Diffusion of "socially active" Brownian particles

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The Brownian diffusion of particles which exhibit a "neighbor-avoidance" tendency in addition to the ordinary random walk is analyzed. The avoidance is modeled by an additional external force $\propto \nabla(P^{-\mu})$ in the stochastic Langevin equation, where $P(t,x)$ is the distribution function formed as a result of the random walk, and μ is an arbitrary exponent. This addition is IR-specific, i.e., alters the infrared asymptotic behavior at long times and at large distances, only if $\mu > 0$. For values of μ in the interval $0 < \mu < 2/d$ (where d is the dimensionality of the space), the equation for the IR asymptotic behavior of P has a normalizable scaling solution $P_{\text{asymp}}(t,x) = t^{-\alpha}\varphi(x^2,t^{-\beta})$, with exponents $\beta = 2/(2-d\mu)$ and $\alpha = d\beta/2$. This solution corresponds to a "dispersal law" $\langle x^2\rangle \sim t^\beta$. The scaling function φ is also determined unambiguously. Such a solution is possible only if the coupling constant has the correct sign, i.e., the sign corresponding to a tendency toward neighbor avoidance, rather than to a tendency toward approaching neighbors.

Ordinary Brownian motion is modeled by the very simple stochastic Langevin equation

$$\dot{x}_i(t)=\eta_i(t),\quad \langle\eta_i(t)\eta_k(t')\rangle=2\nu\delta_{ik}\delta(t-t'), \tag{1}$$

where x_i are the coordinates of the particle undergoing the random walk, η_i is a Gaussian random force with $\langle\eta_i\rangle = 0$ and with the correlation function indicated in (1), and the constant ν is the diffusion coefficient. A generalization of (1) is the problem

$$\dot{x}_i(t)=u_i(t,\ x(t))+\eta_i(t) \tag{2}$$

with the same random force η as in (1) and with some given function $u_i(t,x)$. From Eq. (2) follows the well-known Fokker-Planck equation for the distribution function $P(t,x)$ of the coordinates x at the time t of a particle undergoing a random walk:

$$\{\partial_t+\partial_i[u_i(t,\ x)-\nu\partial_i]\}P(t,\ x)=0. \tag{3}$$

This equation is customarily supplemented with the initial condition $P(0,x) = \delta(x)$, which corresponds to the case in which all the particles start from the origin of coordinates at the time $t = 0$. From Eq. (3) and the initial data we find the normalization condition

$$\int dxP(t,\ x)=1,\quad \forall t\geqslant 0. \tag{4}$$

The solution of Eq. (3) for the very simple problem (1), in a d-dimensional space, is well known:

$$P_0(t,x)=(4\pi\nu t)^{-d/2}\exp\left(-\frac{x^2}{4\nu t}\right). \tag{5}$$

This solution is in a scaling form and corresponds to the "dispersal law" $\langle x^2\rangle \propto t$. The problem (3), which is nontrivial, has been studied in detail in two versions: the problem of a "random random walk" (RRW; a random walk in a random medium) and the problem of a "true self-avoiding random walk" (TSAW). We are using the conventional terminology. The TSAW model was proposed in Ref. 2 (see also the bibliography in Ref. 3). In the RRW case the vector field u_i in (3) is assumed to be a Gaussian random quantity with a given correlation function, and an average is taken over it.[1] In the latter case, u_i is assumed to be the gradient of a potential which is specified as a (point) δ-function on the path traced out by the particle (in this case there is perfect self-avoidance) or as a δ-function which has been smeared out (this is the case of a tendency toward self-avoidance). In either case, the problem can be reformulated as a quantum field model, and then the renormalization-group method can be used to study the IR asymptotic behavior, as in the theory of critical behavior.[4] As a result, a scaling is always predicted. The consequence $\langle x^2\rangle \propto t^\beta$ of this scaling, with a nontrivial critical exponent β, is calculated in the form of some ε expansion (although the legitimacy of this derivation is questionable in the case of a TSAW with an exact δ-function[3]).

