Effective action in the Nambu–Jona-Lasinio gauge model

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We study the Nambu–Jona-Lasinio (NJL) gauge model, which lies at the basis of the modern mechanisms of dynamical breaking of the electroweak symmetry without invoking fundamental Higgs fields. By making use of an approach based on the formalism of Green's functions of composite operators we obtain the effective low-energy action of the NJL gauge model in the form of a series in powers of derivatives of composite fields. An explicit formula for the effective action is produced in the case of a weak ($\alpha \ll 1$) gauge interaction. The structure of the effective action with respect to scale transformations in the region of coupling constants $\alpha \leqslant \alpha_c = \pi/3$ is established.

1. INTRODUCTION

The dynamics of bound states is one of the central problems in particle physics, starting with the description of hadrons in quantum chromodynamics (QCD) and ending with the dynamic breaking of the electroweak symmetry and composite models of leptons and quarks. The basic problem here is the derivation of the low-energy effective action for bound states. The best known example of this approach is the σ model,[1] which provides a low-energy description of bound states of the Nambu–Jona-Lasinio (NJL) model[2] with dynamical breaking of chiral symmetry. The derivation of the low-energy action for the NJL model has been considered in a number of papers.[3] The problem, however, is substantially more complex if the starting microscopic Lagrangian includes a gauge interaction. This is precisely the situation typical in particle physics, when the fundamental Lagrangian is chosen to be that of QCD or the model of the electroweak interaction with dynamical symmetry breaking.

In the present work we study this problem for the choice of microscopic theory in the form of the NJL gauge model with dynamical symmetry breaking.[4] This model is of particular interest since it lies at the basis of various scenarios of dynamical breaking of the electroweak symmetry (see the recent review in Ref. 5): technicolor with slowly changing coupling constant,[6] extended technicolor with strong coupling constant[7,8,9] and finally, the standard model without a Higgs sector, realizing the idea of broken electroweak symmetry by the formation of a condensate of t quarks.[9–11]

Although, just like in the classical NJL model,[2] the four-fermion interaction forms an important part, there is also an important difference: the dynamics is formed here in the critical region near a second-order phase-transition point with breaking of chiral symmetry, where $m_d/\Lambda \ll 1$ [m_d is dynamical mass of the fermion, and Λ is the ultraviolet cutt-off (the scale of the "new physics")].[4,12,13]

The NJL gauge model is described by the Lagrangian

$$L = -{}^1\!/_4 F_{\mu\nu}{}^2 + \bar\Psi i\gamma D\Psi + {}^1\!/_2 G_0[(\bar\Psi\Psi)^2 + (\bar\Psi i\gamma_5\Psi)^2], \quad (1)$$

where the gauge interaction ($D_\mu = \partial_\mu - ieA_\mu$) is treated in the ladder approximation and the four-fermion interaction is treated in the Hartree-Fock approximation[2] [for simplicity we consider only the case of $U(1)$ gauge symmetry and $U_L(1) \times U_R(1)$ chiral symmetry].

We introduce the chiral fields σ and π and write the Lagrangian (1) in the form

$$L = -\frac{1}{4} F_{\mu\nu}{}^2 + \bar\Psi i\gamma D\Psi - \bar\Psi(\sigma + i\gamma_5\pi)\Psi - \frac{1}{2G_0}(\sigma^2 + \pi^2). \quad (2)$$

One readily verifies the equivalence of the Lagrangians (1) and (2) by making use of the Euler-Lagrange equations

$$\sigma = -G_0\bar\Psi\Psi, \quad \pi = -G_0\bar\Psi i\gamma_5\Psi. \quad (2a)$$

The effective action can be obtained by integration in the functional integral over the fermion and gauge fields. Precisely here a significant difference arises between the model under consideration and the standard NJL model. In the pure NJL model there are no gauge fields and the problem reduces to the calculation of the fermionic determinant in the external fields σ and π. The presence of quantized gauge fields considerably complicates the problem.

The problem of evaluation of the effective action in a gauge NJL model was considered recently by Bardeen and Love[14] (see also Ref. 15). In this work an alternative method is developed based on the formalism of Green's functions for composite operators.[16] This approach is applicable to a broad class of models with dynamical symmetry breaking. We will obtain the effective action for the NJL gauge model in the form of a series in powers of derivatives of composite fields and will discuss the mechanism of scale symmetry breaking in the model. It is relevant that for $\alpha \neq 0$ the dynamics of the model leads to composite σ and π fields with noncanonical dynamical dimensions $d_\sigma \neq 1$. As was already noted in Refs. 8–10, 15, 17, 18, and 5, the spinless bound states play an important role in the scenarios of dynamical electroweak symmetry breaking based on the dynamics of NJL gauge models. This is related to the specific structure of these states (strongly bound states[5,18]). In particular, they turn out to represent essential degrees of freedom even at energies significantly exceeding the scale of the electroweak interactions $F_{ew} \sim 250$ GeV, in contrast to QCD where the bound states become irrelevant for $q \gg \Lambda_{\rm QCD}$.

It is therefore to be expected that in the case under consideration the effective Lagrangian plays a more fundamental role than in QCD. The present investigation is a step in the direction of explicit realization of the dynamical picture with such bound states.

2. EXPANSION OF THE EFFECTIVE ACTION IN POWERS OF DERIVATIVES

The expansion inpowers of derivatives is widely utilized in models with spontaneous symmetry breaking.[19] Let us

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show that for the NJL gauge model the derivation of the low-energy action in the form of a series in powers of derivatives of the composite σ and π fields reduces to the calculation of the Greens functions $G^{(n)}(q_1,q_2,...,q_n)$ of these fields near the point $q_1=q_2=...=q_n$ in QED in the ladder approximation.

We consider small fluctuations of the fields $\tilde{\sigma}(x)$, $\tilde{\pi}(x)$ about the configuration with the constant values σ_0, π_0:

$$\begin{aligned} \sigma(x)&=\sigma_0+\tilde{\sigma}(x),\\ \pi(x)&=\pi_0+\tilde{\pi}(x). \end{aligned} \tag{3}$$

We then obtain from the QED Lagrangian [compare with (2)]

$$\bar{L}=-{}^1/_4F_{\mu\nu}{}^2+\bar{\Psi}i\gamma D\Psi-\bar{\Psi}(\sigma+i\gamma_5\pi)\Psi \tag{4}$$

the following expansion for the effective action $\bar{\Gamma}(\Phi^{(i)})$, $i=s,p$, $\Phi^{(s)}=\sigma$, $\Phi^{(p)}=\pi$:

$$\begin{aligned} \bar{\Gamma}(\Phi^{(i)}) &= \bar{\Gamma}(\Phi_0^{(i)}) + \int d^4x \frac{\delta\bar{\Gamma}}{\delta\Phi^{(i)}(x)}\bigg|_{\Phi^{(i)}=\Phi_0^{(i)}} \cdot \tilde{\Phi}^{(i)}(x)\\ &+\frac{1}{2}\int d^4x d^4y \frac{\delta^2\bar{\Gamma}}{\delta\Phi^{(i)}(x)\delta\Phi^{(j)}(y)}\bigg|_{\Phi^{(i)}=\Phi_0^{(i)}}\\ &\times\tilde{\Phi}^{(i)}(x)\tilde{\Phi}^{(j)}(y)+\cdots . \end{aligned} \tag{5}$$

