Scalar waves in an inhomogeneous medium: spatial dispersion in the pair-interaction approximation

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We use the pair-interaction approximation, also called the Bourret approximation,[1] to give a solution of the Helmholtz equation for the average field (averaged over an ensemble of realizations) in an unbounded randomly inhomogeneous medium described by a normalized binary correlation function $\varphi(\mathbf{r}_1,\mathbf{r}_2) = \exp(-r/a)$, where a is the spatial scale of the correlations and $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$. Taking the macroscopic spatial dispersion (caused by the inhomogeneity of the medium) into account enabled us to extend the region of the applicability of the Bourret approximation, including in it the UHF band. We evaluate the scattering indexes, $\gamma_\pm$, and the phase and group velocities, $v_\pm$ and $c_\pm$, for the propagation of plane monochromatic waves corresponding to the two roots $x_\pm$ of the dispersion equation (where x is the dimensionless wave number). We study in their dependence on the wavelength the conditions for the applicability of the pair-interaction approximation for each root. We write the field of a point charge as a superposition of spherically symmetric diverging waves with parameters which are determined by the roots x_+ and x_-. The absolute magnitudes of the amplitudes of these waves reach maximum values on the boundary of the short and very short wave bands. One observes anomalous properties of $\gamma_\pm$, $v_\pm$, and $c_\pm$ in the same region.

1. INTRODUCTION

The problems connected with the wave propagation in randomly inhomogeneous media has greatly attracted the researchers. This is due on the one hand, to the considerable difference between such media and homogeneous and regularly inhomogeneous media and, on the other hand, to the possibility of obtaining results which are important for applications. It is often possible to reduce the solution of the wave equation for the initial (in the general case, tensor) field to the solution of an equation for the scalar field which corresponds to it in some approximation.[1,2]

We take into account below the macroscopic spatial dispersion to solve the problem of the propagation of scalar waves in an unbounded nonabsorbing randomly inhomogeneous medium. We shall mainly pay attention to the evaluation of the parameters of the average field in the following cases: 1) a plane monochromatic wave; 2) a point source. It turns out to be possible to write the whole of the wavelength spectrum as a superposition of several bands described by characteristic asymptotic expressions for the parameters of the average field. For each of the bands we study the criteria for the applicability of the pair-interaction approximation.

2. HELMHOLTZ EQUATION FOR THE AVERAGE FIELD

We consider a scalar monochromatic field $E(\mathbf{r},t) = E(\mathbf{r})e^{-i\omega t}$, described by the Helmholtz equation

$$(\Delta+k_c^2\bar{\varepsilon})E(\mathbf{r})=0, \tag{2.1}$$

$$k_c^2\equiv\varepsilon_c\omega^2c_0^{-2},\quad \bar{\varepsilon}\equiv\varepsilon\varepsilon_c^{-1},\quad \varepsilon\equiv\varepsilon(\mathbf{r}). \tag{2.2}$$

For definiteness we shall call the random scalar field ε the permittivity of the inhomogeneous medium. E then has the meaning of the electric field strength connected with the induction D through the relation $D=\varepsilon E$; k_c is the wave number in the uniform medium (comparison medium) with a permittivity equal to ε_c while c_0 is the light velocity *in vacuo*.

Carrying out the statistical averaging (over an ensemble of realizations) of (2.1), denoted by angular brackets, and introducing the integral effective permittivity operator $\hat{\varepsilon}_*$ through the relation

$$\langle D\rangle=\langle\varepsilon E\rangle\equiv\hat{\varepsilon}_*\langle E\rangle \tag{2.3}$$

we obtain a Helmholtz equation

$$\hat{L}_*\langle E\rangle=0,\quad \hat{L}_*\equiv\Delta\hat{I}+k_c^2\hat{\bar{\varepsilon}}_*,\quad \hat{I}F\equiv F \tag{2.4}$$

for the average field $\langle E\rangle$. Here $\hat{I}$ is the unit operator. The solution of Eq. (2.4) thus needs first finding $\hat{\bar{\varepsilon}}_*$ which is reduced by virtue of (2.3) to establishing the connection between E and $\langle E\rangle$ leading to the equations

$$E=\hat{A}\langle E\rangle,\quad \hat{\varepsilon}_*=\langle\varepsilon\hat{A}\rangle,\quad \langle\hat{A}\rangle=\hat{I}, \tag{2.5}$$

the first of which implicitly presupposes the solution of Eq. (2.1). Various methods based, as a rule, on introducing a small parameter (or several) have been applied to reach this goal.[1,2]

Below we consider an approach using the idea of an auxiliary medium (comparison medium).[3–6] Using this we write together with (2.1) and (2.4)[4–6]

$$E=\langle E\rangle+\hat{R}\hat{Q}_c\bar{\varepsilon}'E, \tag{2.6}$$

$$\hat{Q}_c\equiv k_c^2\hat{H}_c,\quad \hat{L}_c\hat{H}_c=-\hat{I},\quad \hat{L}_c\equiv(\Delta+k_c^2)\hat{I}. \tag{2.7}$$

$$\hat{R}F\equiv F-\langle F\rangle\equiv F'',\quad \hat{R}^2=\hat{R},\quad F'\equiv F-F_c. \tag{2.8}$$

Here $\hat{H}_c$ is the Green operator of the Helmholtz equation for the comparison medium and $\hat{R}$ is the operator which splits off the random component from the whole expression standing behind it; the double prime indicates the centering operator which differs from $\hat{R}$ because of the equations

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$$(\hat{R}F)^2=\hat{R}F\hat{R}F=\hat{R}F(F-\langle F\rangle)=F^2-F\langle F\rangle-\langle F^2\rangle+\langle F\rangle^2,$$
$$(F'')^2=(F-\langle F\rangle)^2=F^2-2F\langle F\rangle+\langle F\rangle^2,$$

while the prime indicates the difference of the fields in the inhomogeneous and the auxiliary media.

