The spectrum of electronic vibrations in a degenerate nonrelativistic plasma

V. S. Krivitskiĭ and S. V. Vladimirov

Institute of General Physics, USSR Academy of Sciences, Moscow

(Submitted 13 December 1990)
Zh. Eksp. Teor. Fiz. **100**, 1483–1494 (November 1991)

We study the spectrum of electronic vibrations in an ideal degenerate cold ($T \ll \mathscr{E}_F$) nonrelativistic plasma. We obtain an expression for Landau damping, which is nonzero in the zero-sound range even at $T = 0$. Under the conditions considered the electronic vibrations spectrum has been found to have a terminal point. Expressions for all characteristic wave vectors are also derived, namely, for k_1 (from which Landau damping begins as $|\mathbf{k}|$ increases), for k_2 (at which the maximum frequency is attained), and for k_3 (corresponding to the terminal point). All these values are found to be of the same order of magnitude (of ω_p/v_F) and differ only in logarithmic factors.

1. INTRODUCTION

The spectrum of electronic vibrations in a degenerate nonrelativistic ideal plasma is assumed to have been well-studied and the results given in standard textbooks (e.g., see Refs. 1–3) are seen as classical. Indeed, the spectrum was calculated quite a long time ago. For instance, in the quasiclassical range of wave vector values $|\mathbf{k}|$ Vlasov[4] found that for $k \ll \omega_p/v_F$, with $\omega_p = (4\pi n q^2/m)^{1/2}$ the plasma electron frequency, $v_F = \hbar(3\pi^2 n)^{1/3}/m$ the electron velocity at the Fermi surface, n the electron number density, q the electron charge, and m the electron mass, the dispersion of longitudinal electronic vibrations assumes the form

$$\omega=\omega_p+{}^3/_{10}k^2v_F{}^2/\omega_p, \tag{1}$$

while Gol'dman[5] found that for $\omega_p/v_F \ll k \ll mv_F/\hbar$,

$$\omega=kv_F[1+2\exp(-2-{}^2/_3k^2v_F{}^2/\omega_p{}^2)] \tag{2}$$

which corresponds to the case of zero sound. Later[6] the dispersion law for the electronic vibrations of a weakly nonideal Fermi gas was rigorously obtained on the basis of the quantum kinetic equation. Note that in the literature cited here the zero-sound region is assumed to extend at least to the wave numbers $k \sim p_F/\hbar$, with $p_F = mv_F$ the electron Fermi momentum, while damping is linked solely with the thermal motion of particles (or their collisions) since [as Eqs. (1) and (2) show] the phase velocity of the excitations considered is always strictly greater than v_F. Note also that for zero sound in a normal Fermi liquid the possibility of collisionless damping at absolute zero was rejected (e.g., see Ref. 7).

Silin and Ursov[8,9] studied the spectrum of longitudinal waves in a relativistic degenerate plasma and pointed to the qualitative change in the spectrum (including the existence of a terminal point) related to the branch points in the dispersion function $\varepsilon(w,\mathbf{k})$, but they rejected the possibility of collisionless damping of waves near the end of the spectrum.

In this paper we show that the dispersion law for longitudinal electronic vibrations of an ideal degenerate plasma as $T\to 0$ in the short-wave range ($k \gtrsim \omega_p/v_F$) differs considerably from the law given in textbooks (see Refs. 1–3). The first difference is that the dispersion curve does not continue to values $k \sim p_F/\hbar$ in accordance with Eq. (2) [there is, however, a wavelength range where (2) is approximately valid]. It has been established that there is a value k_3 for the wave vectors where at $k > k_3$ no vibrations are possible. Also, even at absolute zero collisionless damping can be finite: there is a threshold value k_1 such that for $k < k_1$ no damping is possible but for $k > k_1$ the damping is finite.