It follows from the Lagrangian (4) that

$$\frac{\delta\bar{\Gamma}}{\delta\Phi^{(s)}(x)}\bigg|_{\Phi^{(i)}=\Phi_0^{(i)}} = -\langle 0|\bar{\Psi}\Psi|0\rangle_{\Phi_0^{(i)}}\equiv\bar{\Delta}_s, \tag{6}$$

$$\frac{\delta\bar{\Gamma}}{\delta\Phi^{(p)}(x)}\bigg|_{\Phi^{(i)}=\Phi_0^{(i)}} = -\langle 0|\bar{\Psi}i\gamma_5\Psi|0\rangle_{\Phi_0^{(i)}}\equiv\bar{\Delta}_p, \tag{7}$$

$$\frac{\delta^2\bar{\Gamma}}{\delta\Phi^{(i)}(x)\delta\Phi^{(j)}(y)}\bigg|_{\Phi^{(i)}=\Phi_0^{(i)}} = \bar{\Delta}_{ij}(x-y), \tag{8}$$

where the connected Green's functions $\bar{\Delta}_{ij}$ are given by the expressions

$$\begin{aligned} \bar{\Delta}_{ss}(x-y)&=i\langle 0|T\bar{\Psi}\Psi(x)\bar{\Psi}\Psi(y)|0\rangle_{\Phi_0^{(i)}},\\ \bar{\Delta}_{pp}(x-y)&=i\langle 0|T\bar{\Psi}i\gamma_5\Psi(x)\bar{\Psi}i\gamma_5\Psi(y)|0\rangle_{\Phi_0^{(i)}},\\ \bar{\Delta}_{sp}(x-y)&=i\langle 0|T\bar{\Psi}\Psi(x)\bar{\Psi}i\gamma_5\Psi(y)|0\rangle_{\Phi_0^{(i)}},\\ \bar{\Delta}_{ps}(x-y)&=i\langle 0|T\bar{\Psi}i\gamma_5\Psi(x)\bar{\Psi}\Psi(y)|0\rangle_{\Phi_0^{(i)}} \end{aligned} \tag{9}$$

(the index $\Phi_0^{(i)}$ indicates that the Green's function is calculated in QED with the "bare" mass $m^{(0)}=\sigma_0=+i\gamma_5\pi_0$).

Making use of the expansion

$$\begin{aligned} \tilde{\Phi}^{(i)}(y)&=\tilde{\Phi}^{(i)}(x)+\frac{\partial\tilde{\Phi}^{(i)}}{\partial x^\mu}(y-x)^\mu\\ &+\frac{1}{2}\frac{\partial^2\tilde{\Phi}^{(i)}}{\partial x^\mu\partial x^\nu}(y-x)^\mu(y-x)^\nu+\dots, \end{aligned} \tag{10}$$

we can write the effective action in the form:

$$\begin{aligned} \bar{\Gamma}(\Phi^{(i)})&=\bar{\Gamma}(\Phi_0^{(i)})+\int d^4x\bigg[\bar{\Delta}_i\tilde{\Phi}^{(i)}(x)\\ &+\frac{1}{4}\frac{\partial^2\Delta_{ij}(q)}{\partial q_\mu\partial q_\nu}\bigg|_{q=0}\frac{\partial\Phi^{(i)}}{\partial x^\mu}\frac{\partial\Phi^{(j)}}{\partial x^\nu}+\dots\bigg], \end{aligned} \tag{11}$$

where $\Delta_{ij}(q)$ is the Green's function in momentum space $[\Delta_{ij}(q)=\int d^4x\exp(iqx)\,\bar{\Delta}_{ij}(x)]$. It follows from (2) that the effective action for the NJL gauge model is

$$\Gamma(\sigma,\pi)=\bar{\Gamma}(\sigma,\pi)-\frac{1}{2G_0}\int d^4x(\sigma^2+\pi^2). \tag{12}$$

The problem reduces in this way to the calculation of the Green's functions of composite operators and their derivatives at the point $q_i=0$ in ladder QED with bare mass $m^{(0)}=\sigma_0+i\gamma_5\pi_0$. The general method of calculation of Green's functions for composite operators was developed in Ref. 16, and in what follows we use the technique described in that work.

3. EFFECTIVE POTENTIAL IN THE NJL GAUGE MODEL

As was already noted in the introduction, from the point of view of application to electroweak symmetry breaking the most interesting dynamical regime in the NJL gauge model is the near-critical regime with $m_d/\Lambda\ll 1$. In particular, we are especially interested in the local limit $\Lambda\to\infty$, with m_d remaining finite in that limit.[4,12,13]

We start with the derivation of the effective potential. Although the expression for the potential for $\alpha<\alpha_c=\pi/3$ has already appeared in the literature,[15] we discuss in more detail its derivation. New in the present discussion is the derivation of the effective potential in the strictly local limit, which turns out to be relevant to the explanation of the mechanism of scale symmetry breaking in the model under consideration. We also obtain the potential in the case of supercritical dynamics with $\alpha\geqslant\alpha_c$.

We recall first the basic properties of the solution of the Schwinger-Dyson (SD) equation for the fermion mass function $\Sigma(p)$ in ladder QED[4,12,20] (see Appendix A). As is usual, we make use of the Landau gauge.

For $\alpha<\alpha_c=\pi/3$ the solution for $\Sigma(p)$ has the ultraviolet asympototic form:

$$\Sigma(p)\underset{p\to\infty}{\approx}\tilde{A}\frac{\Sigma_0^2}{p}\frac{1}{\omega}\text{sh}\Big[\omega\Big(\ln\frac{p}{\Sigma_0}+\delta\Big)\Big], \tag{13}$$

where $\Sigma_0\equiv\Sigma(0)$, $\omega=(1-\alpha/\alpha_c)^{1/2}$, and $\tilde{A}(\alpha)$, $\delta(\alpha)$ are functions of the coupling constant α. We note that in the so-called linearized approximation for the SD equation (which approximates the nonlinear equation well for the entire range of momenta) the functions $\tilde{A}(\alpha)$ and $\delta(\alpha)$ reduce to

$$\begin{aligned} \tilde{A}(\alpha)&=2\bigg[\frac{\Gamma(1+\omega)\Gamma(1-\omega)}{\Gamma([3+\omega]/2)\Gamma([3-\omega]/2)\Gamma([1+\omega]/2)\Gamma([1-\omega]/2)}\bigg]^{1/2}\\ \delta(\alpha)&=\frac{1}{2\omega}\ln\bigg[\frac{\Gamma(1+\omega)\Gamma([3-\omega]/2)\Gamma([1-\omega]/2)}{\Gamma(1-\omega)\Gamma([3+\omega]/2)\Gamma([1+\omega]/2)}\bigg]. \end{aligned} \tag{14}$$

In the region $\alpha<\alpha_c$ the solutuion of the SD equation for $\Sigma(p)$ can be obtained from (13) with the help of the replacement $\omega\to i\tilde{\omega}$, $\tilde{\omega}=(\alpha/\alpha_c-1)^{1/2}$:

$$\Sigma(p)\underset{p\to\infty}{\approx}\tilde{A}\frac{\Sigma_0^2}{p}\frac{1}{\tilde{\omega}}\sin\Big[\tilde{\omega}\Big(\ln\frac{p}{\Sigma_0}+\delta\Big)\Big]. \tag{15}$$