Assuming the operator $\hat{R}\hat{Q}_c\bar{\varepsilon}'$ to possess the necessary properties[1,5] we can write the solution of Eq. (2.6) in the form (2.5), where $\hat{A}$ is given by

$$\hat{A}=(\hat{I}-\hat{R}\hat{Q}_c\bar{\varepsilon}')^{-1}=\sum_{n=0}^{\infty}(\hat{R}\hat{Q}_c\bar{\varepsilon}')^n. \tag{2.9}$$

If the fluctuations are so small that we can retain in the expansion only the first and second terms, we are led to the approximation

$$\hat{A}=\hat{I}+\hat{Q}_c\bar{\varepsilon}'',\quad \hat{R}\bar{\varepsilon}'=\bar{\varepsilon}''. \tag{2.10}$$

Substituting (2.10) into (2.5) we get for $\hat{\bar{\varepsilon}}_*$ the expression

$$\hat{\bar{\varepsilon}}_*=\langle\bar{\varepsilon}\rangle\hat{I}+\langle\bar{\varepsilon}''\hat{Q}_c\bar{\varepsilon}''\rangle, \tag{2.11}$$

the use of which together with (2.4) enables us to evaluate the required parameters of the average field.

It is clear from (2.11) that to find $\hat{\varepsilon}_*$ we need only information about pair (two-particle) interactions between the inhomogeneities described by the random field $\varepsilon(\mathbf{r})$. The approximation (2.10), written in the language of the operators $\hat{L}_*$ or $\hat{H}_*\equiv-\hat{L}_*^{-1}$,[7,8] is often called the Bourret approximation.[1,5] It is, however, well known that it was used long before Bourret by I. M. Lifshitz and coworkers for calculating the macroscopic elasticity coefficients in static[9] and dynamic[10] cases and that their results were applied by É. A. Kaner[11] for solving the problem of propagation of electromagnetic and scalar elastic waves in an inhomogenous medium. The stimulus for developing and applying the approximation (2.10) was the work by Keller, Tatarskiĭ, and others (see the surveys in Refs. 12 and 13).

3. DISPERSION EQUATION

The transition from the integral equation (2.4) to the corresponding dispersion equation

$$x^2-q^2\bar{\varepsilon}_*(\mathbf{x},q)=0;\ \mathbf{x}=a\mathbf{k}_*,\ q=ak_c \tag{3.1}$$

is accomplished by a direct substitution into (2.4) of the average field $\langle E\rangle$ in the form of a uniform plane wave

$$\langle E(\mathbf{r})\rangle=E_0\exp(i\mathbf{k}_*\cdot\mathbf{r}), \tag{3.2}$$

$$\mathbf{k}_*=k_*\mathbf{n},\quad k_*=k_1+ik_2. \tag{3.3}$$

It is assumed here and henceforth that the function $\bar{\varepsilon}_*(\mathbf{x},q)$ is the Fourier transform in dimensionless variables of the kernel $\bar{\varepsilon}_*(\mathbf{r},\omega)$ and of the operator $\hat{\varepsilon}_*$, and the parameter a is the spatial scale of the correlations, which is determined by the spatial dependence of the binary correlation function

$$\langle\bar{\varepsilon}''(\mathbf{r}+\mathbf{r}_1)\bar{\varepsilon}''(\mathbf{r}_1)\rangle=D_{\bar{\varepsilon}}\varphi(r/a),\quad D_{\bar{\varepsilon}}=\langle(\bar{\varepsilon}'')^2\rangle \tag{3.4}$$

of the random field $\bar{\varepsilon}(\mathbf{r})$, are statistically isotropic.[1]

The following considerations are based on a study of the roots of the dispersion equation (3.1) which determine the parameters (3.3) of the normal plane wave (3.2). We have for the function $\bar{\varepsilon}_*(\mathbf{x},q)$ in the approximation (2.11)

$$\bar{\varepsilon}_*(\mathbf{x},q)=\langle\bar{\varepsilon}\rangle+D_{\bar{\varepsilon}}F(\mathbf{x},q), \tag{3.5}$$

$$8\pi^3F(\mathbf{x},q)=\int\varphi(\mathbf{x}-\mathbf{y})Q_c(\mathbf{y},q)\,d\mathbf{y}. \tag{3.6}$$

Here $\mathbf{y}$ is the dimensionless wave vector and the Fourier transform $Q_c(\mathbf{y},q)$ of the kernel of the operator $\hat{Q}_c$ from (2.7) has the form

$$Q_c(\mathbf{y},q)=\frac{q^2}{y^2-q^2}. \tag{3.7}$$

For the immediate calculations we use the normalized binary correlation function[7] introduced in (3.4)

$$\varphi(\rho)=\exp(-\rho),\quad a\rho\equiv r. \tag{3.8}$$

The Fourier transform of the function (3.8), denoted by the same symbol, is equal to

$$\varphi(\mathbf{y})=8\pi(1+y^2)^{-2}. \tag{3.9}$$

Substituting (3.9) and (3.7) into (3.6) we find

$$F(\mathbf{x},q)=Q_c(\mathbf{x},q+i)=\frac{q^2}{x^2-(q+i)^2}. \tag{3.10}$$

The choice of the arbitrary parameter ε_c is determined by the nature of the problem to be solved.[3,5,7,14] In our case it is found from the condition

$$\langle\bar{\varepsilon}\rangle=1\Rightarrow\varepsilon_c=\langle\varepsilon\rangle,\quad D_{\bar{\varepsilon}}=D_\varepsilon\langle\varepsilon\rangle^{-2}\equiv D, \tag{3.11}$$

leading to a simplification of the expressions (3.5) for $\bar{\varepsilon}_*$ and of the roots of Eq. (3.1). It is expedient to use the change (3.11) when considering weak fluctuations[14] when D plays the role of small parameter ($D<1$). If $D>1$ (strong fluctuations) ε_c is chosen such that convergence of the series (2.9) is guaranteed.[5,14] Below we study the first case in detail and we discuss the second case.

