 p^3}\frac{(p^2-\gamma^2)^{3/2}}{(q^2+\gamma^2)^{1/2}}. \qquad (34)$$

Since $p \propto \varepsilon$ and $\gamma < p$, we have $P \propto \varepsilon^3 \ll 1$. Knowing the final expression [see Eqs. (32)–(34)] for the solution at the third harmonic, it is easy to calculate the energy flux from the surface into the crystal and the attenuation of the nonlinear surface wave. The energy flux density (per unit area of the surface) is determined by the expression $Q = \langle 2(\partial u/\partial t)(\partial u/\partial z)\rangle$, where the angle brackets denote averaging over a period of the wave. Thus, the damping rate of the nonlinear surface wave equals

$$\frac{dE}{\partial t} = -\left\langle 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\right\rangle = -3VkqP^2 = 2Vq\gamma^2\varepsilon^6(p^2-\gamma^2)^3[81\varkappa^2k^4p^6(q^2+\gamma^2)]^{-1/2}. \qquad (35)$$

If we set $\varkappa = 1/12$ and $\mu = 2$ (i.e., the case of an isotropic crystal), then for small values of k, when $\omega \approx k$ and $q \approx 6^{1/2}k^2$, Eq. (35) simplifies (in addition we will assume that $\gamma \ll \varepsilon k^2$):

$$\frac{dE}{dt} = \frac{32}{9\cdot 6^{1/2}}\frac{\gamma^2}{k^{12}}[\omega^2(k)-\omega^2]^3, \qquad (36)$$

where $\omega(k)$ is the dispersion law (7b).

Note that our treatment here deals only with those nonlinear surface wave energy losses connected with radiation of internal waves into the interior of the crystal. Along with this there undoubtedly exist dissipative losses, which we will not discuss in this paper.

Let us compare the damping (36) of the surface wave with its intrinsic energy. To lowest order in the parameter ε the expression for the energy of a nonlinear surface wave passing through a unit area of the boundary (in the customary dimensionless variables) is

$$E = \int_0^\infty \frac{dz}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}$$

and for small values of k

$$E \approx {}^4/_3[\omega^2(k)-\omega^2]^{1/2}/k^3. \qquad (37)$$

As time passes, the energy and amplitude of the wave decrease; however, in this case it is not clear whether the frequency alone increases $[\omega\to\omega(k)]$ or if there a simultaneous change in the wavelength. Therefore, we present only an estimate for the characteristic attenuation time of the nonlinear surface wave: $T\sim E/(\partial E/\partial t)\sim k^4/\gamma^2\varepsilon^5$. The corresponding distance over which the surface wave can propagate in this time is in order of magnitude

$$L\sim ak^4/\gamma^2\varepsilon^5. \quad (38)$$

Thus, even for the maximum attainable values of $\gamma - \varepsilon k^2$ the distance L the wave propagates is proportional to ε^{-7} and is large for small-amplitude surface waves.

In conclusion, we emphasize once more the importance of including the spatial dispersion of the nonlinear elastic medium. Dispersion reveals itself to be important in two ways: first of all, the region in which the nonlinear surface exists with respect to amplitude and frequency is finite only for finite values of the dispersion $\varkappa$ $(\varkappa\gg\varepsilon^2/k^2)$. Second, the appearance of internal radiation when capillary effects are taken into account is also associated with dispersion, in that $q\sim Q\sim\varkappa^{1/2}$. (Note that as $\varkappa$ decreases the expressions we have derived lose their meaning as soon as $\varkappa\sim\varepsilon^2/k^2$.)

4. SURFACE WAVES IN A "DEFOCUSING" MEDIUM

Finally, let us briefly touch on the case of a "defocusing" elastic medium, in which we have $C_\alpha + 4C_\beta + 4C_\gamma > 0$ and $\delta = -1$ (this sign of the anharmonic terms is usually encountered in discussions of soliton dynamics of one-dimensional nonlinear chains[11]). In this case all the anharmonic terms in Eqs. (3), (8), (14)–(18), and (28) change sign, and the character of the nonlinear elastic waves changes considerably. Now the expression for the strain in the nonlinear internal wave has the form (22), in which the Jacobi sine $\mathrm{sn}(\varphi,r)$ is replaced by an elliptic cosine $\mathrm{cn}(\varphi,r)$. In this case the nonlinear dispersion law of these waves also change significantly:

$$\omega^2=k^2-\varkappa k^4[4K^2(r)(1-2r^2)/\pi^2]. \quad (39)$$

For small wave amplitudes $(r\ll1)$ Eq. (39) simplifies

$$\omega^2\approx k^2-\varkappa k^4+3A^2k^4/4, \quad (40)$$

where A is the amplitude of the wave displacement. Comparing this equation with (21) and (25), we see that the frequency of the nonlinear internal waves now lies above the dispersion law for linear waves, and according to the Lighthill criterion the internal wave is unstable against longitudinal modulation and decays into a train of solitons localized along the x axis. It is not difficult to evaluate the growth rate of this instability. For small wave amplitudes A the time τ for their decay into solitons is a quantity $\sim(kA^2)^{-1}$. On the other hand, nonlinear internal waves are stable against transverse compression. Therefore, for the simplest boundary conditions (5) the nonlinear surface wave does not form when $C_\alpha + 4C_\beta + 4C_\gamma > 0$. However, the situation is changed when we take into account capillary effects. In this case propagation of nonlinear surface waves whose frequency as before lies below the spectrum of nonlinear elastic waves once more becomes possible. [That is, in contrast to the previous case, where for $\omega<\omega(k)$ both surface and internal waves existed, now the surface waves exist for $\omega<\omega(k)$, while the internal waves (which decay into solitons localized along the x axis) exist for $\omega>\omega(k)$].

We will limit our discussion only to the lowest-order approximation for the nonliner surface wave. The equation for the lowest order approximation in ε, substituting Eq. (14), now has the form

$$(1-\varkappa\mu k^2)f_{11}''-k^2f_{11}-{}^3/_4k^4f_{11}^3=0, \quad (41)$$

and this equation has the following solution:

$$f_{11}=\frac{2^{1/2}}{3^{1/2}k}\,\mathrm{sh}^{-1}[p(z-z_0)]. \quad (42)$$

The constant z_0 in his expression is found from the boundary condition (6):

$$z_0=\frac{1}{p}\,\mathrm{Arccth}\frac{\gamma}{p}. \quad (43)$$

Since the solution (42) (is singular), it satisfies our problem only when $z_0>0$. Consequently, in that case surface waves exist only for $\gamma>0$. Furthermore, it follows from (43) that $\gamma>p$. Since $p=\varepsilon k(1-\varkappa\mu k^2)^{-1/2}$, it is necessary to satisfy the inequality $\gamma>\varepsilon k=(3^{1/2}/2^{3/2})Gk^2$, where G is the amplitude of the surface waves; i.e., for $C_\alpha + 4C_\beta + 4C_\gamma > 0$ $(\delta = -1)$ there exist only surface waves with amplitudes smaller than the capillary parameter. The profile of such waves has the form of curve *1* in Fig. 2.

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