2. THE DIELECTRIC CONSTANT

Let us consider an almost ideal degenerate cold nonrelativistic plasma. We allow for a contribution to the dielectric constant only from the (degenerate) electron component; we also assume the ion mass to be infinitely large and ignore the effect of ion mass on the electrodynamics. In what follows (except Sec. 5) we assume that the electron temperature is zero. The weak nonideality of a plasma means that electron interaction is weak:

$$q^2n^{1/3}\ll\mathscr{E}_F,$$

which is equivalent to

$$q^2/\hbar v_F\ll 1, \tag{3}$$

and the nonrelativistic nature of particle motion, $v_F \ll c$, means that in addition to (3) we have

$$q^2/\hbar v_F\gg q^2/\hbar c\approx{}^1/_{137}. \tag{4}$$

When $q^2/\hbar v_F \gtrsim 1$, we have an electron Fermi liquid (strong nonideality; e.g., see Refs. 10–12), while at $v_F \sim c$ the degenerate electron gas becomes relativistic (e.g., see Refs. 13–15). In metals inequality (3) usually does not hold (as a rule $q^2/\hbar v_F \simeq 2.2$ in metals). In white dwarfs, where (3) holds, condition (4) breaks down. Both conditions (3) and (4) may be met in some semiconductors (in germanium, for instance).

The evolution of the electron distribution, which in the field of the longitudinal wave is assumed spin-independent, is described by the following equation for the one-particle density matrix $\rho = \rho(t,\mathbf{r}_1,\mathbf{r}_2)$:

$$i\hbar\frac{\partial\rho}{\partial t}=(\hat{H}_1-\hat{H}_2^*)\rho, \tag{5}$$

where the Hamiltonian of the electron in the external field of longitudinal potential vibrations is spin-independent and has the form

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2-q\varphi(t,\mathbf{r}), \tag{6}$$

and $\hat{H}_{1,2}$ in Eq. (5) acts on the variable $\mathbf{r}_{1,2}$, respectively.

 0038-5646/91/110821-06$03.00

Combining Poisson's equation with Eq. (5), we arrive at an expression (derived in Ref. 6) for the linear dielectric constant $\varepsilon(\omega,\mathbf{k})$:

$$\varepsilon(\omega,\mathbf{k})=1+\frac{4\pi q^2}{\hbar\mathbf{k}^2}\int\frac{2d^3p}{(2\pi\hbar)^3}\frac{1}{\omega-(\mathbf{kv})+i0}\hat{D}_\mathbf{k}n_\mathbf{p}$$
$$=1-\frac{4\pi q^2}{\hbar\mathbf{k}^2}\int\frac{2d^3p}{(2\pi\hbar)^3}n_\mathbf{p}\hat{D}_\mathbf{k}\frac{1}{\omega-(\mathbf{kv})+i0}, \quad (7)$$

where $n_\mathbf{p}$ is the unperturbed electron distribution function, and the difference operator $\hat{D}_\mathbf{k}$ acts according to the rule

$$\hat{D}_\mathbf{k}n_\mathbf{p}\equiv n_{\mathbf{p}+\hbar\mathbf{k}/2}-n_{\mathbf{p}-\hbar\mathbf{k}/2} \quad (8)$$

and in the limit of small **k** transforms into differentiation:

$$\hat{D}_\mathbf{k}\to\hbar\mathbf{k}\cdot\partial/\partial\mathbf{p}.$$

The procedure for finding $\varepsilon(\omega,\mathbf{k})$ specified by (7) is standard and is given in textbooks (e.g., see Ref. 1).

Note that in deriving (7) we assumed the field of the electron wave to be nonquantized; actually, this means that we are considering only the case where the number of quanta of such waves is high, which in turn allows us to ignore the spontaneous processes as compared to stimulated. The appearance of an infinitesimal positive imaginary term in the denominator of (7) is related in the usual manner to the causality principle (the Landau rule of pole bypass).