Substituting (3.5), (3.10), and (3.11) into (3.1) and introducing the notation

$$m\equiv qM,\quad M=M(\mathbf{x},q)\equiv DF(\mathbf{x},q),\quad \bar{q}^2\equiv q^2D, \tag{3.12}$$

we bring the dispersion equation (3.1) to the form

$$m^2-2Rm-\bar{q}^2=0,\quad R\equiv i-(2q)^{-1}. \tag{3.13}$$

The roots of Eq. (3.13) are equal to

$$m_\pm=R\pm Y,\quad Y^2\equiv R^2+\bar{q}^2, \tag{3.14}$$

where the function Y is written in the form

$$Y\equiv Y_1+iY_2,\quad 2^{1/2}\bar{q}Y_{1,2}=\pm(f\pm g)^{1/2},$$
$$f^2\equiv g^2+4\bar{q}^2\bar{q}^{04},\quad g\equiv\bar{q}^4-\bar{q}^2+\bar{q}_0^4,\quad 4\bar{q}_0^4\equiv D. \tag{3.15}$$

The "+" and "−" signs define here, respectively, the functions Y_1 and Y_2.

Returning to the original variables of Eq. (3.1) and using (3.5) and (3.12) we have for the dimensionless wave number x

$$x^2=q^2+qm,\quad x\equiv x_1+ix_2. \tag{3.16}$$

We use in the present paper as a criterion for the applicability of the approximation (2.11) the inequality[15]

$$|M(x,q)|\ll1,\quad M(\mathbf{x},q)=M(x,q), \tag{3.17}$$

by virtue of which it follows from (3.16) that

$$x=q+{}^{1}/{}_{2}m. \tag{3.18}$$

We shall show in Sec. 4.1 that condition (3.17) is not satisfied in the long-wavelength band for the root x_-.

The dependence of the function M on the wave number x is a manifestation of the spatial dispersion caused by the inhomogeneity of the medium[14] which can, if the inequality

$$N(q,q)\ll 1,\qquad N(x,q)\equiv q^2\left|\frac{dM(x,q)}{dx^2}\right| \tag{3.19}$$

is satisfied,[15] be neglected by carrying out the substitution[8,11,12]

$$M(x,\ q)\to M(q,\ q). \tag{3.20}$$

We prefer to use in the inequality (3.19) instead of $N(q,q)$ the functions

$$N_\pm(q)\equiv N(x_\pm,\ q), \tag{3.21}$$

corresponding to the roots $m_\pm$ of (3.14). The violation of the inequalities

$$N_\pm(q)\ll 1, \tag{3.22}$$

which are similar to (3.19), can be interpreted as the necessity to take the spatial dispersion into account when solving Eq. (3.1).

4. CLASSIFICATION OF THE WAVELENGTH BANDS

In the general case the spatial dependence of the binary correlation function (3.4) is rather complicated for finding an analytical solution of the dispersion equation (3.1). It is therefore expedient to study approximations of those solutions for a few characteristic wavelength bands. Using as an example the function (3.8) we shall consider the following bands:

1) l-band (long wavelengths): $q\ll 1$, (4.1)

2) ls-transition region: $q\approx q_{ls}\equiv 1$, (4.2)

3) s-band (short wavelengths): $1<q<D^{-1/2}$, (4.3)

4) su-transition region: $1\ll q\approx q_{su}\equiv D^{-1/2}$, (4.4)

5) u-band (ultrashort wavelengths): $1<D^{-1/2}<q$. (4.5)

To obtain the asymptotic expressions we interpret inequalities $f_1\ll f_3$ and $f_1<f_2<f_3$ to mean $f_1\leqslant 10^{-1}f_3$ and $f_1\leqslant 10^{-1/2}f_2\leqslant 10^{-1}f_3$, respectively. Thus, it follows from (4.3) that $D\leqslant 10^{-2}$ whereas from (4.5) we get $D\leqslant 10^{-1}$. These inequalities themselves have the meaning that there exist bands which have the appropriate asymptotic properties. In particular, if $10^{-2}\leqslant D\leqslant 10^{-1}$, violation of inequalities (4.3) means only, because $q_{ls}<q_{su}$, that there is no s-band together with the analytical properties of the dynamic characteristics of the medium which are characteristic for it.

4.1. Long-wave band

When the inequalities $D\ll 1$ and (4.1) are satisfied the following holds for the function g of (3.15)

$$g=\bar{q}_0{}^4(1-\alpha),\quad \alpha\equiv(\bar{q}^2-\bar{q}^4)/\bar{q}_0{}^4\ll 1, \tag{4.6}$$

and using this the real Y_1 and imaginary Y_2 parts of the function Y of (3.15) take the form

$$Y_1=\frac{\bar{q}_0{}^2}{\bar{q}}\left(1+\frac{1}{2}\frac{\bar{q}^4}{\bar{q}_0{}^4}\right),\qquad Y_2=\frac{1}{2}\frac{\bar{q}^4}{\bar{q}_0{}^4}-1. \tag{4.7}$$

