Supersymmetry of a two-level system in a variable external field

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This paper shows how the method of supersymmetric quantum mechanics can be employed to obtain the exact solutions to a broad spectrum of problems describing a two-level system in an alternating field.

1. INTRODUCTION

Exact methods of determining the behavior of a two-level system in an alternating field are of considerable interest because they reveal the physical aspects of the interaction of laser radiation with matter in atomic-collision theory and are used to build models of various physical situations. The problem cannot always be solved analytically, however. The exact solutions via a hypergeometric differential equation can be found in Refs. 1 and 2.

This paper considers the problem of the behavior of a two-level system in an alternating field from the angle of supersymmetric quantum mechanics.[3,4] The supersymmetry method enables establishing the exactly solvable cases of this problem, on the other hand, and finding the solutions via algebraic calculations, on the other.

2. THE TWO-LEVEL SYSTEM IN AN ALTERNATING FIELD OF VARIABLE AMPLITUDE

The behavior of a two-level system in an external alternating field is described by the following system of differential equations:[1,2]

$$\begin{cases} i\dot{a}_1(t)=V(t)e^{-i\varepsilon t}a_2(t) \\ i\dot{a}_2(t)=V^*(t)e^{i\varepsilon t}a_1(t) \end{cases}, \tag{1}$$

where ε is the resonance detuning, $\hbar V(t)$ the energy of the interaction of the external field with the two-level system, and $a_{1,2}(t)$ the population amplitudes of the ground $|1\rangle$ and excited $|2\rangle$ states. Below we assume that $V(t) = V^*(t)$. For a two-level atom in an external electromagnetic field in the resonance approximation ε is qual to $\omega_{21} - \omega$, where ω_{21} is the *atomic transition frequency*, and ω the *laser field frequency*.

Let us assume that before the external field was switched on the system was in the $|1\rangle$ state, that is, we subject system (1) to the following initial conditions

$$\begin{cases} a_1(t\to-\infty)=1 \\ a_2(t\to-\infty)=0 \end{cases}. \tag{2}$$

Obviously, as $t\to+\infty$ the population amplitudes $a_{1,2}(t)$ acquire the following form:

$$a_1(t\to+\infty)=A_1\exp\left\{-\frac{i}{2}(\varepsilon-2\lambda)t\right\}$$

$$+A_2\exp\left\{-\frac{i}{2}(\varepsilon+2\lambda)t\right\},$$

$$a_2(t\to+\infty)=A_1\frac{\varepsilon-2\lambda}{2V_0}\exp\left\{\frac{i}{2}(\varepsilon-2\lambda)t\right\}$$

$$+A_2\frac{\varepsilon+2\lambda}{2V_0}\exp\left\{\frac{i}{2}(\varepsilon+2\lambda)t\right\}, \tag{3}$$

where

$$\lambda=\left[\frac{\varepsilon^2}{4}+V_0^2\right]^{1/2}, \qquad V_0=V(t\to+\infty). \tag{4}$$

Thus, to calculate the probability of the system's transition from state $|1\rangle$ to state $|2\rangle$ we must only find the coefficients A_1 and A_2.

We introduce the function

$$b(t)=a_1(t)\exp\left\{\frac{i\varepsilon t}{2}\right\}+a_2(t)\exp\left\{-\frac{i\varepsilon t}{2}\right\}. \tag{5}$$

Clearly, $b(t)$ satisfies the following second-order ordinary differential equation:

$$\frac{d^2b(t)}{dt^2}+\left(\frac{\varepsilon^2}{4}+V^2(t)+i\dot{V}(t)\right)b(t)=0. \tag{6}$$

This equation resembles the time-independent Schrödinger equation in which t acts as the spatial coordinate and the difference between the total and potential energies is $\varepsilon^2/4 + V^2(t) + i\dot{V}(t)$. As $t\to\pm\infty$, the function $b(t)$ satisfies the following conditions:

$$b(t)=\exp\left(\frac{i\varepsilon t}{2}\right), \quad t\to-\infty,$$
$$b(t)=B_1\exp(i\lambda t)+B_2\exp(-i\lambda t), \quad t\to+\infty. \tag{7}$$