In the case considered, $T=0$, the electron distribution function has the shape of a step,

$$n_\mathbf{p}=\begin{cases}1, & |\mathbf{p}|<p_F,\\ 0, & |\mathbf{p}|>p_F,\end{cases} \quad (9)$$

so that (7) can easily be integrated:

$$\varepsilon(\omega,\mathbf{k})=1+(3\omega_p^2/2k^2v_F^2)\{1+g(\omega_-)-g(\omega_+)\}, \quad (10)$$

where

$$\omega_\pm=\omega\pm\hbar k^2/2m,$$
$$g(\omega)=\mathscr{E}_F\frac{\omega^2-k^2v_F^2}{\hbar k^3v_F^3}\ln\frac{\omega+kv_F}{\omega-kv_F}. \quad (11)$$

In accordance with the Landau rule of pole bypass, we have

$$\ln a\to\ln|a|-i\pi.$$

for α negative.

For further analysis it is expedient to go over to dimensionless variables

$$\Omega\equiv\omega/\omega_p,\quad K\equiv kv_F/\omega_p.$$

Then, combining (10) and (11), we obtain

$$\varepsilon(\Omega,K)=1+\frac{3}{2K^2}\{1+g(\Omega_-)-g(\Omega_+)\}, \quad (12)$$

$$g(\Omega)=M\frac{\Omega^2-K^2}{K^3}\ln\frac{\Omega+K}{\Omega-K}, \quad (13)$$

with

$$\Omega_\pm=\Omega\pm K^2/4M,\quad M\equiv\mathscr{E}_F/\hbar\omega_p.$$

Inequalities (3) and (4) imply that

$$1<M<\left(\frac{3\pi}{16}\frac{\hbar c}{q^2}\right)^{1/2}\sim10. \quad (14)$$

For the imaginary part of the dielectric constant we obtain (assuming, for definiteness, that both Ω and K are positive)

$$\mathrm{Im}\,\varepsilon(\Omega,K)=-\frac{3\pi}{2K^5}M\times\begin{cases}0, & |\Omega_-|>K,\ |\Omega_+|>K,\\ [\Omega_-^2-K^2], & |\Omega_-|<K<|\Omega_+|,\\ [\Omega_-^2-K^2]-[\Omega_+^2-K^2], & |\Omega_-|<K,\ |\Omega_+|<K.\end{cases} \quad (15)$$

Thus, the imaginary part of the dielectric constant vanishes only when $\Omega>K+K^2/4M$ or when $\Omega<K^2/4M-K$ (if $K>4M$). In all other cases it is finite.

Concluding this section, we note that all the results discussed here are well known (a fairly complete study of the behavior of $\varepsilon(\Omega,K)$ at $T=0$, including the case where $v_F\sim c$, which means that pair production[13] contributes to Im ε, has been done by Kosachev and Trubnikov[14]). Nevertheless, nowhere in the literature, to our knowledge, has there been a definite calculation for noncollisional attenuating of electrical oscillations in the degenerate plasma.

3. THE DISPERSION LAW

Let us consider the solution of the equation

$$\mathrm{Re}\,\varepsilon(\Omega,K)=0, \quad (16)$$

without allowing, for the time being, for a small damping of the vibrations. For $K\ll1$ and $\Omega\sim1$ we have the well-known solution (1):

$$\Omega\approx1+{}^3/_{10}K^2. \quad (17)$$

Next, when $\Omega\gtrsim1$, $K\gtrsim1$, and $\Omega-K\gg K^2/4M$, we arrive at (2):

$$\Omega\approx K[1+2\exp(-2-2K^2/3)]. \quad (18)$$

These solutions to Eq. (16), as noted earlier, are well known.

The $\Omega(K)$ Ω plot that is the solution to Eq. (16) for $\Omega(K)>1$ intersects the curve $\Omega_-=K$ at a certain point $K=K_1$. This point K_1 is the threshold from which Landau damping manifests itself (see below). One can easily establish that at the intersection point K_1 the curves representing $\Omega=\Omega(K)$ and $\Omega=K+K^2/4M$ have the same derivatives, $d\Omega/dK=1+K_1/2M$. Thus, the difference $K-\Omega_-$ in the neighborhood of point K_1 is $o(K-K_1)$. We note also that K_1 is an inflection point of the function $\Omega(K)$: the second derivative $d^2\Omega(K)/dK^2$ is positive for $K<K_1$, negative for $K>K_1$, and vanishes at $K=K_1$.