Substituting (4.7) into (3.14) and using (3.12) we get

$$m_+=q^3D(1+2iq),\quad m_-=i(2-2q^4D)-(q^{-1}+q^3D). \tag{4.8}$$

Hence we find for M

$$M_+=q^2D(1+2iq),\quad M_-=i(2q^{-1}-2q^3D)-(q^{-2}+q^2D). \tag{4.9}$$

One sees easily that M_+ and M_- satisfy the inequalities

$$|M_+|=q^2D(1+4q^2)^{1/2}\ll 1, \tag{4.10}$$

$$|M_-|=D|M_+|^{-1}\gg 1. \tag{4.11}$$

By virtue of (3.17) it follows from (4.11) that the approximation considered is inapplicable for the root m_-, given by (3.14), of Eq. (3.13). The function $N(x,q)$ defined by Eq. (3.19) has, if we use (3.10), (3.12), (3.13), and (3.16), the form

$$N(x,q)=\frac{\bar{q}^2}{|m-2R|^2}. \tag{4.12}$$

Substituting here the values of the roots $x_\pm$ or, what is the same, of $m_\pm$ from (3.14) we find for the functions $N_\pm(q)$ from (3.21)

$$N_\pm(q)=\frac{\bar{q}^2}{|m_\mp|^2}=|M_\pm|^2D^{-1}. \tag{4.13}$$

Using (4.10) we get from this for the root x_+ the inequality

$$N_+(q)=q^4D(1+4q^2)\ll 1, \tag{4.14}$$

and if this is satisfied it means according to (3.22) that it was possible to neglect spatial dispersion and use the substitution (3.20).

4.2. *ls* transition region

In the wavelength band defined by (4.2), we have $\bar{q}\sim\bar{q}_0^2$, and for the function f of (3.15) the following approximation holds

$$f=\bar{q}^2+\bar{q}_0{}^4-2\beta(\bar{q}^2-\bar{q}_0{}^4),\qquad 2\beta\equiv\frac{\bar{q}^4}{\bar{q}^2+\bar{q}_0{}^4}\ll 1, \tag{4.15}$$

the use of which leads for $Y_{1,2}$ instead of (4.7) to

$$Y_1=\frac{\bar{q}_0{}^2}{\bar{q}}(1+\beta),\quad Y_2=\beta-1;\quad \bar{q}\ll 1\approx q. \tag{4.16}$$

Similar to (4.8) we have for $m_\pm$

$$m_+=\frac{q^3D}{1-2iq},\quad m_-=i(2-\beta)-\frac{1}{q}\left(1+\frac{1}{2}\beta\right). \tag{4.17}$$

Hence we find for $|M_\pm|$

$$|M_+|=\frac{\bar{q}^2}{(1+4q^2)^{1/2}}\ll 1,\qquad |M_-|=\frac{(1+4q^2)^{1/2}}{q^2}\sim 1. \tag{4.18}$$

One sees easily that the criterion (3.17) is not satisfied for the root x_-. Substituting (4.17) into (4.13) we get for the root x_+ the inequality

$$N_+(q)=\frac{q^4D}{1+4q^2}\sim D\ll 1, \tag{4.19}$$

signifying that the substitution

$$M(x,q)\to M(x_+,q). \tag{4.20}$$

is allowed.

4.3. Short-wave band

By virtue of (4.3) we can write for the function g of (3.15)

$$g=-\bar{q}^2\left(1-\frac{\bar{q}^4+\bar{q}_0^4}{\bar{q}^2}\right),\quad \bar{q}<1<q. \tag{4.21}$$

Using (4.21) we find from (3.15)

$$Y_1=(2q)^{-1}+{}^1/_4qD,\quad Y_2={}^1/_2q^2D-1. \tag{4.22}$$

Substituting (4.22) into (3.14) and using (3.12) we get

$$m_+=\frac{1+2iq}{4}qD,\quad m_-=i\left(2-\frac{1}{2}q^2D\right)+\left(q^{-1}+\frac{1}{4}qD\right). \tag{4.23}$$

Hence we have for $|M_\pm|$

$$|M_+|=\frac{D}{4}(1+4q^2)^{1/2}\approx\frac{1}{2}\bar{q}D^{1/2}\ll 1,$$
$$|M_-|=D|M_+|^{-1}=4(1+4q^2)^{-1/2}\approx 2/q\ll 1. \tag{4.24}$$

Starting with the s-band the solution corresponding to the second root of the dispersion equation thus satisfies the condition for the applicability of the pair interaction approximation (2.11). As to the criterion (3.22), substitution of (4.24) into (4.13) gives

$$N_+(q)=\frac{1}{16}D(1+4q^2)\approx\frac{1}{4}\bar{q}^2\ll 1, \tag{4.25}$$

$$N_-(q)=1/N_+(q)\gg 1. \tag{4.26}$$

Hence, the solution using the root x_- needs taking the spatial dispersion into account. For a comparison we give the value of the function $N(q,q)$ introduced by Finkel'berg.[15] Using (4.12) and (3.19) we write

$$N(q,q)=\frac{\bar{q}^2}{4|R|^2}=\frac{\bar{q}^2q^2}{1+4q^2}\approx N_+(q)\ll 1. \tag{4.27}$$

The principal differences between (4.26) and (4.27) reflect the differences between the criteria (3.19) and (3.22), the first of which does not take into account the value of the root considered.