Combining (5), (2), and (7), we obtain a relation that links $A_{1,2}$ and $B_{1,2}$:

$$A_{1,2}=\frac{2V_0}{2V_0+\varepsilon\mp2\lambda}B_{1,2}. \tag{8}$$

The structure of Eq. (6) has a remarkable property: it can always be factorized, that is, can be represented in the form

$$Q^+Q^-b(t)+\frac{\varepsilon^2}{4}b(t)=0, \tag{9}$$

where

$$Q^\pm=\pm i\frac{d}{dt}+V(t). \tag{10}$$

This property can be used to solve Eq. (9) or (1) subject to

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conditions (7) or (2), respectively, by the supersymmetry method known from quantum mechanics.[3,4] Following this method, we consider the supersymmetric counterpart of Eq. (9),

$$Q^-Q^+b_1(t)+\frac{\varepsilon^2}{4}b_1(t)=0 \tag{11}$$

with the following conditions:

$$b_1(t)=\exp\left(\frac{i\varepsilon t}{2}\right),\quad t\to-\infty,$$

$$b(t)=B_1^{(1)}\exp(i\lambda t)+B_2^{(1)}\exp(-i\lambda t),\quad t\to+\infty. \tag{12}$$

From Eqs. (11) and (9) it follows that if the particular solution $b_1(t)$ of Eq. (11) is known, then

$$b(t)=\alpha Q^+b_1(t) \tag{13}$$

is the solution to Eq. (9), with α an arbitrary constant.

Now let us suppose that the function $V(t)$ is the solution of the following functional differential equation:

$$V^2(a_n,t)-i\dot V(a_n,t)=V^2(a_{n+1},t)+i\dot V(a_{n+1},t)+C(a_n,a_{n+1}), \tag{14}$$

where

$$C(a_n,a_{n+1})=V_+^2(a_n)-V_+^2(a_{n+1})=V_-^2(a_n)-V_-^2(a_{n+1}), \tag{15}$$

with the a_n comprising the set of parameters of the external field, and $V_\pm(a_n)=V(a_n,t\to\pm\infty)$. Note that Eq. (14) is invalid for real a_n since $V(t)$ is assumed to be a real function. Using Eq. (13), we arrive at a relation that links $B_{1,2}$ and $B_{1,2}^{(1)}$:

$$B_{1,2}(a_0)=\frac{V_+(a_0)\mp\lambda(a_0)}{V_-(a_0)-\varepsilon/2}B_{1,2}^{(1)}(a_0). \tag{16}$$

Assuming that condition (14) holds true for $V(a,t)$, we obtain the following recurrence formula:

$$B_{1,2}(a_0)=\frac{V_+(a_0)\mp\lambda(a_0)}{V_-(a_0)-\varepsilon/2}B_{1,2}(a_1). \tag{17}$$

The procedure described can be repeated, replacing at each step a_n with a_{n+1} in $B_{1,2}(a_n)$ and $V_\pm(a_0)$. As a result we get

$$B_{1,2}(a_0)=B_{1,2}(a_N)\prod_{n=0}^{N-1}\frac{V_+(a_n)\mp\lambda(a_0)}{V_-(a_n)-\varepsilon/2}. \tag{18}$$

Let us now see how $B_{1,2}(a_N)$ can be determined. Problem (6) subject to conditions (7) is known to be exactly solvable for an arbitrary $V(t)$ if $\varepsilon=0$. Equation (9) illustrates this fact in a straightforward manner. The case where $\varepsilon\neq0$ can also be reduced by the recurrence transformations (18) to a known solution. Indeed, after the Nth step we arrive at an equation of the following type:

$$Q^+(a_N)Q^-(a_N)b(a_N,t)+\frac{\varepsilon_{\text{eff}}^2(a_N)}{4}b(a_N,t)=0. \tag{9$'$}$$

where

$$Q^\pm(a_N)=\pm i\frac{d}{dt}+V(a_N,t). \tag{19}$$

$$\frac{\varepsilon_{\text{eff}}^2(a_N)}{4}=\frac{\varepsilon^2}{4}+V_+^2(a_0)-V_+^2(a_N). \tag{20}$$

The condition $\varepsilon_{\text{eff}}(a_N)=0$ is satisfied for complex N, that is, we must perform an analytic continuation of the product (18) defined on the set of natural numbers N to complex-valued N. Then the particular solution of Eq. (9′) has the form

$$b(a_N,t)=C(a_N)\exp\left(-i\int^t V(a_N,\tau)d\tau\right). \tag{21}$$

where a_N is determined by the equation $\varepsilon_{\text{eff}}(a_N)=0$. Imposing the conditions at infinity, $t\to\pm\infty$, we can find $B_{1,2}(a_N)$.

