Theory of quasiequilibrium effects in a system of magnons excited by incoherent pumping

Yu. D. Kalafati

Institute of Radio Engineering and Electronics, USSR Academy of Sciences;

V. L. Safonov

Kurchatov Institute of Atomic Energy

(Submitted 24 December 1990)
Zh. Eksp. Teor. Fiz. **100**, 1511–1521 (November 1991)

A new type of spin-wave instability is considered under conditions of incoherent external pumping. This instability is manifested by an abrupt increase of the number of magnons on the bottom of a spectral band when a certain threshold power is reached. Equations for the threshold are obtained in approximations in which the temperature difference between the magnons and the heat bath are assumed to be negligibly small as well as noticeable. Variants of above-threshold states in which Bose condensation of magnons is possible are discussed. It is proposed to detect the Bose condensation by conversion of wide-band pumping into coherent radiation at frequencies equal to, or multiples of, the frequency of the bottom of the spin-wave spectral band. The intensity of the coherent radiation is calculated.

INTRODUCTION

Action of intense coherent alternating magnetic fields on a system of spin waves in a magnet is known to lead to various types of parametric instability, accompanied by an exponential increase of the number of magnons in frequency and momentum-space regions determined by the synchronism conditions (see, e.g., Ref. 1). This results in strong disequilibrium states describable by using the dynamic and kinetic[1] or thermodynamic[2] approaches of the theory of paramagnetic resonance in distributed media.

As the spectra of initially coherent microwave-field sources broaden, the threshold pump amplitudes at which spin-wave parametric instabilities are produced increase quite rapidly and become possibly inaccessible to experiment. This raises the natural question of the behavior of a strong-disequilibrium magnon system excited predominantly by an incoherent pump source. In the present paper (we published a brief report in Ref. 3) we point out that in a noise-irradiated ordered spin system an instability can result from a change of the distribution function of all the magnons and from accumulation of magnons on the bottom of the spectral band. In the ideal-gas approximation this would mean theoretically the onset of Bose–Einstein condensation in the system. Allowance for nonlinear magnon–magnon interactions (which play an important role in magnets) can either modify the magnon-condensation conditions or produce other cooperative states in the system.

It should be noted that the interest in the possibility of observing Bose condensation under conditions of incoherent energy pumping from the outside dates back to the early Sixties (see Refs. 4 and 5). Much was hoped from exciton and biexciton systems in semiconductors. However, notwithstanding the many efforts in this field, no direct confirmation of the theoretical predictions has been obtained. Theoretical research along these lines continues.[6,7] The difficulties encountered by experimental investigations of Bose condensation in a system of nonequilibrium excitons of a semiconductor (short lifetime of the quasiparticles and the peculiarities of their interactions at short distances) necessitate a search for other physical objects to observe this phenomenon. We assume that magnons, which are readily excited by microwave fields, may serve as the object of his research.

From the theoretical standpoint, the problem of instability in a system of incoherently pumped magnons incorporates, besides features unique to the magnetic system, a number of features not previously encountered. Thus, in the model used in Refs. 5–7, the main nonlinearity was taken to be due to three-particle decay processes (an initial boson is converted into a lower-energy boson and a phonon). External pumping changes in this case only the effective chemical potential of the Bose system. As to magnons, in view of the nonlinear character of the exchange interaction, it is apparently impossible to confine oneself to the above three-particle interaction processes, since magnon–magnon exchange scattering makes a substantial (and sometimes the principal) contribution to the kinetics of the entire system.[1,2] The study of instabilities in a system of magnons excited by an external incoherent pump is therefore of independent interest.

To estimate the threshold of the instability of interest to us we start from a kinetic equation for the magnon occupation numbers $n_{\mathbf{k}}$ in the form

$$\frac{d}{dt} n_{\mathbf{k}} = I^{(4)}\{n_{\mathbf{k}}\} + f_{\mathbf{k}} + r_{\mathbf{k}}, \tag{1}$$

where $f_{\mathbf{k}}$ is the arrival term describing the probability of creation of a magnon by external action; $r_{\mathbf{k}}$ is the relaxation term responsible for the interaction of the magnons with the heat bath (magnons from other modes, photons, etc.); $I^{(4)}(n_{\mathbf{k}})$ is the four-magnon collision integral:

$$I^{(4)}\{n_{\mathbf{k}}\} = (2\pi)^{-5} \int d^3k_1\, d^3k_2\, d^3k_3 \,|\,\Phi(\mathbf{k}, \mathbf{k}_1; \mathbf{k}_2, \mathbf{k}_3)\,|^2$$
$$\times [(n_k+1)(n_1+1)n_2 n_3 - n_k n_1 (n_2+1)(n_3+1)]$$
$$\times \delta(\varepsilon_k + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\Delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3),$$