Let us find the expression that defines the position of point K_1. The simplest way to do so is to substitute $\Omega=K+K^2/4M$ into Eq. (16). This leads to $g(\Omega_-)=0$ (since $x\ln x\to0$ as $x\to0$), and for K_1 we have

$$\frac{2K_1^2}{3}+1=\left(1+\frac{K_1}{4M}\right)\ln\left(\frac{4M}{K_1}+1\right). \quad (19)$$

Solving (19) yields approximately

$$K_1\approx[{}^3/_2\ln(4M/e)]^{1/2} \quad (20)$$

where $e=2.718...$ is the base of natural logarithms, while the electron charge is denoted by q.

Let us find the point K_2 at which the function $\Omega(K)$ attains its maximum. The point is determined by two equations:

$$\frac{d\Omega(K)}{dK}=0\Rightarrow\frac{\partial\varepsilon(\Omega,K)}{\partial K}\bigg|_{\Omega=\Omega(K)}=0. \qquad (21)$$

It can easily be shown that Eqs. (21) yield the following equation:

$$1=\frac{M}{K^5}\Omega_+\Omega_-\ln\left|\frac{\Omega_-+K}{\Omega_--K}\frac{\Omega_+-K}{\Omega_++K}\right|, \qquad (22)$$

whose approximate solution is

$$K_2\approx[{}^3/_2\ln(8M/e)]^{1/2}. \qquad (23)$$

It should be noted, that the condition (21) is not enough for the presence of the maximum function $\Omega(K)$. We have

$$d^2\Omega/dK^2<0,$$

and $K=K_2$ point actually is the maximum point. For the maximum frequency $\Omega_{\max}$ that the longitudinal electron waves may have we arrive at the following formula:

$$\Omega_{max}=\Omega(K_2)\approx K_2+(2K_2+1)\left(\frac{K_2^2}{4M}\right)^2\approx K_2. \qquad (24)$$

As K increases further, the plot of $\Omega(K)$ bends downward: the group velocity of the waves becomes negative. At $K=K_3$ there emerges a singularity corresponding to

$$\frac{\partial\Omega(K)}{dK}=\infty\Rightarrow\frac{\partial\varepsilon(\Omega,K)}{\partial\Omega}\bigg|_{\Omega=\Omega(K)}=0. \qquad (25)$$

The following equation corresponds to conditions (25):

$$\Omega_-\ln\left|\frac{\Omega_-+K}{\Omega_--K}\right|=\Omega_+\ln\left|\frac{\Omega_++K}{\Omega_+-K}\right|. \qquad (26)$$

The solution to Eq. (26) at $\Omega=\Omega(K)$ is

$$K_3\approx[{}^3/_2\ln(8M/e)]^{1/2}\approx K_2. \qquad (27)$$

A more exact calculation shows that

$$K_3-K_2\approx2\left(1-\frac{K_2}{3}\right)(2K_2^2+K_2-1)\left(\frac{K_2^2}{4M}\right)^2. \qquad (28)$$

This expression is always positive for all admissible values of K_2 [defined by inequality (14)]. The vibration frequency at $K=K_3$ is

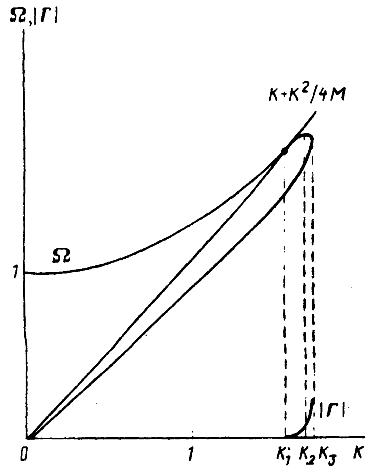

FIG. 1.

