Integrals of the drift motion of particles with finite Larmor radius

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The problem is studied of the presence of integrals in the generalized drift-kinetic equation (DKE) which takes into complete account all the finite Larmor radius (FLR) corrections to first order in an expansion in the inverse cyclotron frequency in the general three-dimensional case. The possibility of macroscopic flow of the plasma is taken into account. It is shown that the magnetic moment, modified with account of the longitudinal motion of the particles and the vortex flow of the plasma is an exact integral of the generalized DKE. It is shown that special calibration of the electric potential leads, in the low-frequency case considered, to an approximate DKE integral of an energy type. A general expression is obtained for the energy density flux connected with FLR corrections in the DKE.

1. INTRODUCTION

The approximation of a collisionless magnetized plasma is frequently used in the description of a wide range of laboratory and cosmic plasma systems. This approximation assumes that the characteristic time and space scales of change in the state of the system exceed the period of cyclotron rotation ($\sim 1/\Omega = mc/eB$) and the Larmor radius ($\sim v_T/\Omega$), respectively, of particles in a magnetic field B, but are significantly smaller than the times and lengths of free flight. The motion of the particles in this case consists of a fast cyclotron rotation, a drift motion across the lines of force of the magnetic field, and a longitudinal motion. The dynamics of slow drift and longitudinal motions of an ensemble of particles is described by the drift kinetic equation (DKE) of Rudakov and Sagdeev,[1] which is obtained from the collisionless equation of Vlasov by averaging over the cylotron rotation. The account of the effects of a finite Larmor radius (FLR) in problems on the equilibrium and stability of a plasma (see, for example, Refs. 2–4) has stimulated numerous attempts to modify the DKE with account of a non-zero value of $1/\Omega$. However, calculations of such a type are rather cumbersome, so that one usually restricts oneself to a two-dimensional case,[2,5,6] to an implicit form of the equation, etc., which narrow down the region of applicability of the results.

A generalization of the DKE, taking completely into account the FLR corrections in first order in expansions in terms of $1/\Omega$ in the general three-dimensional case, has been obtained in Ref. 8. In the present work we consider the problem of the presence of integrals of the motion of the plasma particles with account of macroscopic flow of the plasma and FLR corrections to the drift motion. It is shown that correct allowance for the longitudinal motion of the particles and for the vortex flow of the plasma modify the equation for the adiabatic invariant (the magnetic moment). It is also shown that, if the electrostatic potential is calibrated in special fashion, then the total energy of the particles in a comparatively slow (but not stationary) drift motion is also conserved. A general expression associated with the FLR correction to the DKE is obtained for the energy flux density.

2. DRIFT KINETIC EQUATION WITH ACCOUNT OF FLR

Following Ref. 8, let us write the Vlasov equation for the one-particle distribution function $f(t,\mathbf{r},\mathbf{v})$ in the form

$$\frac{\partial f}{\partial t}+\mathbf{v}\nabla f+\frac{\mathbf{F}}{mn}\frac{\partial f}{\partial \mathbf{v}}=\Omega[(\mathbf{V}-\mathbf{v})\times\mathbf{b}]\frac{\partial f}{\partial \mathbf{v}}. \tag{1}$$

In contrast with the traditional approach,[7] we now introduce the volume density $\mathbf{F}$ of electromagnetic force, while the electric field $\mathbf{E}$ is eliminated by the relation $\mathbf{E}=-[\mathbf{V}\times\mathbf{B}]/c+\mathbf{F}/en$. Most of the notation is standard, m and e are the mass and charge of the particles of the described component of the plasma, and $\mathbf{V}$ is the mean mass velocity of this component. The transition to the quantities $\mathbf{F}$ and $\mathbf{V}$ in Eq. (1) in place of $\mathbf{E}$ and $\mathbf{V}_E=c\mathbf{E}\times\mathbf{B}/B^2$ allows us to avoid the explicit expansion of the electromagnetic field in powers of $1/\Omega$, and the appearance of products of the electron drift velocity $\mathbf{V}_E$ in the FLR terms (see Ref. 7). By integrating (1) with the weights 1 and $\mathbf{v}$, we obtain then the equation of continuity in ordinary hydrodynamic form without any corrections

$$d_t n+n\operatorname{div}\mathbf{V}=0 \tag{2}$$

and the expression for $\mathbf{F}$ in terms of the macroscopic parameters of the plasma

$$\mathbf{F}=mnd_t\mathbf{V}+\nabla\cdot\mathbf{P}, \tag{3}$$

where $\mathbf{P}$ is the pressure tensor and $d_t=\partial_t+\mathbf{V}\nabla$. The dynamics of the magnetic field, which enters into Eq. (1) only through Ω and the unit vector $\mathbf{b}=\mathbf{B}/B$, is determined by the exact equation

$$\frac{\partial_t\mathbf{B}}{B}=\frac{\operatorname{rot}[\mathbf{V}\times\mathbf{B}]}{B}-\frac{1}{\Omega}\operatorname{rot}\frac{\mathbf{F}}{mn}. \tag{4}$$