in which $\Phi(\mathbf{k}, \mathbf{k}_1;\mathbf{k}_2, \mathbf{k}_3)$ is the scattering amplitude and $\varepsilon_\mathbf{k}$ is the magnon energy (we use here and below a system of units with $\hbar = k_B = 1$). We assume that three-magnon collisions are forbidden by the energy and momentum conservation laws, and the contribution of three-magnon anharmonicities is taken into account in the effective four-magnon vertex. In addition, we assume that the characteristic time scale τ_m of the magnon–magnon collisions is much shorter than the characteristic time $\tau = \min\{\tau_r,\tau_f\}$ where τ_r is the time of relaxation of a number of magnons into the lattice and τ_f is the characteristic magnon-excitation time. In other words, it is assumed that magnon–magnon scattering is the principal process in the system, while interaction of magnons with the heat bath and with the pump field are considerably less probable to the extent of the smallness of the parameter $\tau_m/\tau \ll 1$. The solution of (1) can then be represented in the zeroth approximation in τ_m/τ by a quasiequilibrium distribution function:

$$n_\mathbf{k}=\{\exp[(\varepsilon_\mathbf{k}-\mu)/T]-1\}^{-1}, \tag{2}$$

where μ is the chemical potential and T is the temperature, as defined by the two integrals

$$N=\tilde{\mathscr{V}}\int d^3k\, n_\mathbf{k}, \quad E=\tilde{\mathscr{V}}\int d^3k\, \varepsilon_\mathbf{k} n_\mathbf{k}. \tag{3}$$

Here N is the total number of magnons, E is the energy of the magnon system, $\tilde{V} \equiv V/(2\pi)^3$, and V is the volume of the crystal.

With interaction between the magnon and the heat bath taken into account, the magnon system tends during a time $\sim\tau_r$ to an equilibrium state with $\mu = 0$ and $T = T_B$, where T_B is the temperature of the heat bath. Application of an external magnon-pumping source violates the thermodynamic equilibrium with the heat bath, but an energy-flux equilibrium is established between the excitation and the relaxation in the magnon system. Accurate to corrections $\sim\tau_m/\tau$, a quasiequilibrium magnon distribution function (2) is obtained, in which the effective chemical potential and the effective temperature are determined by the external conditions. Note that certain effects connected with quasiequilibrium in magnon systems have already been considered in Refs. 8 and 9.

It is easily seen that the growth of $n_\mathbf{k}$ in (2) occurs when μ and (or) T increases. A singularity in μ exists then at the point where the chemical potential is tangent to the bottom of the magnon band, $\mu \to \min \varepsilon_\mathbf{k}$, where the occupation number becomes infinite (cf. the case of equilibrium Bose condensation). From there on, the magnon system can no longer be described in the framework of Eqs. (1) and (2). We need a detailed analysis of the structure of the magnon–magnon interactions.

The present paper consists of two principal sections. In the first we analyze the subcritical region, both in an approximation in which the effective magnon temperature is equal to the heat-bath temperature T_B and when T becomes different from T_B. The second section is devoted to an analysis of the possible above-threshold states of a Bose system. The article ends with a discussion of a method for observing the predicted Bose-condensation effect through coherent radiation.

1. SUBCRITICAL REGION

We consider first the state of the magnon system prior to the critical point.

We introduce the notation

$$F_N=\tilde{\mathscr{V}}\int d^3k\, f_\mathbf{k}, \quad R_N=\tilde{\mathscr{V}}\int d^3k\, r_\mathbf{k}$$
$$F_E=\tilde{\mathscr{V}}\int d^3k\, \varepsilon_\mathbf{k} f_\mathbf{k}, \quad R_E=\tilde{\mathscr{V}}\int d^3k\, \varepsilon_\mathbf{k} r_\mathbf{k}. \tag{4}$$

From Eq. (1) with allowance for (3) and (4) we easily obtain the balance equations for the total number of particles and the energy of the magnon system:

$$dN/dt=F_N+R_N, \tag{5a}$$

$$dE/dt=F_E+R_E, \tag{5b}$$

which determine implicitly the evolutions of μ and T.

A rigorous analysis of the set of equations (5) is quite complicated and depends substantially on the specific forms of the amplitudes of the interaction between the magnons and the heat bath in the relaxation terms R_N and R_E. For simplicity and to preserve the general character, we confine ourselves hereafter to the τ approximation:

$$R_N=-\tau_N^{-1}(\mu,T)[N(\mu,T)-N(0,T)],$$
$$R_E=-\tau_E^{-1}(\mu,T)[E(\mu,T)-E(\tilde{\mu},T_B)], \tag{6}$$

where the parameter τ_N^{-1} is indicative of the rate of change of the total number of magnons, and τ_E^{-1} the rate of change of the Bose-system energy. The deviations of the number of particles and of the energy in (5) are reckoned from the quasiequilibrium state of a strongly excited Bose system, and the formal parameter $\tilde{\mu}$ is determined from the condition

$$N(\tilde{\mu},T_B)=N(\mu,T). \tag{7}$$

The cause of this equality is that the magnon temperature is established in a time τ_m much shorter than τ_N. It is easy to verify that in the absence of an external pump Eqs. (5)–(7) lead to the solution $\mu = 0$, $T = T_B$. Using (6) and (7) we can in principle analyze the states of a Bose system with appreciable deviations of the particle number and of the temperature from the initial values $N(0,T_B)$ and T_B.