It is well known[4,12,20,21] that for $\alpha<\alpha_c$ there is no solution in ladder QED corresponding to spontaneous chiral symmetry breaking. This means that the minimum of the effective potential $\bar{V}(\Phi_0^{(i)})[-\int d^4x\bar{V}(\Phi_0^{(i)})=\bar{\Gamma}(\Phi_0^{(i)})]$ lies at the symmetric point $\bar{\Phi}_0^{(i)}=0$. However inclusion of the four-fermion interaction changes the situation. In that case the potential takes the form [see (12)]

$$V(\Phi_0^{(i)})=\overline{V}(\Phi_0^{(i)})+\frac{1}{2G_0}\Phi_0^{(j)}\Phi_0^{(j)} \quad (16)$$

and can have a minimum at the nonsymmetric point $\overline{\Phi}_0^{(i)}\neq 0$ even for $\alpha<\alpha_c$. The expression for the critical line (where $\overline{\Phi}_0^{(i)}/\Lambda\to 0$) dividing the phases with broken and unbroken chiral symmetry was first obtained in Ref. 13:

$$g=g_c(\alpha)={}^1/_4[1+(1-\alpha/\alpha_c)^{1/2}]^2,\quad \alpha<\alpha_c, \quad (17)$$

where $g\equiv G_0\Lambda^2/4\pi^2$. For values $\alpha<\alpha_c$ spontaneous symmetry breaking takes place for all $g>g_c(\alpha)$, and the effective potential $V(\overline{\Phi}_0^{(i)})$ (16) describes the fluctuations near the nonsymmetric vacuum $\overline{\Phi}_0^{(i)}\neq 0$. The situation is different for $\alpha>\alpha_c$. While for $\alpha<\alpha_c$ it is mainly the four-fermion interaction that is responsible for the spontaneous breaking of chiral symmetry, for $\alpha>\alpha_c$ the main role is played by the electromagnetic interaction. In this region the critical line has the simple form:

$$\alpha=\alpha_c,\quad g\leqslant {}^1/_4. \quad (18)$$

In that case spontaneous breaking of chiral symmetry $\overline{\Phi}_0^{(i)}$ $\neq 0$) takes place for all $\alpha>\alpha_c$ and, in particular, for $\alpha>\alpha_c$, $g=0$ (pure QED).[4,12,20,21]

The critical line (17), (18) is a line of chiral second-order phase transition: the order parameter $\overline{\Phi}_0^{(i)}$ tends to zero while g and α approach the critical line from the side of the phase with spontaneously broken chiral symmetry.

Two methods are available for the derivation of the effective potential. In the first[14,15] use is made of expression (6) so that $\overline{V}(\sigma_0)$ can be written as

$$\overline{V}(\sigma_0)=\int^{\sigma_0}\langle 0|\overline{\Psi}\Psi|0\rangle_\chi d\chi=\int^{\Sigma_0(\sigma_0)}\langle 0|\overline{\Psi}\Psi|0\rangle_\chi\frac{d\chi}{d\Sigma_0}d\Sigma_0 \quad (19)$$

(it is clear that it is sufficient to consider the configuration $\sigma_0\neq 0$, $\pi_0=0$ for the evaluation of the effective potential). The second method makes use of the fact that $\overline{V}(\sigma_0)$ coincides with the minimum of the Cornwall-Jackiw-Tomboulis[22] potential in ladder QED with bare mass $m^{(0)}=\sigma_0$. The expression for $\overline{V}(\sigma_0)$ in ladder QED can be calculated exactly.[23] Here we make use of the first approach.

We start with the case $\alpha<\alpha_c$. It can be shown from the SD equation that the expression for the chiral condensate has the form (see Appendix A):

$$\langle 0|\overline{\Psi}\Psi|0\rangle_\infty=\frac{1}{3\pi\alpha}\left[p^4\frac{d\Sigma(p^2)}{dp^2}\right]\Big|_{p^2=\Lambda^2}$$

$$=\frac{\tilde{A}\Sigma_0^2\Lambda}{6\pi\alpha}\left(\operatorname{ch}\theta-\frac{\operatorname{sh}\theta}{\omega}\right), \quad (20)$$

where $\theta=\omega\ln(\Lambda e^\delta/\Sigma_0)$ and we have used the asymptotic behavior (13) of $\Sigma(p)$.

The derivative $d\sigma_0/d\Sigma_0$ can be found from the boundary condition for the SD equation (A5):

$$\sigma_0=\frac{d}{dp^2}[p^2\Sigma(p^2)]\,|_{p^2=\Lambda^2}=\frac{\tilde{A}\Sigma_0^2}{2\Lambda}\left(\frac{\operatorname{sh}\theta}{\omega}+\operatorname{ch}\theta\right), \quad (21)$$

$$\frac{d\sigma_0}{d\Sigma_0}=\frac{\tilde{A}\Sigma_0}{2\Lambda}\left[\left(\frac{2}{\omega}-\omega\right)\operatorname{sh}\theta+\operatorname{ch}\theta\right]. \quad (22)$$

Using the expressions (12), (19)–(22) we find the effective potential as a function of Σ_0:

$$V(\Sigma_0)=\frac{\tilde{A}^2}{16\pi^2\omega^2}\Sigma_0^4\left\{\frac{\pi^2}{G_0\Lambda^2}[(1+\omega^2)\operatorname{ch}2\theta+2\omega\operatorname{sh}2\theta+\omega^2-1]+\frac{1}{1-\omega^2}-\operatorname{ch}2\theta\right\},\quad \alpha<\alpha_c, \quad (23a)$$

$$V(\Sigma_0)=\frac{\tilde{A}^2\Sigma_0^4}{8\pi^2}\left[\frac{1}{2}+\frac{\pi^2}{G_0\Lambda^2}(L+1)^2-L^2\right],\quad \alpha=\alpha_c \quad (23b)$$

[here $L=\ln(\Lambda e^\delta/\Sigma_0)$]. From the equation $dV/d\Sigma_0=0$ we can find the value $\overline{\Sigma}_0\equiv m_d$, corresponding to the minimum of the potential V:

$$\frac{1}{4g}=\frac{\pi^2}{G_0\Lambda^2}=\frac{2\operatorname{ch}2\bar\theta-\omega\operatorname{sh}2\bar\theta+2(\omega^2-1)^{-1}}{2\operatorname{ch}2\bar\theta+(3\omega-\omega^3)\operatorname{sh}2\bar\theta+2(\omega^2-1)}, \quad (24a)$$

$$\frac{1}{4g}=\frac{\bar L^2-{}^1/_2\bar L-{}^1/_2}{\bar L^2+{}^3/_2\bar L+{}^1/_2} \quad (24b)$$

for values $\alpha<\alpha_c$ and $\alpha=\alpha_c$ respectively [$\bar\theta=\omega\ln(\Lambda e^\delta/\overline{\Sigma}_0)$, $\bar L=\ln(\Lambda e^\delta/\overline{\Sigma}_0)$].

In the local limit $\Sigma_0/\Lambda\to 0$ we note that the Eqs. (24) determine the critical line (17). It follows from (21) that for $\Sigma_0/\Lambda\ll 1$ the field σ_0 is connected with Σ_0 by the expression

$$\sigma_0=\tilde{A}\frac{\Sigma_0^2}{\Lambda}\frac{\omega+1}{4\omega}\left(\frac{\Lambda e^\delta}{\Sigma_0}\right)^\omega,\quad \alpha<\alpha_c, \quad (25a)$$

$$\sigma_0=\tilde{A}\frac{\Sigma_0^2}{\Lambda}(L+1),\quad \alpha=\alpha_c. \quad (25b)$$

It is convenient to introduce the renormalized fields $\sigma_2=Z_m^{-1}\sigma$, $\pi_2=Z_m^{-1}\pi$ so that $\bar\sigma_{0r}$ now coincides with $\overline{\Sigma}_0$. Then we find from (25) for the constant Z_m:

$$Z_m\approx\tilde{A}\frac{\overline{\Sigma}_0}{\Lambda}\frac{\omega+1}{4\omega}\left(\frac{\Lambda e^\delta}{\overline{\Sigma}_0}\right)^\omega,\quad \alpha<\alpha_c, \quad (26a)$$

$$Z_m\approx\tilde{A}\frac{\overline{\Sigma}_0}{2\Lambda}(L+1),\quad \alpha=\alpha_c. \quad (26b)$$