Thus, if $V(a,t)$ satisfies the functional differential equation (14), the problem of interaction of an external field of variable amplitude with a two-level system is reduced to solving the recurrence equations (18).

We illustrate this method with an example[1,2] where

$$V(t)=\frac{V_0}{2}(\text{th}\,\gamma t+1). \tag{22}$$

As $\gamma\to0$ we have the limiting case of adiabatic switch-on, while for $\gamma\to\infty$ the field is suddenly switched on at $t=0$. With respect to its parameter, (22) does not satisfy Eq. (14). For the equation to be valid in this case, we represent $V(t)$ as follows:

$$V(t)=V(a,t)=\frac{V_0}{2}\left(a\,\text{th}\,\gamma t+\frac{1}{a}\right),$$

where a is a parameter. It is easy to verify that with respect to parameter a function $V(a,t)$ satisfied Eq. (14). For $V(a,t)$ we have $a_n=(2i\gamma/V_0)n+1$ and

$$C(a_n,a_{n+1})=\frac{V_0^2}{4}\left(a_n+\frac{1}{a_n}\right)^2-\frac{V_0^2}{4}\left(a_{n+1}+\frac{1}{a_{n+1}}\right)^2,$$

with $n=0,1,\ldots$.

After simple calculations we arrive at the following relations for $B_{1,2}(a_0)$:

$$B_1(a_0)=\prod_{n=0}^{N-1}\frac{V_+(a_n)-\lambda(a_0)}{V_-(a_n)-\varepsilon/2}B_1(a_N)$$
$$=\frac{\Gamma(\delta-\alpha+N)\Gamma(\beta-N+1)}{\Gamma(\delta-\alpha)\Gamma(\beta+1)}B_1(a_N), \tag{23}$$

$$B_2(a_0)=\prod_{n=0}^{N-1}\frac{V_+(a_n)+\lambda(a_0)}{V_-(a_n)-\varepsilon/2}B_2(a_N)$$
$$=\frac{\Gamma(\delta-\beta+N)\Gamma(\alpha-N+1)}{\Gamma(\delta-\beta)\Gamma(\alpha+1)}B_2(a_N), \tag{24}$$

where $\delta=i\varepsilon/2\gamma$, and

$$\alpha=\frac{i}{4\gamma}(\varepsilon+2V_0-[\varepsilon^2+4V_0^2]^{1/2}),$$

$$\beta=\frac{i}{4\gamma}(\varepsilon+2V_0+[\varepsilon^2+4V_0^2]^{1/2}).$$

To find, say, $B_1(a_N)$ we must solve the equation $\varepsilon_{\text{eff}}=0$, that is,

$$\frac{\varepsilon^2}{4}+\sum_{n=0}^{N-1}C(a_n,a_{n+1})=\frac{\varepsilon^2}{4}+V_0^2-\frac{V_0^2}{4}\left(a_N+\frac{1}{a_N}\right)^2=0,$$

and allow for the conditions (7). This yields $N=\beta-\delta$ and $B_1(a_N)=1$. Substituting these into (23), we find that

$$B_1(a_0)=\frac{\Gamma(\beta-\alpha)\Gamma(\delta+1)}{\Gamma(\delta-\alpha)\Gamma(\beta+1)}$$

or, allowing for (8), we arrive at an expression for A_1:

$$A_1=\frac{\Gamma(\beta-\alpha)\Gamma(\delta)}{\Gamma(\delta-\alpha)\Gamma(\beta)}.$$

In a similar way we can find $B_2(a_0)$ and A_2 if we substitute $N=\alpha-\delta$ and $B_2(a_N)=1$ into (24). The final expression for a_1 is

$$a_1(t\to+\infty)=\frac{\Gamma(\beta-\alpha)\Gamma(\delta)}{\Gamma(\delta-\alpha)\Gamma(\beta)}\exp\left\{-\frac{i}{2}(\varepsilon-2\lambda)t\right\}+\frac{\Gamma(\alpha-\beta)\Gamma(\delta)}{\Gamma(\delta-\beta)\Gamma(\alpha)}\exp\left\{-\frac{i}{2}(\varepsilon+2\lambda)t\right\},$$

which coincides with the results of Refs. 1 and 2, where this problem was solved be means of a hypergeometric equation.

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