Stationary solutions for T and μ, and their dependences on the pump intensity, are obtained from the integral equations (5)–(7) under the condition $d/dt = 0$. We investigate below these solutions.

1.1. The case $T=T_B$

In the simplest case (at $\tau_E \ll \tau_N$) the Bose-system temperature variation up to the instability threshold can be neglected. Equation (5a) suffices then to calculate the change of the chemical potential. The inequality above holds in magnets in which the predominant interactions between the spin waves and the heat bath are those in which the total number of magnons in the investigated spectrum branch is preserved. Such a process is, for example, magnon decay into a magnon and phonon (in YIG ferromagnet or in antiferromagnets $FeBO_3$ and $CsMnCl_3$).

The simplest method of exciting spin waves in magnets is via decay of a microwave-field quantum (Ω) into a pair of magnons of half the frequency and with equal but opposite wave-vector directions ($\varepsilon_{\mathbf{k}}+\varepsilon_{-\mathbf{k}}$). In this case the arrival term in (1) is given by

$$f_{\mathbf{k}}(\mu,T)=2\pi\int d\Omega\,\phi(\Omega)h^2|V_{\mathbf{k}}|^2[2n_{\mathbf{k}}(\mu,T)+1]\delta(\Omega-2\varepsilon_{\mathbf{k}}),\quad(8)$$

or, if $\Omega\geqslant 2\varepsilon_0$

$$f_{\mathbf{k}}(\mu,T)=2\pi\phi(2\varepsilon_{\mathbf{k}})h^2|V_{\mathbf{k}}|^2[2n_{\mathbf{k}}(\mu,T)+1],$$

where h is the characteristic oscillation amplitude of the alternating magnetic microwave field, $\phi(\Omega)$ is the noise-pump-source line shape normalized to unity, and $V_{\mathbf{k}}$ is the coefficient of magnon coupling with the external field.

In weakly anisotropic antiferromagnets, the low-activation magnons, whose spectrum is given by

$$\varepsilon_{\mathbf{k}}=[\varepsilon_0^2+(sk)^2]^{1/2},\quad(9)$$

are excited by the alternating field practically isotropically.[10] The coupling coefficient is then[11]

$$V_{\mathbf{k}}=g^2(H+{}^1/_2H_D)\varepsilon_{\mathbf{k}}^{-1}.\quad(10)$$

here H is the stationary external magnetic field, H_D is the Dzyaloshinskiĭ field, and g is the gyromagnetic ratio.

In ferromagnets, on the other hand, the situation is entirely different. The decisive role in magnon excitation is played here by the dipole–dipole energy, which leads to an isotropic coupling coefficient of the form (see, e.g., Ref. 1)

$$V_{\mathbf{k}}=2\pi g^2M\varepsilon_{\mathbf{k}}^{-1}\sin^2\theta_{\mathbf{k}}\exp(2i\varphi_{\mathbf{k}}),\quad(11)$$

where M is the magnetization, $\theta_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are the polar and azimuthal angles of the vector **k** in a spherical coordinate system whose axis is oriented along **M**. The magnon spectrum is then (for $\varepsilon_0\gg 2\pi gM$)

$$\varepsilon_{\mathbf{k}}\approx\varepsilon_0+\omega_{ex}(ak)^2+2\pi gM\sin^2\theta_{\mathbf{k}}.\quad(12)$$

Here ω_{ex} is the exchange frequency and a is the lattice constant.

It is easy to discern from (11) and (12) a specific feature of the effect of Bose condensation of magnons in a ferromagnet, viz., the external field excites mainly magnons with wave vectors perpendicular to the magnetization, $\mathbf{k}\perp\mathbf{M}$, and the quasiparticle accumulation takes place at points with $\mathbf{k}\|\mathbf{M}$ $(k\to 0)$.

Thus, Eq. (5a) which defines implicitly the effective chemical potential as a function of the external pump, takes in the case of interest to us the form

$$F_N(\mu,T_B)=\tau_N^{-1}[N(\mu,T_B)-N(0,T_B)].\quad(13)$$

To be specific, we consider next the action of an incoherent pump having an equal alternating-magnetic-field amplitude h in the frequency band from $\Omega_{\rm ext}$ to $\Omega_{\rm ext}+\Delta\Omega$ $(\Omega_{\rm ext}>2\varepsilon_0)$. We have then

$$\phi(\Omega)=\begin{cases}\Delta\Omega^{-1}, & \Omega\in[\Omega_{ext},\Omega_{ext}+\Delta\Omega],\\ 0, & \Omega\notin[\Omega_{ext},\Omega_{ext}+\Delta\Omega].\end{cases}\quad(14)$$