Then, in the local limit ($\Lambda\to\infty$) we have

$$\sigma_{0r}=\overline{\Sigma}_0(\Sigma_0/\overline{\Sigma}_0)^{2-\omega}. \quad (27)$$

We now express the potential V (23) in terms of the renormalized fields σ_2 and π_r. Substituting the expression (24) for the coupling constant g into (23) and using (25), (26), and (27) we obtain the following expression for the effective potential in the local limit:

$$V=\frac{\tilde{A}^2(\alpha)}{16\pi^2\omega(1-\omega^2)}\overline{\Sigma}_0^4\left[(2-\omega)\left(\frac{\rho_r^2}{\overline{\Sigma}_0^2}\right)^{2/(2-\omega)}-2\left(\frac{\rho_r}{\overline{\Sigma}_0}\right)^2\right] \quad (28)$$

for $\alpha<\alpha_c$ and $g>1/4$ [see Eq. (17)],

$$V=\frac{\tilde{A}^2(\alpha_c)}{16\pi^2}\overline{\Sigma}_0^2\rho_r^2\left[\ln\frac{\rho_r^2}{\overline{\Sigma}_0^2}-1\right] \quad (29)$$

for $\alpha=\alpha_c$ and $g=1/4$. Here $\rho_r^2=\sigma_r^2+\pi_r^2$ and in the derivation of $V(\rho_r)$ we used the fact that, as a consequence of the $U_L(1)\times U_R(1)$ symmetry, it can be obtained from $V(\sigma_{0r})$ by the replacement of σ_{0r} by the chiral invariant ρ_r. We note that the fermion dynamical mass $\overline{\Sigma}_0$ appears in the potential V as a result of the dimensional transmutation phenomenon: from the requirement that the value $V(\Sigma_0=\overline{\Sigma}_0)$ be a minimum the constant $g=G_0\Lambda^2/4\pi^2$ is determined as a function of $\overline{\Sigma}_0$, Λ, and α. As a result, in the local limit $\Lambda\to\infty$ the dimensionless coupling constant g is replaced by the dimensional parameter Σ_0. The appearance in the potential (28) of a term proportional to $(\rho_r^2/\bar{\Sigma}_0^2)^{2/(2-\omega)}$ is important. It can be shown that the contribution of this term to the effective action is scale-invariant. Indeed, under scale transformations Σ_0 transforms as $\Sigma_0 \to s\Sigma_0$ and it then follows from (27) that the dynamical dimension of the composite field σ_2 equals $d_\sigma = 2 - \omega$.[8,15] This in turn means that the dynamical dimension of the term $(\rho_r^2/\bar{\Sigma}_0^2)^{2/(2-\omega)}$ equals four. Consequently, in the potential V (28) only the second term (the mass term) violates scale symmetry, and the corresponding violation is soft [the dynamical dimension of the mass term obeys $2(2-\omega) < 4$]. We also note the negative sign of the mass term, which is the reason for the appearance of a nontrivial vacuum expectation value for the field ρ_r and, consequently, for spontaneous breaking of chiral symmetry in the manner analogous to the standard Goldstone mechanism.

For $\alpha = \alpha_c$ scale symmetry is violated in the potential (29) by the logarithmic term. The form of the potential (29) is reminiscent of the one-loop effective potential in the Coleman-Weinberg (CW) model.[24] However there are also substantial differences between them: while in the CW potential the power of the scalar field is equal to four, in our case this power is equal to two. This, of course, reflects the fact that for $\alpha = \alpha_c$ the dynamical dimension of the composite field σ_r equals $d_\sigma = (2-\omega)|_{\alpha=\alpha_c} = 2$.

For a more detailed discussion of scale symmetry breaking in the NJL gauge model see Sec. 6.

We now discuss the effective potential in the region $\alpha > \alpha_c$, $g < 1/4$. Using expression (15) for the asymptotic behavior of the mass function in this region we perform all calculations similarly to the case $\alpha < \alpha_c$. As a result we obtain

$$V(\Sigma_0(\rho)) = \frac{\tilde{A}^2\Sigma_0^4}{16\pi^2\tilde{\omega}^2}\left[\left(1+\frac{\tilde{\omega}^2-1}{4g}\right)\cos 2\theta + \frac{\tilde{\omega}}{2g}\sin 2\theta + \frac{1+\tilde{\omega}^2}{4g} - \frac{1}{1+\tilde{\omega}^2}\right], \qquad (30)$$

where $\theta = \tilde{\omega}\ln(\Lambda e^\delta/\Sigma_0)$, while the function $\Sigma_0(\rho)$ is determined from the equation [compare with (21)]

$$\sigma_0 = \frac{\tilde{A}\Sigma_0^2}{2\Lambda}\left[\frac{\sin\theta}{\tilde{\omega}} + \cos\theta\right] \qquad (31)$$

with the replacement $\sigma_0 \to \rho$.

The equation $dV/d\Sigma_0 = 0$ now takes the form

$$\frac{1}{4g} = \frac{2\cos 2\bar{\theta} + \tilde{\omega}\sin 2\bar{\theta} - 2(1+\tilde{\omega}^2)^{-1}}{2\cos 2\bar{\theta} + (3\tilde{\omega}+\tilde{\omega}^3)\sin 2\bar{\theta} - 2(\tilde{\omega}^2+1)}, \qquad (32)$$

where $\bar{\theta} = \tilde{\omega}\ln(\Lambda e^\delta/\bar{\Sigma}_0)$. From (32) we find the solution for $\bar{\theta}$ near $\alpha \sim \alpha_c$:

$$\bar{\theta} = \pi - \operatorname{arctg}\left(\tilde{\omega}\frac{1+4g}{1-4g}\right) \underset{\tilde{\omega}\to 0}{\approx} \pi - \tilde{\omega}\frac{1+4g}{1-4g}. \qquad (33)$$

Writing θ as $\bar{\theta} = \bar{\theta} - \tilde{\omega}\ln(\Sigma_0/\bar{\Sigma}_0)$ we find in the local limit the following expression for the potential (30) on the critical line (18) ($\tilde{\omega} \to 0$):

$$V(\Sigma_0(\rho)) = \frac{\tilde{A}^2\Sigma_0^4}{16\pi^2}\left[\frac{2}{g}\left(\frac{1}{4} - g\right)\ln^2\frac{\Sigma_0}{\bar{\Sigma}_0} + 4\ln\frac{\Sigma_0}{\bar{\Sigma}_0} - 1\right], \quad g < 1/4. \qquad (34)$$

Equation (31) also simplifies in that limit:[1)]

$$\sigma_0 = \frac{\tilde{A}^2\Sigma_0^4}{2\Lambda}\left(\ln\frac{\Sigma_0}{\bar{\Sigma}_0} + \frac{8g}{1-4g}\right). \qquad (35)$$

Choosing as before the renormalization constant Z_m so that $\bar{\sigma}_{0r} = Z_m^{-1}\bar{\sigma}_0 = \bar{\Sigma}_0$, we find

$$Z_m = \frac{4g\tilde{A}}{1-4g}\frac{\bar{\Sigma}_0}{\Lambda}, \qquad (36)$$

$$\sigma_{0r} = \Sigma_0\frac{\Sigma_0}{\bar{\Sigma}_0}\left[1 + \frac{1-4g}{8g}\ln\frac{\Sigma_0}{\bar{\Sigma}_0}\right]. \qquad (37)$$

The presence of the singularity at $g = 0$ (pure QED) in relations (34) and (37) is connected with the fact that the method of introducing the auxiliary fields σ and π into the Lagrangian (2) fails in the case of pure QED. In particular, the Lagrange-Euler equation (2a) gives in that case $\sigma = \pi = 0$ for $g = 0$. The dynamics of pure QED for $\alpha > \alpha_c$ is discussed in detail in Ref. 25.