$$\Omega(K_3)\approx K_3+\left(\frac{K_3^2}{4M}\right)^2\left(1+\frac{2K_3^2}{3}\right)\approx K_3. \qquad (29)$$

The plot of Eq. (16) then corresponds to a decrease in Ω with K, and as $K\to0$ we find that

$$\Omega=\alpha K, \qquad (30)$$

where $\alpha\simeq0.84$ is the solution to the equation

$$2=\alpha\ln\left|\frac{\alpha+1}{\alpha-1}\right|.$$

One must bear in mind that actually the solution of the (30) type for Eq. (16) corresponds to no waves, since on this branch $\mathrm{Im}\,\varepsilon\gtrsim1$ (see the next section). Figure 1 depicts the solution $\Omega=\Omega(K)$ of Eq. (16).

4. LANDAU DAMPING

Let us now discuss the wave damping γ due to the imaginary part of the linear dielectric constant, (15). For the upper part of the curve in Fig. 1, $\Omega_+>K$ up to values $k=k_s$. At the same time it is easy to see that for $K<K_1$ we have $|\Omega_-|>K$ and $\mathrm{Im}\,\varepsilon(\Omega,K)=0$, while for $K>K_1$ we have $|\Omega_-|<K$ and $\mathrm{Im}\,\varepsilon(\Omega,K)\neq0$. Thus, only waves with wave vectors greater than K_1 are damped. Calculating the damping constant by the formula

$$\Gamma\equiv\frac{\gamma}{\omega_p}=-\frac{\mathrm{Im}\,\varepsilon(\Omega,K)}{\partial\,\mathrm{Re}\,\varepsilon(\Omega,K)/\partial\Omega}\bigg|_{\Omega=\Omega(K)}, \qquad (31)$$

we arrive at the following expression for Γ near the threshold (expanding in powers of $K-K_1\ll1$):

$$\Gamma\simeq-\frac{\pi6^{1/2}}{2}\frac{\ln(4M/e^{1/2})\ln^{1/2}(4M/e)}{M}\frac{K-K_1}{\ln^2[6^{1/2}(K-K_1)]}. \qquad (32)$$

The quantity Γ (32) is negative, which corresponds to damping.

We see that the damping constant Γ is not an exponentially small quantity; at the same time near the threshold $\Gamma\ll1$. Equation (32) shows that Γ increases with $K-K_1$ slower than linearly but faster than $(K-K_1)^{1+\zeta}$ for any positive ζ.

For wave vectors close to k_3 the absolute value of Γ starts to increase rapidly, owing to the fact that the derivative of the dielectric constant tends to zero: $\partial\,\mathrm{Re}\varepsilon(\Omega,K)/\partial\Omega\to0$ as $K\to K_3$. One can estimate the values of K at which the damping constant becomes of the order of the frequency $\Omega(K)$. Calculations yield $|\Gamma|\sim\Omega$ at $K=K_3-\delta K$, with $\delta K\simeq[\ln(8M)/8M]^2$. Heavy damping of waves whose wave vectors are in the neighborhood of the terminal point of the spectrum makes the statement that there is a point K_3 of "strict termination" of the dispersion curve meaningless to a certain extent; the end section of the upper branch of the graph of Ω plotted against K in Fig. 1 should be designated by a dotted line, as is customary for large values of $|\gamma|$.

The lower part of the curve depicted in Fig. 1 has no vibrations since in this case $\mathrm{Im}\,\varepsilon\gtrsim1$. Indeed, for the values of Ω under consideration we have $|\Omega_-|<K$ and $\Omega_+<K$, that is,

$$\mathrm{Im}\,\varepsilon(\Omega,K)=\frac{3\pi M(\Omega_+^2-\Omega_-^2)}{2K^5}=\frac{3\pi\Omega}{2K^3}.$$

according to (15). Estimating $\Omega(K)$ in this segment by Eq. (30) as $\Omega \simeq \alpha K$, we find that $\mathrm{Im}\,\varepsilon \gtrsim 1$, and, hence, this branch, which is the solution to the equation $\mathrm{Re}\,\varepsilon = 0$, corresponds to no vibration mode, in fact.