For an antiferromagnet, the functions of the total number of magnons and of the integral-pumping, with allowance for relations (8)–(10) and (14), can be written in the form

$$N(\mu,T)=\frac{\mathcal{V}T^3}{2\pi^2s^3}I_{1a}\left(\frac{\varepsilon_0}{T},\frac{\mu}{T}\right),$$

$$I_{1a}(x,y)\equiv\int_x^\infty dt\,t(t^2-x^2)^{1/2}\frac{1}{e^{t-y}-1}$$

and

$$F_N(\mu,T)\approx\frac{\mathcal{V}T}{\pi\Delta\Omega}\left[g^2h\left(H+\frac{H_D}{2}\right)\right]^2I_{2a}\left(\frac{\varepsilon_0}{T},\frac{\Omega_{ext}}{2T},\frac{\Delta\Omega}{2T}\right),$$

$$I_{2a}(x,y,z)\equiv\int_y^{y+z}dt\,(t^2-x^2)^{1/2}t^{-1}\left(\frac{2}{e^t-1}+1\right).$$

The last approximate equality is valid under the condition $\Omega_{\rm ext}\gg\mu$, when $f(\mu,T)\approx f(0,T)$. Equation (13), which determines implicitly the effective chemical potential of the system, can thus be represented in the form

$$(gh)^2=\frac{\Delta\Omega\tau_N^{-1}T_B^2}{2\pi g^2(H+H_D/2)^2}\frac{I_{1a}(\varepsilon_0/T_B,\mu/T_B)-I_{1a}(\varepsilon_0/T_B,0)}{I_{2a}(\varepsilon_0/T_B,\Omega_{ext}/2T_B,\Delta\Omega/2T_B)}.\quad(15)$$

Expression (15) is noticeably simplified at low temperatures, $T_B\ll\varepsilon_0$ and under the condition $\Omega_{\rm ext}\gg\Delta\Omega,\varepsilon_0$:

$$(gh)^2\approx\frac{\tau_N^{-1}(\varepsilon_0T_B)^{3/2}}{(2\pi)^{1/2}g^2(H+H_D/2)^2}\times\left\{\sum_{n=1}^\infty\frac{\exp[-n(\varepsilon_0-\mu)T_B]}{n^{3/2}}-\exp(-\varepsilon_0/T_B)\right\}.\quad(16)$$

It can be shown using (16) that initially, at $\mu\ll T_B$, the effective chemical potential increases quadratically, $\mu\propto h^2$, and then acquires at $\varepsilon_0\gg\mu\gg T_B$ the logarithmic dependence $\mu\propto\ln h$. As already mentioned, the Bose system becomes unstable when the point $\mu=\varepsilon_0$ is reached. From (16) we obtain for the critical amplitude $h_*^{(a)}$ the expression

$$gh_*^{(a)}\approx\left[\zeta\left(\frac{3}{2}\right)\right]^{1/2}\frac{\tau_N^{-1/2}(\varepsilon_0T_B)^{3/4}}{(2\pi)^{1/4}g(H+H_D/2)},\quad(17)$$

where $\zeta(3/2)\approx 2.612$ is a Riemann zeta function.

The estimate (17) yields for the antiferromagnet $FeBO_3$ ($H_D\approx 100$ kOe) at $\varepsilon_0=2\pi\cdot 10$ GHz

$$gh_*^{(a)}[\text{Oe}]\approx 27(\tau_N^{-1}[\text{kHz}])^{1/2}(T_B[\text{K}])^{3/4},$$

which attests to the possibility of observing the effect in the temperature region $T_B\leqslant 1$ K.

For a ferromagnet we have

$$N(\mu,T)=\frac{\mathcal{V}}{(2\pi)^2}I_{1f}(T,\mu),$$

$$I_{1f}(T,\mu)\equiv\int_0^\infty dk\int_{-1}^1 d\cos\theta_{\mathbf{k}}\,k^2\times\left\{\frac{1}{\exp[(\varepsilon_{\mathbf{k}}-\mu)/T]-1}-\frac{1}{\exp(\varepsilon_{\mathbf{k}}/T)-1}\right\}$$

and

$$F_N(\mu,T)\approx\frac{\mathcal{V}}{2\pi\Delta\Omega}[g^2h\cdot 2\pi M]^2I_{2f}(T,\Omega_{\rm ext},\Delta\Omega),$$

$$I_{2f}(T,\Omega,\Delta\Omega)\equiv\int dk\int d\cos\theta_{\mathbf{k}}k^2(T/\varepsilon_{\mathbf{k}})^2\left\{\frac{2}{\exp(\varepsilon_{\mathbf{k}}/T)-1}+1\right\},$$

$$\Omega/2\leqslant\varepsilon_{\mathbf{k}}\leqslant(\Omega+\Delta\Omega)/2.$$

As a result, Eq. (13) takes the form

$$(gh)^2=\frac{\Delta\Omega\tau_N^{-1}T_B^2}{2\pi(2\pi gM)^2}\frac{I_{1f}(T_B,\mu)-I_{1f}(T_B,0)}{I_{2f}(T_B,\Omega_{ext},\Delta\Omega)}.\quad(18)$$

In the low-temperature limit $T_B \ll \varepsilon_0$ and at $\Omega_{ext} \gg \varepsilon_0$, $\Delta\Omega$ we obtain from (18), neglecting in (12) the anisotropy of the magnon spectrum ($\varepsilon_0 \gg 2\pi gM$).