4. THE EQUATION FOR THE KINETIC TERM IN THE EFFECTIVE ACTION

The $U_L(1) \times U_R(1)$ symmetry leads to the following general form for the kinetic term of the effective Lagrangian in the local ($\Lambda \to \infty$) limit:

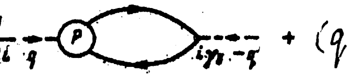
$$L_k = \frac{F_1}{2}\partial^\mu\Phi_r^{(i)}\partial_\mu\Phi_r^{(i)} + \frac{F_2}{\Phi_r^{(i)}\Phi_r^{(i)}}(\Phi_r^{(k)}\partial^\mu\Phi_r^{(k)})(\Phi_r^{(j)}\partial_\mu\Phi_r^{(j)}), \qquad (38)$$

where F_1 and F_2 are in the general case functions of α, $\rho_r \equiv (\Phi_r^{(i)}\Phi_r^{(i)})^{1/2}$ and $\bar{\rho}_r$ (the value $\bar{\rho}_r$ minimizes the potential). It is therefore sufficient for the determination of L_k to evaluate $\partial^2\Delta_{ss}/\partial q^\mu\partial q_\mu|_{q=0}$ and $\partial^2\Delta_{pp}/\partial q^\mu\partial q_\mu|_{q=0}$ in the case $\sigma_0 \neq 0$, $\pi_0 = 0$. In the following we discuss only the evaluation of $\partial^2\Delta_{pp}/\partial q^\mu\partial q_\mu|_{q=0}$ (the evaluation of $\partial^2\Delta_{ss}/\partial q^\mu\partial q_\mu|_{q=0}$ is similar; some specific features of that calculation will be noted below).

The expression for $\Delta_{pp}(q)$ in ladder QED has the form[16]

$$\Delta_{pp}(q) = \frac{1}{2i}\int\frac{d^4k}{(2\pi)^4}\operatorname{tr}[G(k)\Gamma_p(k,k+q)G(k+q)i\gamma_5] + (q \to -q), \qquad (39)$$

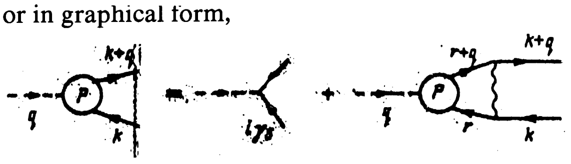
where the fermionic propagator is $G(k) = i[\hat{k} - \Sigma(k^2)]^{-1}$ (we use the Landau gauge) and Γ_p is the amputated vertex connecting the composite operator $\bar{\Psi}_i\gamma_5\Psi$ with the fields $\bar{\Psi}$ and Ψ. Graphically (39) is represented in the form:

$\Delta_{pp}(q) = \frac{1}{2i}$  $+ (q \to -q)$. (39a)

The vertex $\Gamma_p(k, k+q)$ satisfies the equation

$$\Gamma_p(k,k+q) = i\gamma_5 + e^2\int\frac{d^4r}{(2\pi)^4}i\gamma^\mu G(r)\Gamma_p(r,r+q) \times G(r+q)i\gamma^\nu D_{\mu\nu}(k-r), \qquad (40)$$

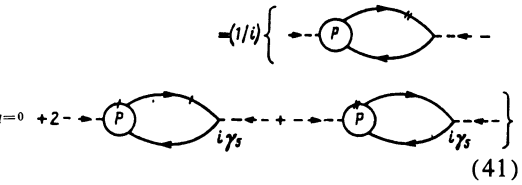
or in graphical form,

 (40a)

where the photon propagator is $D_{\mu\nu}(k) = (1/ik^2) \times (g_{\mu\nu} - k_\mu k_\nu/k^2)$. From (39) we obtain the equation for

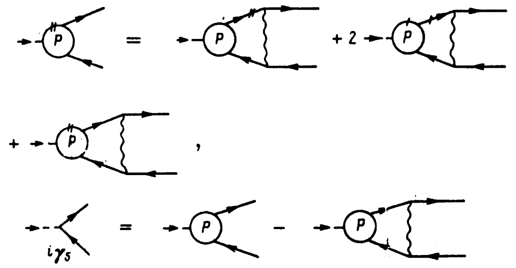
$\partial^2\Delta_{pp}/\partial q^\mu \partial q_\mu |_{q=0}$:

$$\frac{\partial^2\Delta_{pp}(q)}{\partial q^\mu \partial q_\mu}\Big|_{q=0}$$



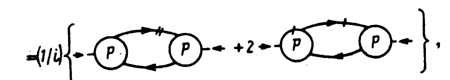
(41)

where the prime denotes derivatives with respect to q. The last expression can be simplified if use is made of the following equations, which result from (40):



We obtain

$$\frac{\partial^2\Delta_{pp}(q)}{\partial q^\mu \partial q_\mu}\Big|_{q=0}$$



(42)

or in analytic form,

$$\frac{\partial^2\Delta_{pp}(q)}{\partial q^\mu \partial q_\mu}\Big|_{q=0} = \frac{1}{i}\Big\{\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}[G(k)\Gamma_p(k,k) \times G_{,\mu}{}^{\mu}(k)\Gamma_p(k,k)] + 2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}[G(k)\Gamma_{p,\mu}(k,k)G_{,}{}^{\mu}(k)\Gamma_p(k,k)]\Big\}. \qquad (43)$$

For the vertex function for coincident momenta, $\Gamma_p(k,k)$, the exact expression

$$\Gamma_p(k,k) = i\gamma_5\frac{\Sigma(k^2)}{\sigma_0} \qquad (44)$$

was obtained in Ref. 16. We note that for the scalar vertex $\Gamma_s(k,k)$, connecting the composite operator $\bar\Psi\Psi$ with $\bar\Psi$ and ψ (and entering into the expression for $\partial^2\Delta_{ss}/\partial q^\mu\partial q_\mu|_{q=0}$) the analogous relation has the form $\Gamma_s(k,k) = i(\partial/\partial\sigma_0)\Sigma(k^2)$ (Ref. 16). For the the derivative $\Gamma_{p,\mu}(k,k)$ we can write a general expansion containing four scalar functions $gi(k^2)$ $(i = 1, 2, 3, 4)$:

$$\Gamma_{p,\mu}(k,k) \equiv \frac{\partial\Gamma_p(k,k+q)}{\partial q^\mu}\Big|_{q=0} = i\gamma_5\Big\{k_\mu g_1(k^2) + \gamma^\nu\Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)g_2(k^2) + \gamma^\nu\frac{k_\mu k_\nu}{k^2}g_3(k^2) + i\sigma_{\mu\nu}k^\nu g_4(k^2)\Big\}. \qquad (45)$$

$\Gamma_{p,\mu}(k,k)$ satisfies the following equation [see (40)]

$$\Gamma_{p,\mu}(k,k) = e^2\int\frac{d^4r}{(2\pi)^4}\,i\gamma^\alpha G(r)\Gamma_p(r,r)G_{,\mu}(r)i\gamma^\beta D_{\alpha\beta}(r-k) + e^2\int\frac{d^4r}{(2\pi)^4}\,i\gamma^\alpha G(r)\Gamma_{p,\mu}(r,r)G(r)i\gamma^\beta D_{\alpha\beta}(r-k). \qquad (46)$$

In the following sections these equations will be used to derive the kinetic term in the effective action.