Introducing the velocity $\mathbf{u}$ and the basis $\{\mathbf{e}_\rho,\mathbf{e}_\varphi,\mathbf{b}\}$ in velocity space by the relation

$$\mathbf{u}=\mathbf{v}+\mathbf{V}=u_\parallel\mathbf{b}+u_\perp\mathbf{e}_\rho,\quad \mathbf{e}_\varphi=[\mathbf{b}\times\mathbf{e}_\rho]. \tag{5}$$

We can now write the right-hand side of (21) in the form Ω $(\partial f/\partial\varphi)$, and seek the distribution function in the form of the expansion

$$f=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}f_n^j;\qquad f_n^j-\Omega^{-n}e^{ij\varphi},\quad i=(-1)^{1/2}. \tag{6}$$

In zeroth order in $1/\Omega$, the distribution function is isotropic in the plane perpendicular to the direction of the magnetic field ($f_0=f_0^0$) and is determined by the equation

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$$
0=\hat{\Phi}^0[f_0^0]=d_*f_0+\frac{\partial f_0}{\partial u_\parallel}\left[\frac{u_\perp^2}{2}\operatorname{div}\mathbf{b}+\mathbf{b}\left(\frac{\mathbf{F}}{mn}-d_*\mathbf{V}\right)\right]
-\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}(\operatorname{div}_\perp\mathbf{V}+u_\parallel\operatorname{div}\mathbf{b}),
$$

$$
d_*=d_t+u_\parallel\mathbf{b}\nabla=\partial_t+\mathbf{V}\nabla+u_\parallel\mathbf{b}\nabla. \tag{7}
$$

In first order, the oscillating part of the distribution function is given by the relations[8]

$$
\begin{aligned}
f_1^1&=-\frac{1}{\Omega}\mathbf{e}_\varphi\left\{u_\perp\nabla f_0+u_\perp\frac{\partial f_0}{\partial u_\parallel}(d_*\mathbf{b}-\mathbf{b}(\nabla)\mathbf{V})
+\frac{\partial f_0}{\partial u_\perp}\left[\frac{\mathbf{F}}{mn}-d_*(\mathbf{V}+u_\parallel\mathbf{b})\right]\right\},\\
f_1^2&=\frac{1}{2\Omega}\left\{\left(u_\parallel u_\perp\frac{\partial f_0}{\partial u_\perp}-u_\perp^2\frac{\partial f_0}{\partial u_\parallel}\right)\left(\mathbf{e}_\varphi(\mathbf{e}_\rho\nabla)\mathbf{b}-\frac{1}{2}\mathbf{b}\operatorname{rot}\mathbf{b}\right)
+u_\perp\frac{\partial f_0}{\partial u_\perp}\left(\mathbf{e}_\varphi(\mathbf{e}_\rho\nabla)\mathbf{V}-\frac{1}{2}\mathbf{b}\operatorname{rot}\mathbf{V}\right)\right\},
\end{aligned}\tag{8}
$$

while the DKE for the isotropic part is modified in the following fashion:[1)]