4.4. *su* transition region

For wave numbers defined by the band (4.4) the inequality $|\bar{q}^4-\bar{q}^2|\ll\bar{q}_0^4$ holds, by virtue of which we can write the functions $Y_{1,2}$ of (3.15) in the form

$$Y_{1,2}=\pm\lambda\bar{q}_0(1\pm\xi_++{}^1/_2\xi_-^2), \tag{4.28}$$

$$\lambda^4\equiv\frac{1+\bar{q}^2}{2\bar{q}^2},\quad \lambda\approx 1,\quad \xi_\pm\equiv\frac{\bar{q}^4-\bar{q}^2\pm\bar{q}_0^4}{[8\bar{q}_0^4(\bar{q}^4+\bar{q}^2)]^{1/2}},\quad |\xi_\pm|\ll 1.$$

Here, as in (3.15), the " + " and " − " signs correspond to the functions Y_1 and Y_2. Substituting (4.28) into (3.14) we find

$$m_\pm=(1\pm Y_2)(i\pm Y_1),\quad |Y_{1,2}|\approx\bar{q}_0<1. \tag{4.29}$$

Using (4.29) we have for $|M_\pm|$

$$|M_\pm|\approx D^{1/2}|1\pm\bar{q}_0|(1+\bar{q}_0^2)^{1/2}\sim D^{1/2}\ll 1, \tag{4.30}$$

i.e., both roots of Eq. (3.1) satisfy the criterion (3.17). Using (4.30) we get from (4.13)

$$N_\pm(q)\approx(1\mp\bar{q}_0)^2(1+\bar{q}_0^2)\sim 1. \tag{4.31}$$

By virtue of (3.22) Eq. (4.31) means that in this wavelength band both roots of the dispersion equation (3.1) must be evaluated taking spatial dispersion into account.

4.5. Ultrashort wavelength band

By virtue of (4.5) we write for the function g of (3.15)

$$g=\bar{q}^4\left(1-\frac{\bar{q}^2-\bar{q}_0^4}{\bar{q}^4}\right),\quad 1<\bar{q}<q. \tag{4.32}$$

Substitution of (4.32) into (3.15) gives

$$Y_1=\bar{q}\left(1-\frac{1}{2\bar{q}^2}\right)\approx\bar{q},\quad Y_2=-\frac{\bar{q}_0^2}{\bar{q}^2}\left(1+\frac{1}{2\bar{q}^2}\right)\approx-\frac{\bar{q}_0^2}{\bar{q}^2}. \tag{4.33}$$

Hence we have, similar to (4.29),

$$m_\pm=\left(1\mp\frac{\bar{q}_0^2}{\bar{q}^2}\right)(i\pm\bar{q}). \tag{4.34}$$

Instead of (4.30) we now get for $|M_\pm|$

$$|M_\pm|\approx D^{1/2}\left|1\mp\frac{\bar{q}_0^2}{\bar{q}^2}\right|\approx D^{1/2}<1. \tag{4.35}$$

Using (4.35) we find from (4.13)

$$N_\pm(q)\approx\left(1\mp\frac{\bar{q}_0^2}{\bar{q}^2}\right)^2\approx 1. \tag{4.36}$$

Concluding the analysis of the various wavelength bands we must note that by virtue of (4.26), (4.31), and (4.36) the violation of the N criterion (3.22) leads in those cases to the need to take spatial dispersion into account. The violation in (4.11) and (4.18) of the M criterion (3.17) means the inadmissibility of the use of the approximation (2.10) and (2.11) for the x_- root [see (3.14) and (3.16)]. The M criterion is satisfied for the x_+ root in the whole range of wavelengths provided $D\ll 1$ which is valid in the case of weak fluctuations of the random field $\varepsilon(\mathbf{r})$. In this connection we call the roots x_+ and x_- in what follows real and virtual, respectively.

5. SCATTERING INDEX AND PHASE AND GROUP VELOCITIES OF A PLANE WAVE

Below we study some characteristics of the field (3.2) calculated on the basis of the dimensionless wave number x of (3.16) satisfying the dispersion equation (3.1) for the effective medium.

As a quantitative measure for the scattering of the waves we introduce the dimensionless scattering index

$$a\gamma\equiv\bar{\gamma}\equiv 2\,\mathrm{Im}\,x\equiv 2x_2, \tag{5.1}$$

where γ is the intensity attenuation coefficient of the average field.[1,11,12] The dimensionless phase ($\bar{v}$) and group ($\bar{c}$) velocities are defined by the equations

$$\bar{v}\equiv\frac{q}{x_1}=\frac{v_*}{v_c},\quad \frac{1}{\bar{c}}\equiv\frac{dx_1}{dq}=\frac{v_c}{c_*},\quad v_c\equiv c_0\varepsilon_c^{-1/2}. \tag{5.2}$$

where v_* and c_* are, respectively, the phase and group velocities for propagation of plane waves in the effective medium. Using the roots x_+ and x_- in (5.1) and (5.2) leads to the parameters $\bar{\gamma}_\pm$, $\bar{v}_\pm$, and $\bar{c}_\pm$ of the real and the virtual waves. Using (3.12) and (3.18) we can rewrite (5.1) and (5.2) in the form

$$\begin{gathered}\bar{\gamma}=\bar{v}m_2, \quad \bar{v}\approx 1-{}^1/_2M_1, \quad \bar{c}=\bar{v}(1-\nu_1)^{-1}; \\ M\equiv M_1+iM_2, \quad \nu_1\equiv d(\lg\bar{v})/d(\lg q).\end{gathered} \tag{5.3}$$

Below we need also the functions

$$\nu\equiv\frac{d(\lg\bar{\gamma})}{d(\lg q)}\equiv\nu_1+\nu_2, \qquad \nu_2\equiv\frac{d(\lg m_2)}{d(\lg q)}. \tag{5.4}$$