5. THE EFFECTIVE ACTION FOR SMALL α

It follows from Eq. (46) that the functions $gi(k^2)$ in (45) are quantities of order α. A detailed analysis of Eq. (46) shows also[26] that the functions $gi(k^2)$ decrease as $k^2\to\infty$. Therefore the dominant contribution to $\partial^2\Delta pp/\partial q^\mu\partial q_\mu|_{q=0}$ for $\alpha \ll 1$ comes from the first term on the right-hand side of Eq. (43). Using formulas (11), (13), (43) and (44) and taking into account that $\tilde{A}(0) = \sqrt{2}$, $\delta(0) = \frac{1}{2}\ln 2$ [see (14)] we find the expression for the kinetic term for $\alpha \ll 1$:

$$L_k = \frac{1}{12\pi\alpha}\Big[\frac{1}{2}(\partial^\mu\sigma_r\,\partial_\mu\sigma_r + \partial^\mu\pi_r\,\partial_\mu\pi_r)\Big] + O(1). \qquad (47)$$

In deriving (47) we used the fact that $Z_m = 1 + O(\alpha)$ for $\alpha \ll 1$ [see (26)].

From (28) and (47) we now find the final expression for the effective Lagrangian for $\alpha \ll 1$:

$$L_{eff} = \frac{1}{12\pi\alpha}\Big[\frac{1}{2}(\partial^\mu\sigma_r\,\partial_\mu\sigma_r + \partial^\mu\pi_r\,\partial_\mu\pi_r) + \bar\Sigma_0^2(\sigma_r^2 + \pi_r^2) - \frac{1}{2}(\sigma_r^2 + \pi_r^2)^2\Big] + \dots, \qquad (48)$$

where the dots denote the presence of terms with more than two derivatives. The terms with higher derivatives and, in particular, the Wess-Zumino terms, can in principle be obtained in a similar manner.

We discuss now in more detail the Lagrangian (48). It is convenient to introduce the interpolating fields

$$\sigma_{int} = \Big(\frac{1}{12\pi\alpha}\Big)^{1/2}\sigma_r, \qquad \pi_{int} = \Big(\frac{1}{12\pi\alpha}\Big)^{1/2}\pi_r \qquad (49)$$

and to rewrite (48) in the form

$$L_{eff} = {}^1/_2(\partial^\mu\sigma_{int}\,\partial_\mu\sigma_{int} + \partial^\mu\pi_{int}\,\partial_\mu\pi_{int}) + \bar\Sigma_0^2(\sigma_{int}^2 + \pi_{int}^2) - 6\pi\alpha(\sigma_{int}^2 + \pi_{int}^2)^2. \qquad (50)$$

The minimum of the effective potential is determined from the equation $\sigma_{\rm int}^2 + \pi_{\rm int}^2 = (1/12\pi\alpha)\bar\Sigma_0^2$. Choosing the vacuum to be the configuration $\sigma_{\rm int} = (1/12\pi\alpha)^{1/2}\bar\Sigma_0$, $\pi_{\rm int} = 0$, we find that the excitation spectrum contains a massless pseudoscalar Goldstone boson and a scalar boson with mass $M_\sigma = 2\bar\Sigma_0$. This result coincides with the analogous result in the pure NJL model.[2]

We now find the decay constant F_π. Since the composite fields $\sigma_{\rm int} \propto \bar\Psi\Psi$ and $\pi_{\rm int} \propto \bar\Psi i\gamma_5\Psi$ transform under chiral rotations $\Psi \to \exp(i\gamma_5\Phi)\Psi$ as

$$\sigma_{int} \to \sigma_{int}\cos 2\Phi + \pi_{int}\sin 2\Phi, \qquad \pi_{int} \to -\sigma_{int}\sin 2\Phi + \pi_{int}\cos 2\Phi, \qquad (51)$$

the axial current is determined by the expression

$$j_{\mu 5} = \frac{\delta L_{eff}}{\delta\partial^\mu\Phi} = 2(\pi_{int}\partial_\mu\sigma_{int} - \sigma_{int}\partial_\mu\pi_{int}), \qquad (52)$$

and in the tree approximation we find

$$iP_\mu F_\pi = \langle 0|j_{\mu 5}(0)|\pi\rangle = 2\Big(\frac{1}{12\pi\alpha}\Big)^{1/2}\bar\Sigma_0 iP_\mu. \qquad (53)$$

Hence

$$F_\pi^2=\bar{\Sigma}_0^2/3\pi\alpha. \quad (54)$$

Now making use of the Goldberger-Treiman relation we obtain the expression for the Yukawa coupling constant g_Y:

$$g_Y=\bar{\Sigma}_0/F_\pi=(3\pi\alpha)^{1/2}. \quad (55)$$

From (50) it also follows that the four-fermion interaction constant λ equals

$$\lambda=6\pi\alpha. \quad (56)$$

Thus in the limit $\alpha\ll 1$ the NJL gauge model describes the dynamics of spontaneously broken chiral symmetry with weakly interacting fermions and spinless bosons. The coupling constants g_Y and λ vanish as $\alpha\to 0$, which corresponds to the familiar fact that in the local limit the NJL model represents a free theory.

We also note that the kinetic term in $L_{\rm eff}$ (48) is scale invariant. As will be shown in the next section this property is valid for all $\alpha\leqslant\alpha_c$. More than that, we will show that also all terms with higher derivatives are scale invariant.[2]

6. STRUCTURE OF THE EFFECTIVE LAGRANGIAN AND BREAKING OF SCALE SYMMETRY IN THE NJL GAUGE MODEL

It was shown in Sec. 2 that the derivation of the low-energy effective action in the form of a series in powers of derivatives of the fields σ and π reduces to the calculation of Green's functions of the composite operators $\bar{\Psi}\Psi$ and $\bar{\Psi}i\gamma_5\Psi$ in ladder QED with bare mass $m^{(0)}=\sigma_0$. We show now that the only terms in the effective action that violate scale symmetry are the mass and logarithmic terms in the effective potential (28) and (29) for $\alpha<\alpha_c$ and $\alpha=\alpha_c$, $g=1/4$, respectively.

Let us recall the basic properties of the Green's functions of the operators $\bar{\Psi}\Psi$ and $\bar{\Psi}i\gamma_5\Psi$ in ladder QED. From the work in Ref. 27 it is known that the anomalous dimension of these operators in ladder QED is equal to $\gamma_m=1-(1-\alpha/\alpha_c)^{1/2}$. In the case of an elementary field $\Phi(x)$ this would mean multiplicative renormalization of the corresponding connected Green's functions

$$\bar{Z}_m{}^n\Delta^{(n)}(q_1,q_2,\ldots,q_{n-1})=\bar{\Delta}_{ren}^{(n)}(q_1,q_2,\ldots,q_{n-1}), \quad (57)$$

where $\bar{Z}_m=C(\Sigma_0/\Lambda)^{\gamma m}$, with C a normalization constant (Σ_0 can be viewed as the renormalized fermion mass in QED). The case of composite operators, however, is considerably more complicated (see, for example, Ref. 28). While multiplicative renormalization takes place for all Green's functions containing only one composite operator and any number of elementary fields, this property can be violated for certain Green's functions with a larger number of composite operators. For example, in spite of the fact that in the free theory the renormalization constant satisfies $Z=1$ for all operators, there is quadratic and logarithmic divergence in $\Delta^{(2)}$ and logarithmic divergence in $\Delta^{(4)}$ ($\Delta^{(2)}$ and $\Delta^{(4)}$ are the Green's functions of the composite operator $\bar{\Psi}\Psi$ or $\bar{\Psi}i\gamma_5\Psi$). This circumstance leads to the violation of multiplicative renormalization of the Green's functions of the operators $\bar{\Psi}\Psi$ and $\bar{\Psi}i\gamma_5\Psi$ in ladder QED. Fortunately, this violation appears only in the propagator $\Delta ij\equiv\Delta_{ij}^{(2)}$ ($i,j=s,p$). In actuality, the renormalization relation (57) is satisfied for all Green's functions with $n>2$. In the case $n=2$ the following modified relation is valid

$$\bar{Z}_m^2(\Delta_{ij}(q)-\Delta_{ij}(0))=\bar{\Delta}_{ij,\,ren}(q)-\bar{\Delta}_{ij,\,ren}(0) \quad (58)$$