$$
\begin{aligned}
&\hat{\Phi}^0[f_1^0]\\
&+\frac{1}{\Omega}\left\{\mathbf{b}\operatorname{rot}((\mathbf{b}\nabla)\mathbf{b})\left[\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\left(u_\parallel^2+\frac{p_\perp-p_\parallel}{mn}\right)-\frac{u_\perp^2}{2}u_\parallel\frac{\partial f_0}{\partial u_\parallel}\right]\right.\\
&+[\mathbf{b}\times(\mathbf{b}\nabla)\mathbf{b}]\\
&\times\left[\left(u_\parallel^2+\frac{p_\perp-p_\parallel}{mn}\right)\left(\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\frac{\nabla B}{B}+\nabla f_0\right)-\frac{u_\perp^2}{2}u_\parallel\frac{\partial f_0}{\partial u_\parallel}\frac{\nabla B}{B}\right.\\
&\left.+\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\nabla\frac{p_\parallel-p_\perp}{mn}\right]\\
&+\operatorname{rot}_\perp\mathbf{V}\cdot\left[-\frac{u_\perp^2\partial f_0}{2\partial u_\parallel}\frac{\nabla B}{B}+(d_t\mathbf{b}+(\mathbf{b}\nabla)\mathbf{V})\left(\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\right.\right.\\
&\left.\left.-u_\parallel\frac{\partial f_0}{\partial u_\parallel}\right)+\frac{\nabla p_\perp}{mn}\frac{\partial f_0}{\partial u_\parallel}\right.\\
&\left.+(\mathbf{b}\nabla)\mathbf{b}\left[u_\perp u_\parallel\frac{\partial f_0}{\partial u_\perp}-\left(\frac{u_\perp^2}{2}+u_\parallel^2+\frac{p_\perp-p_\parallel}{mn}\right)\frac{\partial f_0}{\partial u_\parallel}\right]\right]\\
&+2[(\mathbf{b}\nabla)\mathbf{V}\times\mathbf{b}]\cdot\left[-\frac{u_\perp}{2}u_\parallel\frac{\partial f_0}{\partial u_\perp}\frac{\nabla B}{B}\right.\\
&\left.-u_\parallel\nabla f_0+(\mathbf{b}\nabla)\mathbf{b}\left(u_\parallel\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}-u_\parallel^2\frac{\partial f_0}{\partial u_\parallel}\right)\right]\\
&+\left[\frac{\nabla p_\perp}{mn}\times\mathbf{b}\right]\cdot\left(\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\frac{\nabla(nB)}{nB}\right.\\
&\left.+\nabla f_0+u_\parallel\frac{\partial f_0}{\partial u_\parallel}(\mathbf{b}\nabla)\mathbf{b}\right)+\mathbf{b}\operatorname{rot}\mathbf{b}\left(\mathbf{b}\nabla\frac{u_\perp^2f_0}{2}\right.\\
&\left.+\frac{u_\perp}{2}\frac{\partial f_0}{\partial u_\perp}\frac{\mathbf{b}\nabla p_\perp}{mn}\right)+u_\parallel u_\perp\frac{\partial f_0}{\partial u_\perp}[\mathbf{b}\operatorname{rot}(\mathbf{b}\nabla)\mathbf{V}-\mathbf{b}\operatorname{rot}\mathbf{b}(\mathbf{b}(\mathbf{b}\nabla)\mathbf{V})]\\
&+\frac{[\mathbf{b}\times\nabla B]}{B}\nabla\left(\frac{u_\perp^2}{2}f_0\right)\\
&+\frac{\partial f_0}{\partial u_\parallel}\left[\frac{u_\perp^2}{4}[\mathbf{b}\operatorname{rot}\mathbf{b}\operatorname{div}_\perp\mathbf{V}-\mathbf{b}\operatorname{rot}\mathbf{V}\operatorname{div}_\perp\mathbf{b}\right.\\
&\left.\left.\left.-2\mathbf{b}\operatorname{rot}(\mathbf{b}\nabla)\mathbf{V}\right]+\Omega\frac{(\nabla\cdot\mathbf{P}^1)}{mn}\mathbf{b}\right]\right\}=0.
\end{aligned}\tag{9}
$$

The moments $p_\parallel=mn\langle u_\parallel^2\rangle$, $p_\perp=mn\langle u_\perp^2\rangle/2$, $q_\parallel=mn\langle u_\parallel^3\rangle$, $q_\perp=mn\langle u_\parallel$ and $u_\perp^2\rangle/2$ in the left side of (9) are determined by the transverse isotropic part of the distribution function, taken in zeroth order and described by Eq. (7). Correspondingly, the calculation of the divergence of the part of the pressure tensor of order $1/\Omega$ is carried out with the help of (9), and leads to the following expression for its longitudinal component:

$$
\begin{aligned}
(\nabla\cdot\mathbf{P}^1)\mathbf{b}=\frac{mn}{\Omega}&\left\{\left[\mathbf{b}(\mathbf{b}\operatorname{rot}\mathbf{b})+\frac{[\mathbf{b}\times\nabla B]}{B}\right]\nabla q_\perp+\nabla q_\parallel\operatorname{rot}_\perp\mathbf{b}\right.\\
&+(2q_\perp-q_\parallel)\left[\operatorname{div}(\mathbf{b}(\mathbf{b}\operatorname{rot}\mathbf{b}))+\frac{\nabla B}{B}\operatorname{rot}_\perp\mathbf{b}\right]-\nabla p_\perp\operatorname{rot}_\perp\mathbf{V}\\
&+2[\mathbf{b}\times(\mathbf{b}\nabla)\mathbf{V}]\nabla p_\parallel+p_\perp\\
&\times\left[\frac{\nabla B}{B}\operatorname{rot}_\perp\mathbf{V}+\mathbf{b}\operatorname{rot}(\mathbf{b}\nabla)\mathbf{V}+2\operatorname{rot}\mathbf{V}(\mathbf{b}\nabla)\mathbf{b}\right.\\
&\left.+\frac{1}{2}\mathbf{b}\operatorname{rot}\mathbf{V}\operatorname{div}\mathbf{b}-\frac{1}{2}\mathbf{b}\operatorname{rot}\mathbf{b}\operatorname{div}_\perp\mathbf{V}\right]+2p_\parallel\left[2\operatorname{rot}\mathbf{b}(\mathbf{b}\nabla)\mathbf{V}\right.\\
&\left.\left.-(\mathbf{b}\operatorname{rot}\mathbf{b})\mathbf{b}(\mathbf{b}\nabla)\mathbf{V}-\mathbf{b}\operatorname{rot}(\mathbf{b}\nabla)\mathbf{V}+\frac{[\mathbf{b}\times\nabla B]}{B}(\mathbf{b}\nabla)\mathbf{V}\right]\right\}.
\end{aligned}
$$

3. CONSERVATION OF THE MAGNETIC MOMENT

Having the exact solution (9), which describes the dynamics of the collisionless distribution function to first order in $1/\Omega$, it is natural to raise the question of the existence of first integrals in this equation. We attempt to find such integrals X in the form of the simplest polynomial expansion

$$
X=\sum_{i=0}^{n}u_\parallel^iX_i. \tag{10}
$$

The action of the functional (7) on (10) leads to the polynomial

$$
\begin{aligned}
\hat{\Phi}^0[X]=\sum_{i=0}^{n+1}u_\parallel^i&\left\{\mathbf{b}\nabla X_{i-1}+\frac{u_\perp}{2}\frac{\partial X_{i-1}}{\partial u_\perp}\frac{\mathbf{b}\nabla B}{B}\right.\\
&+d_tX_i-iX_i\mathbf{b}(\mathbf{b}\nabla)\mathbf{V}\\
&\left.-\frac{u_\perp}{2}\frac{\partial X_i}{\partial u_\perp}\operatorname{div}_\perp\mathbf{V}+(i+1)X_{i+1}\left(\frac{u_\perp^2}{2}\operatorname{div}\mathbf{b}+\mathbf{b}\frac{\nabla\cdot\mathbf{P}^1}{mn}\right)\right\},
\end{aligned}\tag{11}
$$

where $X_j=0$ at $(j>n)\cup(j<0)$. Successively equating to zero terms with different powers of $u_\parallel$, we obtain a chain of equations for the variable X_i (in the next higher order of expansion in $1/\Omega$). Thus, for the highest mode X_n, we have

$$
\mathbf{b}\nabla X_n+\frac{u_\perp}{2}\frac{\partial X_n}{\partial u_\perp}\frac{\mathbf{b}\nabla B}{B}=0. \tag{12}
$$

The general solution of (12) takes the form

$$
X_n=\psi+y(\mu_0),
$$

where $\psi(\mathbf{r},t)$ is such that $\mathbf{b}\nabla\psi=0$, and y is an arbitrary function of argument $\mu_0=u_\perp^2/B$. Furthermore, considering the terms $\sim u_\parallel^n$, we obtain

$$
\begin{aligned}
\mathbf{b}\nabla X_{n-1}+\frac{u_\perp}{2}\frac{\partial X_{n-1}}{\partial u_\perp}\frac{\mathbf{b}\nabla B}{B}+\frac{dy}{d\mu_0}\mu_0\left(-\frac{d_tB}{B}-\operatorname{div}_\perp\mathbf{V}\right)&\\
+d_t\psi-nX_{n-1}\mathbf{b}(\mathbf{b}\nabla)\mathbf{V}=0.&
\end{aligned}\tag{13}
$$

The term with y in (13) vanishes in accord with (4), while the last nondivergent term in the general case $\mathbf{b}(\mathbf{b}\cdot\nabla)\mathbf{V}\neq0$ is eliminated only in the case $n=0$. Thus, the traditional drift-kinetic equation $\Phi^0[X]=0$ has the simple integrals ψ and μ_0. The role of ψ can be played by any quantity fixed in the plasma, constant along the lines of force of the magnetic field, while $\mu \approx \mu_0$ is proportional to the magnetic moment of the charged particle.