Substituting (3.14)–(3.16) into (5.3) and (5.4) leads to very cumbersome formulas, so that we shall only write down asymptotic expressions for the quantities $\bar{\gamma}$, $\bar{v}$, $\bar{c}$, ν_1, and ν. Labeling the wave number q which is the argument of the functions γ, $\bar{v}$, $\bar{c}$, ν_1, and ν by the letters indicating the corresponding wavelength bands and using the results of § 4 we get, respectively,

$$\begin{gathered}\bar{v}_+(l)=1-{}^1/_2\bar{q}^2, \quad \bar{\gamma}_+(l)=2q^4D, \quad \bar{q}<q\ll 1, \\ \bar{c}_+(l)=1-{}^3/_2\bar{q}^2, \quad \nu_1{}^+(l)=-\bar{q}^2, \quad \nu_+(l)=4\end{gathered} \tag{5.5}$$

in the long wavelength band (4.1);

$$\begin{gathered}\bar{v}_+(ls)=1-\frac{{}^1/_2q^2D}{1+4q^2}, \quad \bar{\gamma}_+(ls)=\frac{2q^4D}{1+4q^2}, \\ \bar{c}_+(ls)=1-\frac{q^2D(3+4q^2)}{2(1+4q^2)}, \quad \bar{q}<q\approx 1, \\ \nu_1{}^+(ls)=-\frac{q^2D}{(1+4q^2)^2}, \quad \nu_+(ls)=4\frac{1+2q^2}{1+4q^2}\end{gathered} \tag{5.6}$$

in the ls transition region (4.2);

$$\begin{gathered}\bar{v}_+(s)\approx\bar{c}_+(s)\approx 1-{}^1/_8D, \quad \bar{\gamma}_+(s)={}^1/_2\bar{v}_+(s)\bar{q}^2, \quad \bar{q}<1<q, \\ \bar{v}_-(s)=1+{}^1/_2q^{-2}, \; \bar{c}_-(s)=1+{}^3/_2q^{-2}, \; \bar{\gamma}_-(s)\approx 2-\bar{\gamma}_+(s), \\ \nu_1{}^+(s)=0, \quad \nu_+(s)=2, \quad \nu_1{}^-(s)=q^{-2}, \quad \nu_-(s)=q^{-2}-{}^1/_2\bar{q}^2\end{gathered} \tag{5.7}$$

in the short wavelength band (4.3);

$$\begin{gathered}\bar{v}_\pm(su)=1+\frac{\bar{q}_0{}^4}{\bar{q}^2}\mp\frac{\bar{q}_0{}^3}{\bar{q}}\lambda, \qquad \bar{c}_\pm(su)=\frac{\bar{v}_\pm(su)}{1-\nu_1{}^\pm(su)}, \\ \nu_1{}^\pm(su)\approx -2\bar{q}_0{}^3\bar{q}\lambda\frac{d\xi_+}{d\bar{q}^2}\approx\mp\bar{q}_0\frac{2\bar{q}^2-1}{2\bar{q}\lambda}\approx\mp\frac{1}{2}\bar{q}_0, \quad 1\approx\bar{q}\ll q, \\ \bar{\gamma}_\pm(su)\approx\left(1\mp\frac{\bar{q}_0{}^3}{\bar{q}}\lambda\right)(1\mp\bar{q}_0\lambda)\approx 1\mp\bar{q}_0\lambda\left(1+\frac{\bar{q}_0{}^2}{\bar{q}}\right), \\ \nu_2{}^\pm(su)\approx\pm\frac{2\bar{q}_0\lambda}{1\mp\bar{q}_0\lambda}\bar{q}^2\frac{d\xi_+}{d\bar{q}^2}\approx\pm(1\mp\bar{q}_0\lambda)^{-1}\frac{2\bar{q}^2-1}{2\lambda\bar{q}_0}\approx\pm\frac{1}{2\lambda\bar{q}_0}\end{gathered} \tag{5.8}$$

in the su transition region (4.4);

$$\begin{gathered}\bar{v}_\pm(u)\approx\bar{c}_\pm(u)\approx\bar{\gamma}_\pm(u)\approx 1\mp{}^1/_2D^{1/2}, \\ \nu_1{}^\pm(u)=\mp\frac{1\pm D^{1/2}}{2q\bar{q}}\approx 0, \quad \nu_2{}^\pm(u)=\pm\frac{1}{q\bar{q}}\approx 0, \quad 1<\bar{q}<q\end{gathered} \tag{5.9}$$

in the ultrashort wavelength band (4.5). In Fig. 1 we show the curves of the functions $\bar{v}_\pm$ and $\bar{c}_\pm$. The numbers at the curves indicate the values of the parameter $\mu\equiv -\log D$. In the su transition region we see the group velocity retardation

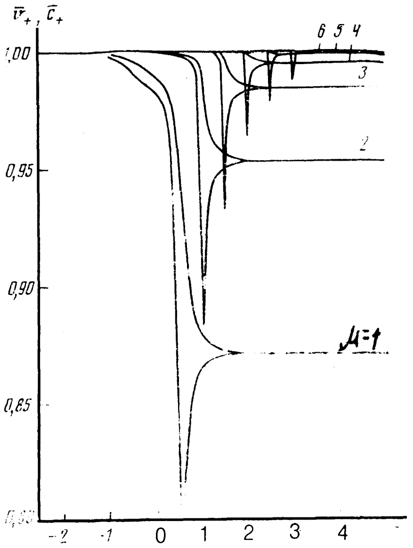

FIG. 1. The dimensionless phase $\bar{v}_+$ (upper curve of each pair) and group $\bar{c}_+$ (lower curve) velocities as functions of the logarithm of the dimensionless wave number q (the numbers at the curves indicate the values of the parameter $\mu = -\log D$ corresponding to each pair of curves.

effect caused by multiple inverse coherent scattering. The depth of the minimum in the $\bar{c}_+$ curve is determined by the parameter ν_1^+. According to (5.8) we have

$$\frac{\bar{v}_+(su)}{\bar{c}_+(su)}-1=-\nu_1{}^+(su)\approx{}^1/_2\bar{q}_0=2^{-3/2}D^{1/4}, \tag{5.10}$$

which gives, respectively, 0.199 and 0.112 for $\mu = 1$ and $\mu = 2$.