[the relation (58) does not, of course, fix the value of $\bar{\Delta}_{ij,\rm ren}(0)$]. This means that multiplicative renormalization takes place both for $\Delta^{(n)}$ with $n>2$ and for all derivatives with respect to q of the propagator $\Delta_{ij}(q)$.[3] This is sufficient for the description of the structure of the effective action. We prove first the scale invariance of the kinetic term. In fact to this end it is sufficient to show that for all $\alpha\leqslant\alpha_c$, $g\geqslant 1/4$ the kinetic term has the form ($\Lambda\to\infty$):

$$L_k=\left(\frac{\rho_r^2}{\bar{\rho}_r^2}\right)^{(\omega-1)/(2-\omega)}\left\{\frac{f_1(\alpha)}{2}(\partial^\mu\sigma_r\,\partial_\mu\sigma_r-\partial^\mu\pi_r\,\partial_\mu\pi_r)\right.$$
$$\left.+\frac{f_2(\alpha)}{\rho_r^2}(\sigma_r\partial^\mu\sigma_r+\pi_r\partial^\mu\pi_r)(\sigma_r\partial_\mu\sigma_r+\pi_r\partial_\mu\pi_r)\right\}, \quad (59)$$

where $f_1(\alpha)$, $f_2(\alpha)$ are finite functions of the coupling constant α. Since the dynamical dimension of the fields σ_r and π_r equals $d\sigma=2-\omega$ [see (27)] it follows from (59) that the dynamical dimension of the kinetic term L_k equals four, i.e., that L_k is scale invariant.

It follows from (11) that L_k is determined by the expression

$$L_k=\frac{1}{4}\frac{\partial^2\Delta_{ij}(q)}{\partial q_\mu\,\partial q_\nu}\bigg|_{q=0}\frac{\partial\Phi^{(i)}}{\partial x^\mu}\frac{\partial\Phi^{(j)}}{\partial x^\nu}$$
$$=\frac{1}{4}\left(\frac{Z_m}{\bar{Z}_m}\right)^2\frac{\partial^2\bar{\Delta}_{ij,ren}(q)}{\partial q_\mu\,\partial q_\nu}\bigg|_{q=0}\frac{\partial\Phi_r^{(i)}}{\partial x^\mu}\frac{\partial\Phi_r^{(j)}}{\partial x^\nu}. \quad (60)$$

In the last equality in (60) we used the relation (58). We choose the normalization constant C in the expression for $\bar{Z}_m$ the same as in the expression for Z_m [see (26a)]:

$$\bar{Z}_m=\tilde{A}\frac{\Sigma_0}{\Lambda}\frac{\omega+1}{4\omega}\left(\frac{\Lambda e^\delta}{\Sigma_0}\right)^\omega. \quad (61)$$

From this and (27) and (60) we find ($\Lambda\to\infty$)

$$L_k=\frac{1}{4}\left(\frac{\Sigma_0^2}{\bar{\Sigma}_0^2}\right)^{\omega-1}\frac{\partial^2\bar{\Delta}_{ij,ren}(q)}{\partial q_\mu\,\partial q_\nu}\bigg|_{q=0}\frac{\partial\Phi_r^{(i)}}{\partial x^\mu}\frac{\partial\Phi_r^{(j)}}{\partial x^\nu}$$
$$=\frac{1}{4}\left(\frac{\sigma_{0r}^2}{\bar{\sigma}_{0r}^2}\right)^{(\omega-1)/(2-\omega)}\frac{\partial^2\bar{\Delta}_{ij,ren}(q)}{\partial q_\mu\,\partial q_\nu}\bigg|_{q=0}\frac{\partial\Phi_r^{(i)}}{\partial x^\mu}\frac{\partial\Phi_r^{(j)}}{\partial x^\nu}. \quad (62)$$

In that expression $\partial^2\bar{\Delta}_{ij,\rm ren}(q)/\partial q_\mu\partial q_\nu$ is the renormalized Green's function in ladder QED with fermion mass Σ_0. Since in the local limit $\Lambda\to\infty$ the mass Σ_0 is the only dimensional parameter of QED, it follows from here and the dimensionless character of the quantity $\partial^2\bar{\Delta}_{ij,\rm ren}(q)/\partial q_\mu\partial q_\nu|_{q=0}$ that

$$\frac{\partial^2\bar{\Delta}_{ij,ren}(q)}{\partial q_\mu\,\partial q_\nu}\bigg|_{q=0}=g^{\mu\nu}\Psi_{ij}(\alpha), \quad (63)$$

where $\Psi_{ij}(\alpha)$ is a finite function of α.

The expression (62) corresponds to the configuration $\sigma_{0r}\neq 0$, $\pi_{0r}=0$. Since this configuration is invariant under the transformation $\sigma_r\to\sigma_r$, $\pi_r\to-\pi_r$, it follows that $\Psi_{ij}(\alpha)=0$ if $i\neq j$. Taking into account the expression (38) for L_K we arrive for an arbitrary configuration σ_{0r}, π_{0r} at (59), where $f_1(\alpha)=\frac{1}{2}\Psi_{pp}(\alpha)$, and $f_2(\alpha)=\frac{1}{4}[\Psi_{ss}(\alpha)-\Psi_{pp}(\alpha)]$.

We emphasize once more that the key considerations in our proof are:

a) multiplicative renormalizability of the function $\partial^2\Delta_{ij}(q)/\partial q_\mu\partial q_\nu$;

b) existence of but one dimensional parameter Σ_0 in ladder QED in the local limit;

c) absence of renormalization of the coupling constant α in ladder QED, which gives rise to power dependence of the renormalization constant $\overline{Z}_m(\Sigma_0/\Lambda)$.

Since multiplicative renormalizability holds for all Green's functions that contribute to the terms with higher derivatives, the above proof can be extended to them as well. This means that all terms with higher derivatives in the effective action are scale invariant.

It is clear, however, that the above proof is not valid for the effective potential $\overline{V}$, which can be expressed in terms of the values of the Green's functions at $q=0$. The reason for this is that in that case there is additional quadratic divergence in the propagator $\Delta_{ij}(q)$ and multiplicative renormalizability is violated. In actuality this quadratic divergence cancels against the corresponding divergence from the term $(2G_0)^{-1}\Phi^{(j)}\Phi^{(j)}$ in the potential (16) near the critical line, with the result that the mass term remains in the potential (28), giving rise to violation of scale symmetry.

Since the effective action in the case $\alpha=\alpha_c$, $g=1/4$ can be obtained as the limit of the corresponding action for $\alpha<\alpha_c$, we find that in the potential (29) only the logarithmic term violates scale symmetry.

We note that the effective action with a potential of the form (29) and a scale-invariant kinetic term has been postulated before for the model under discussion (for $\alpha=\alpha_c$) in Ref. 23. The soft breaking of scale invariance means that the hypothesis of partial conservation of the dilatation current (PCDC) is a quite reasonable assumption for the description of the dynamics of the σ-scalar.[23,29] The main point of the PCDC approach is precisely the inclusion of the noncanonical dynamical dimension of the spinless fields. Consequences of the PCDC hypothesis in the NJL gauge model were discussed in more detail in Refs. 23 and 29.