In a real situation, the stochastic behavior stems from collisions with neighbors, and $P(t,x)$ describes the distribution of the cloud of diffusing particles at time t if these particles were all at the origin of coordinates at $t = 0$. A common thread running through all these problems is "autonomy": The behavior of an individual particle in the course of its random walk is totally unrelated to the presence or absence of neighbors (in the most complicated TSAW case, the particle "remembers" only its own path).

It is therefore interesting to look at a problem without autonomy, i.e., a problem in which a particle "senses" the neighbors around itself. For example, the particle may exhibit a tendency to approach its neighbors or to move away from them. This formulation of the problem is clearly of interest from many points of view, extending all the way to sociology.

In this paper we propose and study a very simple model of this sort: Eq. (3) with $u_i = c\partial_i(P^{-\mu})$, where c and μ are constants, and P is the same distribution function as is formed in the course of a random walk, i.e., the solution of (3). Equation (3) with such a u_i can be rewritten as

$$[\partial_t-\nu\partial^2]P=g\partial^2P^{1-\mu}, \tag{6}$$

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where $\partial^2 \equiv \partial_i \partial_i$ is the Laplacian, and $g = c\mu/(1-\mu)$ is a coupling constant.

We are interested in the IR asymptotic behavior (i.e., large values of t and x) of the solution of Eq. (6). In a normal situation we would have $P \to 0$ in the IR limit (spreading of the cloud). In this regime, with $\mu > 0$, we can ignore the free contribution $\partial^2 P$ in Eq. (6) in comparison with the nonlinearity. In the case $\mu < 0$, we can do the opposite. In the latter case the asymptotic behavior is obviously free. We therefore assume $\mu > 0$ in the discussion below. We then find the following simplified equation for the IR asymptotic expression P_{asymp} of the exact function P:

$$\partial_t P_{\text{asymp}} = g \partial^2 P_{\text{asymp}}^{1-\mu}, \quad \mu > 0. \tag{7}$$

A value $g > 0$ here corresponds to repulsion of particles, and $g < 0$ to an attraction. The same conclusions regarding the role of the nonlinearity can be reached through a conventional analysis of the canonical dimensionalities, precisely as in the theory of critical behavior.

It is natural to seek a solution of Eq. (7) in a scaling form similar to (5):

$$P_{\text{asymp}}(t,x) = t^{-\alpha} \varphi(z), \quad z \equiv x^2 t^{-\beta}. \tag{8}$$

This expression corresponds to a dispersal law $\langle x^2 \rangle \propto t^\beta$. Substituting (8) into (7), we find the equation

$$-t^{-1-\alpha}\left(\alpha + \beta z \frac{d}{dz}\right)\varphi(z) = g t^{-\alpha(1-\mu)-\beta}\left[2d\frac{d}{dz} + 4z\frac{d^2}{dz^2}\right]\varphi^{1-\mu}(z), \tag{9}$$

from which we in turn find the relationship $1 + \alpha = \alpha(1-\mu) + \beta$. A second relationship between the indices $\alpha = d\beta/2$ can be found by substituting (8) into the normalization condition (4). To replace the exact function P in (4) by its asymptotic expression P_{asymp} is a legitimate step if at large values of t the overwhelming majority of the particles are at large values of x, where the asymptotic expression (8) prevails. For the free solution (5), this condition always holds. Mutual repulsion of the particles should accelerate the spreading of the cloud even more, so in this case ($g > 0$) the substitution $P \to P_{\text{asymp}}$ in the normalization condition can be judged legitimate. From the two relationships given above, we can determine the exponents α and β unambiguously:

$$\beta = \frac{2}{2 - d\mu}, \quad \alpha = \frac{d\beta}{2}. \tag{10}$$

The natural requirement $\beta > 0$ imposes an upper limit on μ:

$$\mu < 2/d. \tag{11}$$

We will assume below that this condition holds.

The discussion above is of course valid only if the equation for the scaling function which follows from (9) has a solution that falls off rapidly enough in the limit $z \to \infty$ that the normalization integral in (4) converges at infinity. If there is no such solution, the derivation of Eqs. (10) is incorrect, since that derivation leaned heavily on the normalization condition (4) for the function (8).