The K-dependence of the damping constant Γ calculated for the upper part of the graph of Ω plotted against K is depicted in Fig. 1.

Let us discuss this result. In Ref. 1 the damping constant Γ for zero sound is said to be strictly zero at $T=0$ since, as Eq. (2) shows, the phase velocity of the wave, ω/k, is higher than v_F, while at $T=0$ the particle velocities v are lower than v_F, with the result that a Cherenkov resonance is impossible. However, the very condition for a Cherenkov resonance requires refining if we allow for the quantum recoil effect. Indeed, the physical mechanism of Landau damping is related to the absorption (or emission) of the wave by a particle. The law of conservation of energy for this process states that

$$\mathscr{E}_{\mathbf{p}}-\mathscr{E}_{\mathbf{p}\pm\hbar\mathbf{k}}=\mp\hbar\omega, \tag{33}$$

with $\mathscr{E}_{\mathbf{p}} = \mathbf{p}^2/2m$ for nonrelativistic particles, that is,

$$\omega=(\mathbf{kv})\pm\hbar\mathbf{k}^2/2m, \tag{34}$$

which corresponds to the poles of the dielectric constant ε(7). Thus, for a resonance to occur the phase velocity ω/k and the particle velocity must not be equal but must differ by a (small) quantity $\hbar k/2m$. Since according to Eq. (2) the ratio ω/k differs from v_F by an exponentially small quantity, the resonance condition (34) is met starting from a threshold value k_1. Thus, for waves with $k>k_1$ Landau damping is finite.

5. SUBSTANTIATION

The above results require substantiation. We start by examining the method of obtaining the dispersion law and wave damping based on Eqs. (16) and (31).

Longitudinal vibrations are described by the equation

$$\varepsilon(\omega+i\gamma,\ \mathbf{k})=0, \tag{35}$$

where ω is the frequency of the propagating wave, and γ the growth rate (the damping decrement), that is, the field of the wave is proportional to $\exp[-i(\omega+i\gamma)t]$. The "standard" method of solving Eq. (35) assumes that the real part of the dielectric constant, the polarizability $\varkappa \equiv (\varepsilon-1)/4\pi$, is much greater than the imaginary part, that is, $\mathrm{Im}\,\varepsilon \ll 1$. Then, in the first approximation, the wave is described by the purely real solutions $\omega=\omega(\mathbf{k})$ of the equation $\mathrm{Re}\,\varepsilon(\omega,\mathbf{k})=0$, and in the next approximation Eq. (35) assumes the form

$$\mathrm{Re}\,\varepsilon(\omega+i\gamma,\ \mathbf{k})+i\,\mathrm{Im}\,\varepsilon(\omega,\ \mathbf{k})=0, \tag{36}$$

from which we can arrive at formula (31) for γ by expanding the first term in powers of the small quantity γ and equating the real and imaginary parts of (36) to zero separately.

But can this method be applied in our case? To answer this question we must examine the analytic properties of the dielectric constant $\varepsilon(\omega)$ (7) of a degenerate plasma.

The common procedure for deriving an expression for $\varepsilon(\omega)$ from Eqs. (5) and (6) yields formula (7) for the dielectric constant with a real value of the independent variable ω. The imaginary part of ε is linked with bypass of the

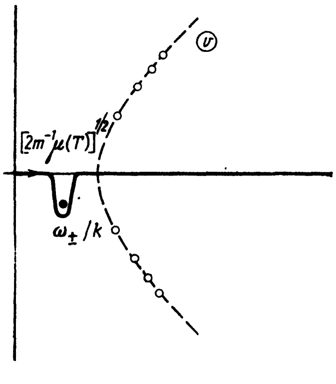

FIG 2.