$$(gh)^2 \approx \frac{\tau_N^{-1}(\Omega_{ext}T_B)^{3/2}}{(2\pi)^{1/2}(4\pi gM)^2} \times \left\{\sum_{n=1}^{\infty} \frac{\exp[-n(\varepsilon_0-\mu)/T_B]}{n^{3/2}} - \exp\left(-\frac{\varepsilon_0}{T_B}\right)\right\}. \tag{19}$$

From (19), finally, follows an expression for the critical amplitude of the noise field:

$$gh_*^{(f)} \approx \left[\zeta\left(\frac{3}{2}\right)\right]^{1/2} \frac{\tau_N^{-1/2}(\Omega_{ext}T_B)^{3/4}}{(2\pi)^{1/4} 4\pi gM}. \tag{20}$$

For YIG ($4\pi M = 1.75$ kOe) at $\Omega_{ext} = 2\pi \cdot 100$ GHz, the estimate (20) yields

$$gh_*^{(f)}[\text{Oe}] \approx 27(\tau_N^{-1}[\text{kHz}])^{1/2}(T_B[\text{K}])^{3/4}.$$

1.2. The case $T \neq T_B$

If the relaxation is slower, the difference between the magnon temperature T and the heat-bath temperature T_B can no longer be neglected. To determine μ and T in this case it is necessary to solve the complete set of equations (5)–(7). It is clear from physical considerations that the magnon system overheats, i.e., $T > T_B$. It follows then from (7) that $\mu < \tilde{\mu}$. It can be readily concluded hence that the changes of μ and T are described by the system (6) correctly only under the condition

$$\tilde{\mu}(\mu, T, T_B) < \varepsilon_0, \tag{21}$$

prior to the condensation of the magnons with $k = 0$. Therefore rigorous calculation of the critical characteristic amplitude $\tilde{h}_*$ of the incoherent pumping, at which the effective chemical potential touches the bottom of the magnon band, cannot be carried out within the framework of the considered approximation. It will be shown below, however, that one can determine the interval in which the value $\tilde{h}_*$ of interest is located. Thus, the equation

$$\tilde{\mu}(\mu, T, T_B) = \varepsilon_0 \tag{22}$$

with $\mu < \varepsilon_0$ can be used in Eqs. (6) to obtain the lower bound h_{*min}: $\tilde{h}_* > \tilde{h}_{*min}$. The upper bound h_{*max}: $h_* < h_{*max}$, however, can be obtained from (6) by putting $\mu = \varepsilon_0$ and assuming the energy $E(\tilde{\mu}, T_B)$ to correspond to the fully condensed state: $E(\tilde{\mu}, T_B) = N\varepsilon_0$.

We confine ourselves to calculation of h_{*min} and h_{*max} in the low temperature approximation $T_B, T \ll \varepsilon_0$ for an antiferromagnet. Simple manipulations yield

$$\tilde{h}_{*min}^{(a)} < h_*^{(a)}, \quad T \approx T_B. \tag{23}$$

Thus, the lower bound calculated from (22) is even lower than the obvious physical estimate

$$h_*^{(a)} \leqslant \tilde{h}_*^{(a)}, \quad T \geqslant T_B. \tag{24}$$

Calculation of the upper bound of the critical field yields

$$\tilde{h}_{*max}^{(a)} \approx h_*^{(a)}(T/T_B)^{3/4}, \tag{25}$$

$$T \approx \Omega_{ext} \frac{\tau_E}{3\tau_N} \frac{\zeta(3/2)}{\zeta(5/2)}. \tag{26}$$

Note that for the low-temperature approximation to be valid the parameters in (26) must satisfy the inequality

$$\tau_E^{-1} \gg \tau_N^{-1} \frac{\Omega_{ext}}{3\varepsilon_0} \frac{\zeta(3/2)}{\zeta(5/2)}. \tag{27}$$

To describe systems with slower energy relaxation (down to $\tau_E^{-1} \sim \tau_N^{-1}$) account must be taken of the superheating of the magnons to temperatures $T \sim \varepsilon_0$ and higher.

2. ABOVE-THRESHOLD STATE

We consider now the state of a magnon system in which, according to the analysis above, the effective chemical potential has already touched the bottom of the spin-wave band. The very fact that a large number of bosons had accumulated in a narrow phase-space region still does not allow us to conclude that they have condensed. The interaction between these particles must be analyzed. We write for the magnon system the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) a_{\mathbf{k}}^+ a_{\mathbf{k}} + \frac{1}{2} \sum_{1,2,3,4} \Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) a_1^+ a_2^+ a_3 a_4 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4), \tag{28}$$

where $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ are the Bose creation and annihilation operators, and $\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)$ is the magnon–magnon scattering amplitude. The terms describing explicitly the noise pump and the energy escape from the system have been omitted. We confine ourselves to an approximation in which the role of the pump reduces only to changing the number of Bose particles, and can be taken into account via the function of the total magnon number

$$N = \sum_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}},$$

a function determined by the balance between the pump and the relaxation. To take into account the constancy of the particle numbers (for a given pump level) we have introduced into (28) the chemical potential μ of the magnons.