To test the internal self-consistency of the solution with the exponents (10), we consider the equation for φ which follows from (9):

$$\left(\alpha + \beta z \frac{d}{dz}\right)\varphi(z) + g\left[2d\frac{d}{dz} + 4z\frac{d^2}{dz^2}\right]\varphi^{1-\mu}(z) = 0. \tag{12}$$

Using the second equality in (10), we can rewrite this equation as

$$\left(\frac{d}{2} + z\frac{d}{dz}\right)\left[\beta\varphi(z) + 4g\frac{d}{dz}\varphi^{1-\mu}(z)\right] = 0.$$

Hence we immediately find the first integral

$$\beta\varphi(z) + 4g\frac{d}{dz}\varphi^{1-\mu}(z) = Cz^{-d/2}, \tag{13}$$

where C is an arbitrary constant.

It is not difficult to see that Eq. (13) cannot have a normalizable solution for φ for any $C \neq 0$, so $C = 0$ is the only possible value. The reason is that for $C \neq 0$ the right side of (13) is a multiple of the power $z^{-d/2} \propto |x|^{-d}$, which corresponds to a logarithmic divergence of the normalization integral in (4) at infinity. By virtue of the condition which we have adopted (normalizability), the function φ on the left side of (13) must fall off more rapidly, so the contribution proportional to $z^{-d/2}$ can (and must, in the case $C \neq 0$) be present in the derivative $d\varphi^{1-\mu}(z)/dz$. However, the relation $d\varphi^{1-\mu}(z)/dz \propto z^{-d/2}$ implies $\varphi \propto z^{(2-d)/2(1-\mu)}$, and under inequality (11) this relation means an even slower (slower than $z^{-d/2}$) decay of φ at infinity. For value $C \neq 0$, Eq. (13) thus cannot have normalizable solutions for φ. We are thus obliged to set $C = 0$. Equation (13) with $C = 0$ can be integrated easily:

$$\varphi(z) = \left[\frac{\beta\mu}{4g(1-\mu)}(z + C_1)\right]^{-1/\mu}, \tag{14}$$

where C_1 is a new integration constant. If inequality (11) holds (and we are assuming that it does), the function (14) falls off more rapidly than $z^{-d/2}$ at infinity, so the solution (14) with $C_1 > 0$ is normalizable. The constant C_1 is found unambiguously from the normalization condition:

$$C_1 = \pi^{\alpha\mu}\left[\frac{4g(1-\mu)}{\beta\mu}\right]^\beta \left[\Gamma\left(\frac{1}{\mu\beta}\right) \Big/ \Gamma\left(\frac{1}{\mu}\right)\right]^{\beta\mu}. \tag{15}$$

The solution in (14) is meaningful only for $g > 0$, i.e., only if there is mutual repulsion between particles.

We thus see that, with $g > 0$ and $0 < \mu < 2/d$, Eq. (12) for the asymptotic behavior of the distribution function P has a unique normalizable solution of the scaling type, with the exponents in (10) and the scaling function in (14). By virtue of this uniqueness we can assume that this solution is the IR asymptotic expression for the exact distribution function which we have been seeking. The sign $g > 0$ corresponds to repulsion between particles, and $\mu > 0$ corresponds to the condition under which the interaction is IR specific (nonlinearity). The equality (11) is limited from above by the "repulsion strength." When this boundary is reached, the function in (14) becomes unrenormalizable. The solution found here corresponds to the dispersal law $\langle x^2 \rangle \propto t^\beta$ with $\beta > 1$ from (10). In other words, the dispersal becomes faster (for free particles we would have $\beta = 1$), as it naturally would in the case of mutual repulsion. The question of the asymptotic behavior of P in the case of strong repulsion ($\mu \geqslant 2/d$) or in the case of IR-specific attraction ($g<0$, $\mu>0$) remains open, since a solution cannot be constructed for those cases by the simple method presented above.

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