poles of (7) lying on the real axis, $\omega_\pm = \mathbf{k}\cdot\mathbf{v}$, which is carried out according to the causality principle: in the integration with respect to v the points $\omega_\pm/k$ must be bypassed from below. Next, to obtain $\varepsilon(\omega)$ for a complex-valued independent variable we continue the function $\varepsilon=\varepsilon(\omega)$ analytically from the real axis, which means the path of integration with respect to v must be shifted downward for $\mathrm{Im}\omega<0$, so that the points $\omega_\pm/k$ are bypassed from below (Fig. 2). (For $\mathrm{Im}\omega>0$ the integration is carried out in the complex v plane along the real axis, which in this case contains no singular points and, therefore, $\varepsilon(\omega)$ is analytic in the upper half-plane; e.g., see Refs. 1 and 16.) The singular points of $\varepsilon(\omega)$ lie in the lower half-plane and correspond to the fact that for complex-valued ω the point ω_+/k (or ω_-/k) coincides with a singular point of the function $n_{\mathbf{p}}$ (this function is multiplied by $[\omega_\pm - \mathbf{k}\cdot\mathbf{v}]^{-1}$ when we integrate with respect to $\mathbf{v}$ in Eq. (7)). Indeed, as point $\omega_\pm/k$ moves toward a singular point v_0 of $n_{\mathbf{p}}$, there is no way in which the path of integration can avoid the point $\omega_\pm/k$, since it is pinched between the points $\omega_\pm/k$ and v_0 (e.g., see Ref. 1).

Let us find the singular points of the Fermi distribution function

$$n_{\mathbf{p}}=\frac{1}{\exp[(\mathbf{p}^2/2m-\mu(T))/T]+1} \tag{37}$$

which tends to (9) as $T\to+0$. It can easily be verified that the singular points of (37) are poles of the first order that lie on a hyperbole intersecting the real axis at points $\pm[2\mu(T)/m]^{1/2}$, with the distance between adjacent points proportional to temperature T and with the distance between far-off neighboring jth and $(j+1)$st poles decreasing like $1/j^{1/2}$ for large j (see Fig. 2). [The Maxwellian distribution function is an entire function, with the result that the dielectric constant of a classical Maxwellian plasma, $\varepsilon(\omega)$, is also an entire function, that is, has no singularities for finite ω.] As $T\to+0$, these poles "fill the entire hyperbola."

Let us now follow the movement in the complex v plane of the point $(\omega+i\gamma)/k$ corresponding to the longitudinal wave in the plasma. (Strictly speaking, after integration (with an isotropic $n_{\mathbf{p}}$) over the angles between $\mathbf{k}$ and $\mathbf{v}$ in (7) is performed, the points $\pm\omega_\pm/k$ become branch points of the respective logarithms $\ln(\omega_\pm \pm kv)$, and for integration in the complex v plane (see Fig. 2) to be meaningful we must select a single-valued regular branch of the logarithm, to which end we cut the complex v plane from point $\omega_\pm/k$

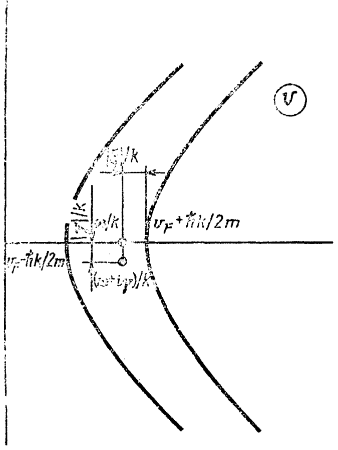

FIG. 3.

along a certain curve originating at point $\omega_{\pm}/k$ "upward" (e.g., along the upward ray of the straight line Re $v = \text{const}$), so that point $\omega_{\pm}/k$ can be traversed from below.)