We now consider how the expression for μ is modified in first order in $1/\Omega$. First, in correspondence with (8), oscillations in the phase angle of the contribution appear:

$$\mu_1{}^1+\mu_1{}^2=\frac{\mu_0}{\Omega}\left\{\mathbf{e}_\varphi\cdot\left[u_\perp\frac{\nabla B}{B}+\frac{2}{u_\perp}\left(\left(u_\parallel{}^2+\frac{p_\perp-p_\parallel}{mn}\right)(\mathbf{b}\nabla)\mathbf{b}-\frac{\nabla p_\perp}{mn}+2u_\parallel(\mathbf{b}\nabla)\mathbf{V}\right)\right]+\hat{e}_{\varphi\rho}[\mathbf{V}+u_\parallel\mathbf{b}]\right\}, \quad (14)$$

where $\hat{e}_{\varphi\rho}[\mathbf{V}]=\mathbf{e}_\varphi(\mathbf{e}_\rho\nabla)\mathbf{V}-b$ curl $\mathbf{V}/2$. Second, substituting μ_0 in place of f_0 in (9) and using the same procedure of expansion as in (11), we find

$$\mu_1{}^0=-\frac{\mu_0}{\Omega}\mathbf{b}\,\mathrm{rot}(\mathbf{V}+u_\parallel\mathbf{b}) \quad (15)$$

(the contributions to Eq. (15) that contain the arbitrary function μ_0 are omitted). Equations (10) and (11) generalize the well-known formulas for the magnetic moment (9) to the case of plasma flow with nonzero mass velocity $\mathbf{V}$.

Thus, the quantity $\mu_0+\mu_1^0$ an exact integral of the general DKE (9), and $\mu\approx\mu_0+\mu_1^0+\mu_1^1+\mu_1^2$ is the solution of the kinetic equation (1) to first order in the expansion in $1/\Omega$. The apparent inconsistency because μ, which is an invariant of the motion of a single particle, depends [according to (14)] on the characteristics of the ensemble ($p_{\parallel,\perp}$) is explained by the fact that the pressure forces can be expressed in terms of the field and the force of inertia by means of the equation of motion (3).

We emphasize again that the expansion of Eq. (1) in powers of $1/\Omega$ leads to the result that the adiabatic invariant of the motion μ becomes an exact integral of the resultant equation.

4. LAW OF CONSERVATION OF ENERGY

4.1. Case of motion of individual particles

If a particle moves in a stationary potential electric field ($\mathbf{E}=-\nabla\Phi, \partial_t\Phi=0$), then, as is well known, its total energy $\varepsilon=e\Phi+mv^2/2$ is conserved (the presence of a slowly changing magnetic field leaves this assertion in force). In other words, the quantity ε is an integral of the kinetic equation (1), as is easy to verify by direct substitution. However, substitution of ε in the DKE (7) does not reduce it to an identity. The explanation of this paradox is connected with the necessity of correctly following the logic of the expansion in powers of $1/\Omega$ and is possible only on the basis of Eq. (9). Actually, in contrast with the quantity $\mathbf{F}$, the electric field at $V_\perp\neq0$ contains a quantity $\sim\Omega$, which should be taken into account in the next higher order expansion. Formally, the stationary quantity $\Phi(\mathbf{r})$ satisfies (7) if $V\nabla\Phi+u_\parallel\mathbf{b}\nabla\Phi=0$, whence, in the next higher order, $\nabla\Phi\approx[\mathbf{V}\times\mathbf{B}]/c$ (the factor B/c assures that Φ has the dimensionality of the electric potential. In the stationary case, such a potential Φ is frozen in the plasma and is constant along the lines of force of the magnetic field, i.e., it can serve as the integral ψ described above. Furthermore, Φ should be substituted in terms $\sim1/\Omega$ of Eq. (9), whence we obtain the result that the quantity

$$\varepsilon=e\Phi+(u_\perp{}^2+V^2+2u_\parallel V_\parallel+u_\parallel{}^2)m/2 \quad (16)$$

is an integral of Eq. (9), along with the magnetic moment. Formally, the kinetic energy in (16) is $\sim1/\Omega$ of the potential.