Following the theory of weakly ideal Bose gas (see, e.g., Ref. 12), developed for gap-free spectra, we single out the classical condensate amplitudes with $k = 0$: $a_0 = a_0^+ = N_0^{1/2}$, $N_0 = N + N'$, $N'/N \ll 1$. We obtain then from (28), accurate to quadratic terms,

$$\mathcal{H} = (\varepsilon_0 - \mu) N_0 + \frac{1}{2} \mathcal{T}_0 N_0^2 + \sum_{\mathbf{k} \neq 0} \left[A_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}} + \frac{1}{2} B_{\mathbf{k}} (a_{\mathbf{k}} a_{-\mathbf{k}} + a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+) \right], \tag{29}$$

$$A_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu + 2\mathcal{T}_{\mathbf{k}} N_0, \quad \mathcal{T}_{\mathbf{k}} \equiv \Phi(\mathbf{k}; 0; \mathbf{k}, 0),$$

$$B_{\mathbf{k}} = \mathcal{D}_{\mathbf{k}} N_0, \quad \mathcal{D}_{\mathbf{k}} \equiv \Phi(0, 0; \mathbf{k}, -\mathbf{k}).$$

Diagonalizing the quadratic form (29) by the linear (u,v) transformation $a_{\mathbf{k}} = u_{\mathbf{k}} b_{\mathbf{k}} + v_{\mathbf{k}} b^{+}_{-\mathbf{k}}$, we readily find that

$$\mathcal{H}=U+\sum_{\mathbf{k}} \tilde{\varepsilon}_{\mathbf{k}} b_{\mathbf{k}}^{+} b_{\mathbf{k}},$$

where

$$U=(\varepsilon_0-\mu)N_0+\frac{1}{2}\mathcal{T}_0 N_0^2+\frac{1}{2}\sum_{\mathbf{k}\neq 0}(\tilde{\varepsilon}_{\mathbf{k}}-A_{\mathbf{k}}) \tag{30}$$

is the energy of the ground state, in which the chemical potential μ, the number of particles N_0, and the spectrum of the new quasiparticles $\tilde{\varepsilon}_{\mathbf{k}}$ are defined by the equations

$$\mu=\varepsilon_0+\mathcal{T}_0 N_0+\sum_{\mathbf{k}\neq 0}\{\mathcal{T}_{\mathbf{k}}[(A_{\mathbf{k}}/\tilde{\varepsilon}_{\mathbf{k}})\mathrm{cth}(\tilde{\varepsilon}_{\mathbf{k}}/2T)-1] - (\mathcal{T}_{\mathbf{k}} N_0^2/2\tilde{\varepsilon}_{\mathbf{k}})\mathrm{cth}(\tilde{\varepsilon}_{\mathbf{k}}/2T)\}, \tag{31a}$$

$$N=N_0+{}^1/_2\sum_{\mathbf{k}\neq 0}[(A_{\mathbf{k}}/\tilde{\varepsilon}_{\mathbf{k}})\mathrm{cth}(\tilde{\varepsilon}_{\mathbf{k}}/2T)-1]. \tag{31b}$$

$$\tilde{\varepsilon}_{\mathbf{k}}=[(\varepsilon_{\mathbf{k}}-\mu+2\mathcal{T}_{\mathbf{k}}N_0)^2-(\mathcal{D}_{\mathbf{k}}N_0)^2]^{1/2}. \tag{31c}$$

In the zeroth approximation in $N'/N \ll 1$ the quasiparticle spectrum can be written in the form

$$\tilde{\varepsilon}_{\mathbf{k}}\approx[N^2(\mathcal{T}_{\mathbf{k}}^2-\mathcal{S}_{\mathbf{k}}^2)+2N\mathcal{T}_{\mathbf{k}}(\varepsilon_{\mathbf{k}}-\varepsilon_0)]^{1/2}. \tag{32}$$

For the stationary state to be stable it is necessary that the radicand in (32) be non-negative. The sufficient conditions for this are the inequalities

$$|\mathcal{T}_{\mathbf{k}}|\geqslant|\mathcal{D}_{\mathbf{k}}|, \tag{33a}$$

$$\mathcal{T}_{\mathbf{k}}>0. \tag{33b}$$

Analysis of the magnon–magnon interaction amplitudes (see Ref. 1) shows that these two inequalities can be satisfied only in a ferromagnet. For weakly anisotropic antiferromagnets the theory yields $\mathcal{S}_0=\mathcal{T}_0<0$.

It follows thus from the above calculation that a quasi-equilibrium magnon Bose-condensation can be expected only in ferromagnetic crystals (for example YIG, EuO). In antiferromagnets, on the other hand, the accumulation of a large number of magnons on the bottom of a band is apparently accompanied by their "sticking" and subsequent collapse,[4] which attests to formation of a spatially inhomogeneous state. If, however, account is taken of the additional nonlinear terms of (28), which limit the increasing density of magnons with $k=0$, spatially homogeneous states can result from a first-order phase transition.[13,14] We confine ourselves below, for simplicity, to Bose condensation in a ferromagnet.