In Fig. 2 we show, using the notation of Fig. 1, the curves of the function ν_+ which has a maximum in the su transition region. According to (5.4) and (5.8) we find for max ν_+

$$\max\nu_+\approx\nu_2{}^+(su)\approx(2\bar{q}_0)^{-1}=2^{-1/2}D^{-1/4}, \tag{5.11}$$

which gives 1.26 and 2.24, respectively, for $\mu = 1$ and $\mu = 2$. One sees easily that the approximate equation

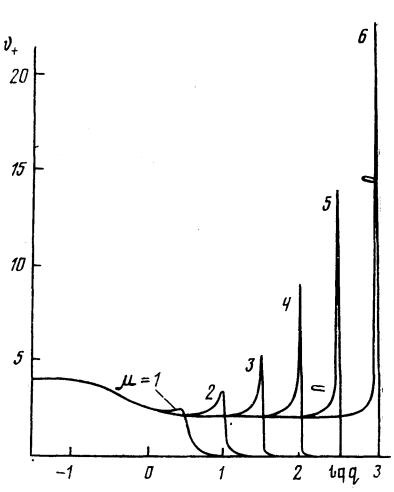

FIG. 2. The parameter ν_+ of (5.4) as function of the logarithm of the dimensionless wave number q (the numbers at the curves indicate the values of the parameter μ).

$$4v_1^+(su)v_2^+(su)\approx -1 \tag{5.12}$$

is satisfied. Equations (5.10) to (5.12) correctly reflect the changes in min $\bar{c}_+$ and max ν_+ when the parameter μ increases. It is clear from Fig. 2 that there is no s band for $\mu<2$ while the ls and su transition regions merge into a single lu transition region.

6. AVERAGE FIELD OF A POINT CHARGE

Below we evaluate for an effective medium with material properties described by the operator (2.11) the field produced by a point source (Green function). Similar to (2.4) the Helmholtz equation in operator form looks like

$$\hat{L}_*\hat{H}_*=-\hat{I}. \tag{6.1}$$

The Fourier transform of the kernel $H_*(\mathbf{r},q)$ of the operator $\hat{H}_*$ can be found from the equation

$$[y^2-q^2\varepsilon_*(\mathbf{y},q)]H_*(\mathbf{y},q)=1, \tag{6.2}$$

where we used the notation of (3.1) and (3.6). In the pair interaction approximation (3.5) for the normalized binary correlation function (3.8), leading to (3.10), the Fourier transform $H_*(\mathbf{y},q)$ of the Green function $H_*(\mathbf{r},q)$ can be written in the form

$$H_*(\mathbf{y},q)=\frac{H_+(q)}{y^2-x_+^2}+\frac{H_-(q)}{y^2-x_-^2}, \tag{6.3}$$

where the $x_\pm$ are the roots of the disperion equation (3.1); their explicit form is found by substituting (3.14) and (3.15) into (3.16) or (3.18). The amplitudes $H_\pm(q)$ of the real and the virtual components of the Green function are equal to

$$H_\pm(q)=\pm\frac{x_\pm^2-(q+i)^2}{x_+^2-x_-^2} \tag{6.4}$$

and satisfy the equation

$$H_+(q)+H_-(q)=1. \tag{6.5}$$

Using (3.14), (3.15), and (4.13) we get from (6.4)

$$\left|\frac{H_-(q)}{H_+(q)}\right|=\left|\frac{m_+(q)}{m_-(q)}\right|=N_+(q)\equiv N(q)\leqslant 1, \tag{6.6}$$

where the last inequality follows from the results of Sec. 4. In the $\bar{q}\to\infty$ limit we have from (6.6) and (4.36): $N(q)=1$.

In view of the complexity of the $H_\pm(q)$ functions we give below as in Sec. 5 the asymptotic expressions for $|H_\pm(q)|$ and $N(q)$. Using the results and notation of Secs. 4 and 5 we get, respectively,

$$|H_+(l)|=1-q^4D\approx 1,\quad |H_-(l)|=q^4D,\quad N(l)\approx q^4D\ll 1 \tag{6.7}$$

in the long wavelength band (4.1);

$$|H_+(ls)|=1+\frac{q^4D}{1+4q^2},\qquad N(ls)\approx|H_-(ls)|=\frac{q^4D}{1+4q^2} \tag{6.8}$$

in the ls transition region (4.2);

$$|H_+(s)|=1+{}^1\!/_4\bar{q}^2,\quad N(s)\approx|H_-(s)|={}^1\!/_4\bar{q}^2 \tag{6.9}$$

in the short wavelength band (4.3);

$$|H_\pm(su)|=\frac{1\pm\lambda\bar{q}_0}{2^{3/2}\lambda\bar{q}_0}\left(\frac{1+\lambda^2\bar{q}_0^2}{1+\xi_+^2+\xi_-^2}\right)^{1/2},\quad N(su)=\frac{1-\lambda\bar{q}_0}{1+\lambda\bar{q}_0} \tag{6.10}$$

in the su transition region (4.4);

$$|H_\pm(u)|=\frac{1}{2}\left(1\pm\frac{\bar{q}_0^2}{\bar{q}^2}\right),\qquad N(u)=\frac{\bar{q}^2-\bar{q}_0^2}{\bar{q}^2+\bar{q}_0^2}\leqslant 1 \tag{6.11}$$

in the very short wavelength band (4.5).

Our study shows that the functions $|H_\pm(q)|$ have anomalous properties in the su transition region, reaching maxima equal to

$$\max|H_\pm(su)|\approx\frac{1\pm\bar{q}_0}{2^{3/2}\bar{q}_0}\approx 2^{-1/2}\max v_+. \tag{6.12}$$

In the $\bar{q}\to\infty$ limit we find from (4.34) and (6.11) for the amplitudes $H_\pm$ that $2H_\pm=1$.