An important feature of the generalized DKE (9), which was obtained as a result of the expansion of the kinetic equation in powers of $1/\Omega$, is the fact that Eq. (6) can be an integral of Eq. (2) even in the case of alternating fields with frequency $\omega\ll\Omega$. For this, it is only necessary to calibrate the scalar potential Φ in corresponding fashion. Actually, let the electric field be given by the general expression $\mathbf{E}=-\nabla\Phi-\partial_t\mathbf{A}/c$, where $\mathbf{A}$ is the vector potential curl $\mathbf{A}=\mathbf{B}$. If the longitudinal electric field is only weakly nonpotential $[\mathbf{b}\cdot\partial_t\mathbf{A}\sim BV(\omega/\Omega)^2]$ then, substituting (16) in (9), we obtain a deviation $\sim1/\Omega^2$ from the next higher terms in (9) ($\sim\Omega mV^2$), if

$$\frac{\partial\Phi}{\partial t}-\frac{1}{c}\mathbf{V}\frac{\partial\mathbf{A}}{\partial t}\approx0. \quad (17)$$

Equation (17) can be regarded as the desired gauge of the potential. Actually, in this case, the scalar potential is determined with accuracy to within an arbitrary function, which moves along with the plasma (i.e., $\partial_t\xi+\mathbf{V}\cdot\nabla\xi=0$). In other words, the substitution $\Phi\to\Phi+\xi$ simultaneously with $\mathbf{A}\to\mathbf{A}+\nabla a$, where $\partial_t a/c=-\xi$, conserves both the electric and the magnetic fields, as well as the gauge (17). It is curious to note that the gauge (17) for the stationary plasma flow ($\partial_t\mathbf{V}=0$) transforms into the Petviashvili-Gordin transformation[10] $\Phi=\mathbf{A}\cdot\mathbf{V}/c$, which locally conserves the magnetic helicity (as $\Omega\to\infty$).

4.2. The case of macroscopic motion of the plasma

In the general case of plasma flow in alternating electric and magnetic fields, the energy of the individual particle is not an integral of the motion. However, it is natural to expect that the total energy of the plasma in the absence of dissipation is conserved, as is the case in magnetohydrodynamics. Nevertheless, up to the present time, a general expression for the change in the energy density, obtained within the framework of drift theory, is still lacking.[11] Such an expression can be obtained on the basis of (9) to first order in $1/\Omega$. Let us first discuss the corresponding derivation for the DKE (7) in zeroth order in $1/\Omega$.

The volume energy density of the thermal motion of the particles is simply $mn\langle u_\perp^2+u_\parallel^2\rangle/2=p_\perp+p_\parallel/2$. Integrating the DKE with weights $u_\perp^2$ and $u_\parallel^2$, and combining the resultant equations, we obtain

$$\partial_t(p_\perp+p_\parallel/2)=-(2p_\perp+p_\parallel/2)\,\mathrm{div}\,\mathbf{V}+(p_\perp-p_\parallel)\mathbf{b}(\mathbf{b}\nabla)\mathbf{V}-\mathrm{div}(\mathbf{b}(q_\perp+q_\parallel/2)).$$

The nondivergent terms on the right-hand side are eliminated by taking into account the energy of the magnetic field and the kinetic energy of the directed motion of the plasma; in sum,

$$\partial_t\left(mn\frac{V^2}{2}+p_\perp+\frac{p_\parallel}{2}+\frac{B^2}{8\pi}\right)=-\mathrm{div}\left\{\mathbf{V}\left(mn\frac{V^2}{2}+2p_\perp+\frac{p_\parallel}{2}+\frac{B^2}{4\pi}\right)+\mathbf{b}\left[q_\perp+\frac{q_\parallel}{2}+\left(p_\parallel-p_\perp-\frac{B^2}{4\pi}\right)\mathbf{V}\mathbf{b}\right]\right\}+\mathbf{V}\left(\mathbf{F}+\frac{1}{4\pi}[\mathbf{B}\times\mathrm{rot}\,\mathbf{B}]\right). \quad (18)$$

This equation has a simple physical meaning. The change in the energy density is connected with the convective energy transport (the first term under the div sign), with the heat flow (the terms with $q_{\perp,\parallel}$), with the convective transport of energy along the lines of force of the magnetic field (with account of the energy of the magnetic field), and also with the work of the electromagnetic forces [the last term in (18)]. If, for example, the flow of current is connected only with the component of the plasma considered, then the last term falls exactly to zero, which is equivalent to a zero electric field in a system of coordinates frozen in the plasma.