To determine N we must write down balance equations [similar to (5)] for the total number of particles and for the system energy. This is easiest to do when the incoherent-action region lies substantially higher than the magnon-activation energy ($\Omega_{\mathrm{ext}} \gg \varepsilon_0$). The only absorption is then by quasiparticles whose spectrum hardly differs from that of the magnons, (i.e., Eq. (8) is perfectly valid for the arrival term). If the low-temperature approximation ($T\approx T_B \ll \Omega_{\mathrm{ext}}$) is furthermore applicable, the effective heating of the quasiparticle system has practically no influence on the absorbed power.

The particles obviously exit from the system both via spin-lattice relaxation of the quasiparticles and via coherent emission of photons by the condensate. However, since it is assumed in the above model that $N-N_0 \ll N$ is small, it can be suggested that the particles leave the Bose system mainly via condensate relaxation. We confine ourselves here to the case of greatest interest, in which the modes with $k=0$ can relax mainly via coherent emission of electromagnetic waves. This approximation is fully justified for magnetic systems in which: 1) decay of a magnon with $k=0$ into elementary crystal excitations with lower energies is forbidden; 2) conversion of magnons with $k=0$ into two phonons with equal but opposite wave vectors is ineffective; 3) interaction of homogeneous magnetic oscillations with impurities is either insignificant or leads rapidly to saturation of the impurity subsystem with increase of the homogeneous-precession amplitude.

The magnetodipole-emission intensity is given by the equation[15]

$$\mathcal{I}=\frac{2}{3c^3}\left(\frac{d^2}{dt^2}\mathbf{M}\right)^2, \tag{34}$$

where $\mathbf{M}$ is the magnetic moment of the sample and c is the speed of light. The nonlinear character of the magnetic-moment motion produces in the emission spectrum, besides the fundamental frequency $\omega=\varepsilon_0$, the harmonics 2ω, 3ω, etc. As a result

$$\mathcal{I}=\mathcal{I}_{\omega}+\mathcal{I}_{2\omega}+\mathcal{I}_{3\omega}+\ldots,$$

where $\mathcal{I}_{j\omega}$ is the jth harmonic radiation intensity.

Under the above assumptions, the value of N can be estimated from the following particle-number balance equation

$$R_N=\sum_{j=1}\mathcal{I}_{j\omega}/j\omega\approx F_N, \tag{35}$$

in which F_N is determined from Eqs. (3) and (8) with $n_{\mathbf{k}}\approx 0$.

We present the explicit form of $\mathcal{I}_{j\omega}$ for a cubic ferromagnet:

$$\mathcal{I}_{\omega}=\alpha\omega^4 S\mathcal{N}N_0,\quad \mathcal{I}_{2\omega}=\alpha v^2(2\omega)^4 N_0^2, \quad \mathcal{I}_{3\omega}=\alpha v^2(3\omega)^2 N_0^3/16S\mathcal{N},\quad \alpha=2^4\mu_B^2/3c^3. \tag{36}$$

Here S is the electron-shell spin, $\mathcal{N}$ is the number of elementary magnetic cells in the sample, μ_B is the Bohr magneton, and $v \ll 1$ is a parameter indicative of the degree of "bending" of the magnetic-moment hodograph over the quantization axis; this parameter is determined by the magnetodipole interaction (see e.g., Ref. 16):

$$|v|=[(A_0-\varepsilon_0)/2\varepsilon_0]^{1/2},$$
$$A_0=2\pi gM[(\mathcal{N}_x+\mathcal{N}_y)-2\mathcal{N}_z], \tag{37}$$
$$\varepsilon_0=[(H_i+2\pi M\mathcal{N}_x)(H_i+4\pi M\mathcal{N}_y)]^{1/2},$$

where $\mathcal{N}_x$, $\mathcal{N}_y$, and $\mathcal{N}_z$ are the demagnetizing factors of the sample, while $H_i=H-4\pi\mathrm{M}\mathcal{N}_z$ is the internal field. In other words, v is coefficient of linear $(u$-$v)$ transformation from local spin deviations to spin waves with $k=0$. It should be noted that Eqs. (36) are valid only if all the magnons are "synchronized," i.e., make up a Bose condensate.

Without this condition the classical Eq. (34) predicts absence of radiation. The system radiation is then fluctuating and a quantum-mechanical calculation is required.

Within the framework of a calculation without allowance for the cavity, a rigorous relation holds between the radiation intensities at multiple frequencies (see Ref. 36). These conditions, however, may be violated if the sample is placed in a cavity tuned to one of the radiation frequencies. It is particularly interesting that the role of this cavity can be assumed by the intrinsic magnetostatic modes of the sample. Thus, by varying the sample geometry it is possible to change strongly the emission intensity at a given frequency.