The transition from the Fourier transform (6.3) to the original gives

$$4\pi rH_*(\mathbf{r},q)=H_+(q)\exp(ix_+r/a)+H_-(q)\exp(ix_-r/a), \tag{6.13}$$

i.e., in the general case the Green function can be written as a superposition of two spherical waves with parameters determined by the complex wave numbers x_+ and x_-. In the long wavelength band ($q\to 0$) we find from (3.18), (4.8), (6.7), and (6.13)

$$4\pi rH_*(\mathbf{r},q)=H_+(l)\exp[ix_+(l)r/a],\quad x_+(l)\approx q+iq^4D. \tag{6.14}$$

In the other limiting case, of ultrashort wavelengths ($\bar{q}\to\infty$), we have from (3.18), (4.34), (6.11), and (6.13)

$$4\pi rH_*(\mathbf{r},q)=\exp\left(\frac{2iq-1}{2a}r\right)\cos\frac{\pi r}{\Lambda},\quad \Lambda\equiv\frac{2\pi a}{q}D^{-1/2}. \tag{6.15}$$

In the $q\to 0$ and $\bar{q}\to\infty$ limits the field of a point source in the effective medium thus has the form of an outgoing spherical wave. However, in contrast to (6.14), there appears in (6.15) a modulating factor with a spatial period $2\Lambda\gg\lambda_c$, where $\lambda_c\equiv 2\pi a/q$ is the wavelength in the comparison medium. In the case considered by us the material properties of this medium are by virtue of (3.11) described by the parameter $\varepsilon_c=\langle\varepsilon\rangle$, obtained by a statistical average of the random field $\varepsilon(\mathbf{r})$.

7. DISCUSSION

The idea of a nonlocal relation between induction and field, first introduced in the theory of excitons,[16] was rather soon afterwards used in the theory of inhomogeneous media.[14,15,17]

The study carried out by us has demonstrated the importance of the spatial dispersion through the example of a well known problem. We obtained in the pair-interaction approximation (Bourret approximation) a solution of the dispersion equation (3.1) valid for all wavelengths. It was possible by the use of the parameters a (the spatial scale of the correlations of (3.1) and (3.2)) and D [the dimensionless dispersion of the field ε given by (3.4) and (3.11)] to write the whole wave spectrum as a superposition of five bands (see Sec. 4). For each of the bands thus introduced we studied the two criteria: 1) applicability of the Bourret approximation (M criterion), and 2) possibility of neglecting the spatial dispersion (N criterion). We showed that provided $D^{1/2} \ll 1$ the M criterion is satisfied for the real root x_+ in the whole of the wave spectrum but for the virtual root x_- only in its short wavelength part, including the s band. Here, however, the N criterion for the root x_+ is violated indicating the need to take spatial dispersion into account.[1)] The anomalous behavior of the dynamic parameters $\bar{v}, \bar{c}$, and $\bar{\gamma}$ (see Figs. 1 and 2) and of the amplitudes $H_\pm$ of the Green function (see Sec. 6) show the effect of taking multiple inverse coherent scattering into account. It seems that this can be explained by the observation of an amplitude modulation of the Green function (6.15) in the very short wavelength band.

[1)] The N criterion for the root x_- is, in contrast to the root x_+, not satisfied for any wavelength. In terms of ε_* this is a consequence of the strong spatial dispersion of its imaginary part.[12,14] However, the fact that the amplitude H_- is small for the long and the short wavelengths (see Sec. 6) means that one can in those bands neglect the root x_-.

[1] S. Rytov, Yu. A. Kravtsov, and V. I. Tatarskiĭ, *Introduction to Statistical Radiophysics, Part II, Random Fields* [in Russian], Nauka, Moscow (1978).

[2] L. M. Brekhovskikh, *Waves in Layered Media*, Academic, New York, (1960).

[3] V. M. Finkel'berg, Zh. Tekh. Fiz. **34**, 509 (1964) [Sov. Phys. Tech. Phys. **9**, 396 (1964)].

[4] A. G. Fokin, Zh. Tekh. Fiz. **41**, 1073 (1971) [Sov. Phys. Tech. Phys. **16**, 849 (1971)].

[5] A. G. Fokin, Phys. Status Solidi B **119**, 741 (1983).

[6] A. G. Fokin and T. D. Shermergor, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **32**, 176 (1989) [Radiophys. Qu. Electron. **32**, 136 (1989)].

[7] R. C. Bourret, Can. J. Phys. **40**, 782 (1962).

[8] R. C. Bourret, Nuovo Cim. **26**, 1 (1962).

[9] I. M. Lifshitz and L. N. Rozentsveĭg, Zh. Eksp. Teor. Fiz. **16**, 967 (1946).

[10] I. M. Lifshitz and G. D. Parkhomovskiĭ, Uch. Zap. A. M. Gor'kov Khar'kov Gos. Univ. **27**, 25 (1948).

[11] É. A. Kaner, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **2**, 827 (1959).

[12] Yu. A. Ryzhov and V. V. Tamoĭkin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **13**, 356 (1970) [Radiophys. Qu. Electron. **13**, 273 (1973)].

[13] Yu. N. Barabanenkov, Yu. A. Kravtsov, and S. M. Rytov, Usp. Fiz. Nauk **102**, 3 (1970) [Sov. Phys. Usp. **13**, 551 (1971)].

[14] D. A. Ryzhov, V. V. Tamoĭkin, and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **48**, 656 (1965) [Sov. Phys. JETP **21**, 433 (1965)].

[15] V. M. Finkel'berg, Zh. Eksp. Teor. Fiz. **53**, 401 (1967) [Sov. Phys. JETP **26**, 268 (1968)].

[16] S. I. Pekar, Zh. Eksp. Teor. Fiz. **33**, 1022 (1957) [Sov. Phys. JETP **6**, 835 (1958)].

[17] Yu. A. Ryzhov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **9**, 39 (1966) [Sov. Radiophys. **9**, 25 (1967)].

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