Applying the analogous procedure to Eq. (9), we obtain an expression for the change in the energy density with account of the FLR terms:

$$
\begin{aligned}
\partial_t\left(mn\frac{V^2}{2}+p_\perp+\frac{p_\parallel}{2}+\frac{B^2}{8\pi}\right)
&= -\operatorname{div}\Bigg\{\mathbf{V}\left(mn\frac{V^2}{2}+2p_\perp+\frac{p_\parallel}{2}+\frac{B^2}{4\pi}\right)\\
&\quad+\mathbf{b}\left[q_\perp+\frac{q_\parallel}{2}+\left(p_\parallel-p_\perp-\frac{B^2}{4\pi}\right)\mathbf{Vb}\right]\\
&\quad+\frac{1}{\Omega}\Bigg\{\left(2R_\perp+\frac{R_\#}{2}\right)\left[\mathbf{b}\times\frac{\nabla B}{B}\right]\\
&\quad+\mathbf{b}(\mathbf{b}\operatorname{rot}\mathbf{b})\left(2R_\perp-\frac{R_\parallel+R_\#}{2}\right)\\
&\quad+\operatorname{rot}\mathbf{b}\left(R_\#+\frac{R_\parallel}{2}\right)+(q_\parallel+4q_\perp)[\mathbf{b}\times\mathbf{b}\nabla\mathbf{V}]\\
&\quad+q_\perp\mathbf{b}(\mathbf{b}\operatorname{rot}\mathbf{V})+B\left[\nabla\frac{q_\perp}{B}\times\mathbf{V}\right]\\
&\quad+\mathbf{b}(\mathbf{b}\operatorname{rot}\mathbf{b})\frac{p_\parallel-p_\perp}{mn}\left(2p_\perp+\frac{p_\parallel}{2}\right)\\
&+\left[\mathbf{B}\times\left(\nabla\left(\frac{p_\perp-p_\parallel}{mnB}\left(2p_\perp+\frac{p_\parallel}{2}\right)\right)-\frac{\nabla p_\perp}{mnB}\left(2p_\perp+\frac{p_\parallel}{2}\right)\right)\right]\\
&\quad+\frac{B^2}{4\pi}\left[\frac{\mathbf{F}}{mn}\times\mathbf{b}\right]-\Omega\mathbf{P}^1\cdot\mathbf{V}\Bigg\}\Bigg\}\\
&\quad+\mathbf{F}\left(\mathbf{V}-\frac{B}{4\pi mn\Omega}\operatorname{rot}\mathbf{B}\right)+\frac{\mathbf{V}}{4\pi}[\mathbf{B}\times\operatorname{rot}\mathbf{B}]. \qquad (19)
\end{aligned}
$$

Here we have already higher-order moments $R_\parallel=mn\langle u_\parallel^4\rangle$, $R_\#=mn\langle u_\parallel^2u_\perp^2/2\rangle$, $R_\perp=mn\langle u_\perp^4/8\rangle$, calculated according to f_0, while the expression for the pressure tensor $\mathbf{P}_1\sim 1/\Omega$ is obtained with the help of the relation (8).[8]

For a one-component plasma (say, hydrogen), Eq. (10) describes the change in the energy density of the ionic component and the magnetic feld. Taking it into account that, in the case of quasi-neutrality, $\mathbf{F}+\mathbf{F}_e\approx-[\mathbf{B}\times\mathrm{curl}\mathbf{B}]/4\pi$, while $\mathbf{V}_e\approx\mathbf{V}=B$ curl $\mathbf{B}/4\pi mn\Omega$ (see Ref. 8), we find that the last (nondivergent) term in (19) drops out if we mean by pressure (on the left-hand side) the total pressure of the ions and electrons (the inertia of the electrons can be neglected).

Thus, as was to be expected, the total integral of the energy over the entire volume occupied by the plasma is conserved and, with account of FLR corrections to the drift motion of the ions, is given by

$$\partial_t\int d^3r\left(mn\frac{V^2}{2}+p_\perp+\frac{p_\parallel}{2}+\frac{B^2}{8\pi}\right)=0.$$

It is easily seen that the analogous derivation is valid also for the case of a multi-component plasma if the energy density of all the components of the plasma is inserted under the integral.

5. CHANGE OF VARIABLES

The surfaces $\varepsilon=$ const and $\mu=$ const in phase space are essentially the characteristics of the in partial differential Eq. (9). This means that, transforming to ε and μ as new independent variables, we can reduce the number of the latter in (9). As a result, knowledge of the integrals of the drift motion allows us to write down the rather unwieldy generalized DKE (9) in a simple and physically lucid form. Actually, the change of variables $\{t,\mathbf{r},u_\parallel,u_\perp\}\to\{t,\mathbf{r},\varepsilon,\mu\}$ in (9) leads to the "basic" equation for the distribution function

$$\left(\frac{\partial f}{\partial t}\right)_{\varepsilon,\mu}+(\mathbf{V}+\mathbf{V}_\perp^*+\mathbf{b}(u_\parallel+V_\parallel+V_\parallel^*))(\nabla f)_{\varepsilon,\mu}\approx0. \qquad (20)$$