DISCUSSION

We have estimated in this paper in the intensities of an incoherent microwave field in which Bose condensation of continuously pumped-in magnons can take place. It should be noted that accumulation of a large number of magnons at the bottom of a spectral band can be observed in principle also by using pulse methods, for example, by parametric excitation of spin waves by a narrow-band high-power pulse. In this case, after the end of the action of a short pulse, the magnon-magnon collisions in the system can establish (under assumptions made in the Introduction) an effective temperature and a chemical potential, defined by the equations

$$E\approx {}^1/_2\Omega N_\Omega, \quad N\approx N_\Omega, \tag{38}$$

where E and N are defined in (3) and N_Ω is the number of magnons parametrically excited by a pump pulse of frequency Ω. If the energy relaxation is much faster that the magnon-number relaxation, the cooling of the excited system is accompanied by an increase of the effective chemical potential (to satisfy the condition $N=\text{const}$), which can reach the bottom of the spin-wave spectral band at $N_\Omega > N(\mu=0, T=T_B)$.

The assumed possibility of Bose condensation of magnons with $k=0$ under pulsed parametric excitation of magnons with $k\neq 0$ by an external coherent field is attested to by experiments (on YIG single crystals[17,18] and with nuclear spin waves in antiferromagnetic $CsMnF_3$ (Ref. 19). Microwave radiation was observed in these experiments of frequencies that are multiples of the frequency of the bottom of the spin-wave band. It should be noted that these phenomena received in Ref. 17 a different theoretical treatment (called "kinetic" instability of spin waves), based on interaction between magnons having $k=0$ and a spectrally narrow packet of parametrically excited magnons with $k\neq 0$. This instability sets in when the rate of damping of spin waves not connected with the pump is decreased to zero. A detailed experimental investigation of these phenomena (Refs. 17–19) is therefore of interest from the viewpoint of separating the validity regions of the two theoretical approaches (strong disequilibrium—Ref. 17—and quasiequilibrium—proposed by us).

To exclude the influence of parametric magnons, the excitation pulses must be made shorter than in Refs. 17–19 ($t<\gamma^{-1}$, where γ is the excited-magnon relaxation parameter), and correspondingly stronger.

The authors thank A. K. Khitrin for helpful discussions and V. I. Ozhogin for reading manuscript and for valuable remarks.

[1] V. S. L'vov, *Nonlinear Spin Waves* [in Russian], Nauka, 1987.
[2] Yu. D. Kalafati and V. L. Safonov, Zh. Eksp. Teor. Fiz. **95**, 2009 (1989) [Sov. Phys. JETP **68**, 1162 (1989).
[3] Yu. D. Kalafati and V. L. Safonov, Pis'ma Zh. Eksp. Teor. Fiz. **59**, 135 (1989) [JETP Lett. **50**, 149 (1989)].
[4] S. A. Moskalenko, *Introduction to the Theory of High-Density Excitons* [in Russian]. ShTINITsA, Kishinev, 1983.
[5] H. Fröhlich, Phys. Lett. A **26**, 402 (1968).
[6] N. G. Duffield, Helvetica Phys. Acta **61**, 363 (1988); J. Phys. A **21**, 625 (1988).
[7] S. G. Tikhodeev, Zh. Eksp. Teor. Fiz. **97**, 681 (1990) [Sov. Phys. JETP **70**, 380 (1990)]. Preprint No. 72, Inst. Gen. Phys., Moscow 1989.
[8] V. B. Cherepanov, Zh. Eksp. Teor. Fiz. **91**, 235 (1986). [So. Phys. JETP **64**, 136 (1986)].
[9] E. B. Fel'dman and A. K. Khitrin, *ibid.* **98**, 967 (1990) [**71**, 538 (1990)].
[10] V. S. Lutovinov and V. L. Safonov, Fiz. Tverd. Tela (Leningrad) **22**, 2640 (1980) [Sov. Phys. Solid State **22**, 1541 (1980)].
[11] V. I. Ozhogin, Zh. Eksp. Teor. Fiz. **58**, 2079 (1970) [Sov. Phys. JETP **31**, 1121 (1971)].
[12] V. V. Tolmachev, *Bose-Gas Theory* [in Russian], Moscow Univ. Press, 1969, Chap. 3.
[13] S. V. Iordanskiĭ, Zh. Eksp. Teor. Fiz. **47**, 167 (1964) [Sov. Phys. JETP **20**, 112 (1964)].
[14] D. M. Chernikova, *ibid.* **57**, 2125 (1969) [**30**, 1154 (1969)].
[15] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, Addison-Wesely.
[16] R. White, *Quantum Theory of Magnetism* [Russ. Transl.], Mir, 1985, Chap. 6.
[17] A. V. Lavrinenko, V. S. L'vov, G. A. Melkov, and V. B. Cherepanov, Zh. Eksp. Teor. Fiz. **81**, 1022 (1981) [Sov. Phys. JETP **54**, 542 (1981)].
[18] I. V. Krutsenko, A. Lavrinenko, and G. A. Melkov, Fiz. Nizk. Temp. **9**, 1289 (1983) [Sov. J. Low-Temp. Phys. **9**, 664 (1983)].
[19] S. A. Govorkov and V. A. Tulin, Zh. Eksp. Teor. Fiz. **95**, 1398 (1989) [Sov. Phys. JETP **68**, 807 (1989)].

Translated by J. G. Adashko