In this work we have discussed the dynamics of the model corresponding to only a part of the critical line ($\alpha<\alpha_c$ and $g>1/4$). The picture of scale-symmetry breaking for $\alpha=\alpha_c$ and $g<1/4$, where the breaking of chiral symmetry is due mainly to the electromagnetic interaction, is somewhat different from the discussed above[23] and is studied in more detail in Ref. 25.

We note, furthermore, that our effective action corresponds to the tree approximation of the effective theory, neglecting the quantum fluctuations of the fields σ and π. In particular, the exact relation for the mass M_σ can be substantially different (especially for large α) from that obtained in our approximation.

7. CONCLUSION

The low-energy effective action in the NJL gauge model corresponds to the σ model with noncanonical dynamical dimension $d_\sigma=2-(1-\alpha/\alpha_c)^{1/2}$ for the composite fields $\sigma\propto\Lambda^{-2}\overline{\Psi}\Psi$, $\pi\propto\Lambda^{-2}\overline{\Psi}i\gamma_5\Psi$. The dynamical dimension d_σ is connected with the ultraviolet behavior of the dynamical mass function of the fermion [see (13)] and amputated Bethe-Salpeter function $\chi_{(\text{amp.})}$ of spinless bound states:

$$\Sigma_d(p)\propto\overline{\Sigma}_0(p/\overline{\Sigma}_0)^{1-d_\sigma}, \qquad (64)$$

$$\chi_{amp}(p)\propto\Sigma_d(p)\propto(p)^{1-d_\sigma}. \qquad (65)$$

Since in the model under consideration $d_\sigma\leqslant 2$, the wave function $\chi_{(\text{amp.})}(p)$ (which determines the form factors of the bound states) decreases for large p much more slowly than in theories of the type of QCD, where $d_\sigma=3$ (the canonical dimension of the $\overline{\Psi}\Psi$ operator). In this case we can speak of the presence of strongly bound states in the NJL gauge model.[5,18] A characteristic feature of such bound states is that, in contrast to QCD, they do not decouple from the dynamics at high energies ($p\gg F_{ew}$ in the case of the electroweak interactions).[15–18,30,5] This property might turn out to be more important for the description of the dynamics of the breaking of the electroweak symmetry in scenarios based on the NJL gauge model. In that case the bound states play the role of Higgs bosons. The properties of such states were recently considered for certain scenarios in Refs. 31 and 32.

Another important application of the NJL gauge model is its use in the analysis of data of computer calculations on the lattice for asymptotically nonfree theories of the type of QED.[33] As was noted in Refs. 15 and 30, the results of lattice calculations for noncompact QED with a sufficiently large bare coupling constant[33] can be understood if it is supposed that in the scaling region of this model an essential role is played by the induced four-fermion interaction (and other formally inessential interactions). The derivation of the effective action in the NJL gauge theory shows clearly how in the scaling region $\overline{\Sigma}_0/\Lambda\ll 1$ these formally inessential interactions are transformed into essential degrees of freedom similar to the spinless fields σ and π. The NJL gauge model might also turn out to be important for the understanding of the dynamics at short distances for a strong coupling constant.

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APPENDIX A

We discuss here the properties of the solution of the SD equation for the fermion mass function $\Sigma(p)$ in ladder QED {the fermion propagator is $G(p)=i[\hat{p}A(p^2)-\Sigma(p^2)]^{-1}$}.

In Euclidean space in the Landau gauge the corresponding equation has the form:[4,12,20]

$$A(p^2)=1, \qquad (A1)$$

$$\Sigma(p^2)=m^{(0)}+\frac{3\alpha}{4\pi}\int_0^{\Lambda^2}dq^2\frac{q^2\Sigma(q^2)}{q^2+\Sigma^2(q^2)}\left[\frac{\theta(p^2-q^2)}{p^2}+\frac{\theta(q^2-p^2)}{q^2}\right]. \qquad (A2)$$

Equation (A2) is equivalent to the differential equation

$$\frac{d}{dp^2}\left[p^4\frac{d\Sigma(p^2)}{dp^2}\right]+\frac{3\alpha}{4\pi}\frac{p^2\Sigma(p^2)}{p^2+\Sigma^2(p^2)}=0 \qquad (A3)$$

with the boundary conditions

$$p^4\frac{d\Sigma(p^2)}{dp^2}\Bigg|_{p^2=0}=0, \qquad (A4)$$

$$m^{(0)}-\frac{d}{dp^2}[p^2\Sigma(p^2)]\,|_{p^2=\Lambda^2}=0. \qquad (A5)$$

Using (A3) and (A4) we find for the condensate $\langle 0|\overline{\Psi}\Psi|0\rangle$:

$$\langle 0|\bar{\Psi}\Psi|0\rangle_{m^{(0)}}=-\frac{1}{4\pi^2}\int_0^{\Lambda^2}dq^2\frac{q^2\Sigma(q^2)}{q^2+\Sigma^2(q^2)}$$

$$=\frac{1}{3\pi\alpha}\left[q^4\frac{d\Sigma(q^2)}{dq^2}\right]\bigg|_{q^2=\Lambda^2}. \qquad (A6)$$

In the so-called linearized approximation,[12] which approximates Eq. (A2) well in the entire range of momenta, the function $\Sigma(p^2)$ in the denominator of Eqs. (A2) and (A3) is replaced by the mass parameter $\Sigma_0\equiv\Sigma(0)$. In that case the solution of Eqs. (A3) and (A4) is expressible in terms of the hypergeometric function

$$\Sigma(p^2)=\Sigma_0F\left(\frac{1+\omega}{2},\ \frac{1-\omega}{2},\ 2;\ -\frac{p^2}{\Sigma_0^2}\right), \qquad (A7)$$

where $\omega=(1-3\alpha/\pi)^{1/2}$. The ultraviolet asymptotic behavior of this function has the following form for $\alpha<\alpha_c=\pi/3$ and $\alpha>\alpha_c$ respectively:

$$\Sigma(p^2)\underset{p\to\infty}{\longrightarrow}\tilde{A}(\alpha)\frac{\Sigma_0^2}{p}\frac{1}{\omega}\mathrm{sh}\left[\omega\left(\ln\frac{p}{\Sigma_0}+\delta(\alpha)\right)\right], \qquad (A8)$$

$$\Sigma(p^2)\underset{p\to\infty}{\longrightarrow}\tilde{A}(\alpha)\frac{\Sigma_0^2}{p}\frac{1}{\tilde{\omega}}\sin\left[\tilde{\omega}\left(\ln\frac{p}{\Sigma_0}+\delta(\alpha)\right)\right], \qquad (A9)$$

where $\tilde{\omega}=(3\alpha/\pi-1)^{1/2}$. The functions $\tilde{A}(\alpha)$ and $\delta(\alpha)$ are given by Eqs. (14) of the text.

The ultraviolet asymptotic behavior of the solution of the exact equation (A2) has the same form as in (A8) and (A9), but with somewhat different $\tilde{A}(\alpha)$, $\delta(\alpha)$ (Refs. 4, 20).

[1)] The singularity for $g=1/4$ in Eq. (35) reflects the fact that this expression is only valid for $1/(1-4g)\ll\ln(\Lambda/\Sigma_0)$. For $1/(1-4g)\gg\ln(\Lambda/\Sigma_0)$ the relation (25b) occurs with the additional logarithmic term.

[2)] An alternative derivation of this fact was given by Bardeen and Love (private communication).

[3)] The last fact can also be demonstrated by using the equations for $\Delta_{ij}(q)$. Indeed, since multiplicative renormalization takes place for the vertex $\Gamma_p(q,k)$, containing only one composite operator, the validity of this property follows from Eq. (43) also for the derivative $\partial^2\Delta_{pp}(q)/\partial q^\mu\partial q_\mu$.

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