Here $V_\parallel^*=u_\perp^2\mathbf{b}$ curl $\mathbf{b}/2\Omega$, while $\mathbf{V}_\perp^*$ represents the velocity of local ion drift:

$$
\begin{aligned}
\mathbf{V}_\perp^*=\frac{1}{\Omega}\Bigg[\mathbf{b}\times\Bigg\{&\frac{u_\perp^2}{2}\frac{\nabla B}{B}+u_\parallel^2(\mathbf{b}\nabla)\mathbf{b}+2u_\parallel(\mathbf{b}\nabla)\mathbf{V}\\
&-\frac{1}{mn}\left(\nabla p_\perp+\mathbf{B}\nabla\left(\frac{p_\parallel-p_\perp}{B}\mathbf{b}\right)\right)\Bigg\}\Bigg]. \qquad (21)
\end{aligned}
$$

As is easily seen, Eq. (21) describes the gradient of the drift, the centripetal drift, due to curvature of the lines of force of the magnetic field, the drift connected with the change in the direction of the magnetic field with time and finally, the drift induced by the forces of the plasma pressure. Thus, Eq. (20) has a simple physical meaning. If the mass velocity in the phase microvolume does not compensate for the local drift of the particle, the resulting phase current produces a change in the particle density in the given microvolume [the quantities $u_{\parallel,\perp}$ in (20) should of course be expressed in terms of ε and μ with the help of (15), (16)].

If the conditions formulated in Sec. 4.1 are not satisfied, then the energy (16) is not conserved, and we must add to the left side of (20) the phase current $(\partial f/\partial\varepsilon)\dot{\varepsilon}$ connected with the change in the energy

$$
\begin{aligned}
\dot{\varepsilon}=e(\partial_t\Phi-\mathbf{V}\partial_t\mathbf{A}/c)&-eu_\parallel\mathbf{b}\partial_t\mathbf{A}/c\\
&+mu_\parallel^2\mathbf{V}_1(\mathbf{b}\nabla)\mathbf{b}+2mu_\parallel\mathbf{V}_1(\mathbf{b}\nabla)\mathbf{V}_1\\
&-\frac{mu_\parallel}{\Omega}\mathbf{V}\operatorname{rot}_\perp\frac{\mathbf{F}}{mn}+\frac{mu_\perp^2}{2}\left(\mathbf{b}\operatorname{rot}\frac{[\mathbf{V}_1\times\mathbf{B}]}{B}+\frac{\mathbf{V}_1\nabla B}{B}\right)\\
&-\frac{1}{n}(\mathbf{V}_1\cdot\nabla\mathbf{P}+\mathbf{V}_\Phi\cdot\nabla\mathbf{P}_1). \qquad (22)
\end{aligned}
$$

Here the mass velocity $\mathbf{V}$ is represented in the form of the sum

$$\mathbf{V}=\mathbf{V}_\Phi+V_\parallel\mathbf{b}+\mathbf{V}_1, \qquad (23)$$

in which the velocity components $\mathbf{V}_\Phi=c[\mathbf{b}\times\nabla\Phi]/B$ and $\mathbf{V}_1=[\mathbf{b}\times(\partial_t\mathbf{A}+cF/en]/B$ associated with the potential and vortical parts, respectively, of the electric field are introduced explicitly. The unity subscript indicates smallness $\sim1/\Omega$. As is easily seen, under the assumption of Sec. 4.1, the quantity $\dot{\varepsilon}$ is found to be of order $\sim(\omega^3/\Omega^2)\varepsilon$ and should be omitted in the considered order of expansion. In the opposite case, the corresponding terms should be taken into account in (20). This, nevertheless, does not complicate the considered equation very much. Thus, the use of the total Eq. (9) turns out to be necessary only in the case in which the partition of (23) is not possible, i.e., $|\mathbf{V}_\Phi| \leqslant |\mathbf{V}_1|$ (for example, in the absence of an electric field).

6. CONCLUSION

The simple consequences of the generalized drift kinetic Eq. (9) that are considered in the present work extend the usual representations of the drift motion of charged particles in a strong magnetic field to include the case of a finite Larmor radius in an abritrary three-dimensional geometry. Use of traditional "hydrodynamic" variables allows us to proceed without *a priori* ordering of the terms of the kinetic equation. The presence of the integrals introduced in the present work in the generalized drift-kinetic equation have made it possible to simplify greatly the considered equation and to describe it in a physically lucid form.

[1)] The advantage of repeating here the generalized DKE is the correction of a large number of typographical errors in the formulas of Ref. 8.

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