For $k < k_1$ damping is nil, $\gamma = 0$, and point ω/k lies on the real axis in the complex v plane to the right of the points of intersection of both hyperbolas $(\text{Re}\, v \pm \hbar k/2m)^2 - (\text{Im} v)^2 = v_F^2$ (as $T \to +0$) with the real axis. At $k = k_1$ point ω/k coincides with the point of intersection of the right hyperbola with the real axis. Finally, for $k > k_1$ point $(\omega + i\gamma)/k$ moves from point $v_F + \hbar k/2m$ to the left and downward (Fig. 3). Expansion in powers of γ in (36) is admissible if $|\gamma|$ is smaller than the radius of convergence of function ε at point ω or, in other words, if $|\gamma/k|$ is smaller than the distance between point ω/k and the nearest singular point of function $n_{\mathbf{p} \pm \hbar \mathbf{k}/2}$, that is (as $T \to +0$), the distance between point ω/k and the hyperbola

$$(\text{Re}\, v - \hbar k/2m)^2 - (\text{Im}\, v)^2 = v_F^2;$$

(the other hyperbola

$$(\text{Re}\, v + \hbar k/2m)^2 - (\text{Im}\, v)^2 = v_F^2$$

lies farther from point ω/k since, as can easily be verified, $\omega(k) > kv_F$ for all values of k up to $k = k_3$ (see Fig. 3).

Thus, for (31) to be valid with $K > K_1$ [which means the validity of (32), too], $|\Gamma|$ must be smaller than $|\xi|$, with $\xi \equiv \Omega_- - K$ ($\xi = 0$ at the threshold point K_1). This condition, as can easily be demonstrated, is met in a small neighborhood of the threshold point for $K > K_1$; indeed, near the threshold (for $K > K_1$) we have

$$\text{Im}\, \varepsilon \sim \xi, \quad \frac{\partial\, \text{Re}\, \varepsilon}{\partial \Omega} \sim \ln\left(\frac{1}{|\xi|}\right),$$

and, hence, according to (31), the ratio

$$\frac{|\Gamma|}{|\xi|} \sim \ln^{-1}\left(\frac{1}{|\xi|}\right)$$

and is small for small $|\xi|$ (the quantity ξ for small positive $K - K_1$ behaves

$$\sim \frac{(K-K_1)}{\ln(1/(K-K_1))}.$$

But then, at a certain $K = K_4 > K_1$, the quantities $|\Gamma|$ and $|\xi|$ become equal and, hence, formula (31) ceases to be valid at $K = K_4$, earlier than K reaches the value K_3 and even earlier than K reaches $K_3 - \delta K$, when $|\Gamma|$ becomes of the order of the vibration frequency Ω. But on the whole (to within the same logarithmic accuracy as above) all these points lie fairly close to each other; the difference $K_4 - K_1$ is proportional to $\ln^{-1/2} M$ (just as the difference $K_3 - K_1$, which is also proportional to $\ln^{-1/2} M$). Thus, the range where (32) is valid is fairly narrow (and, for that matter, so is the range of zero sound).

The study conducted in the present paper refers to the case where $T = 0$. At a low but nonzero temperature the additional contribution to Landau damping caused by the thermal spread of the distribution of $n_\mathbf{p}$ is proportional to $\exp(-1/T)$ and, hence, becomes negligibly small as $T \to +0$. One can fairly easily find the "critical" temperature T_0 at which the contribution to the collisionless damping γ from the thermal spread of the $n_\mathbf{p}$-distribution becomes commensurate with the value of γ at absolute zero. A rough estimate yields

$$T_0 \sim \mathscr{E}_F \ln^{1/2} M/M.$$

The above results are valid at temperatures much lower than T_0.

Let us also discuss the applicability of formula (31) and the result $\Gamma = 0$ for the threshold point proper, where $K = K_1$. As $T \to +0$, the hyperbola, "the carrier of the singular points of function ε" (more precisely, one of the two of such hyperbolas, situated to the right), moves in the complex Ω plane to the right and intersects the real axis at $T = 0$ exactly at point $K_1 + K_1^2/4M$ (Fig. 4). [The singular points of $\varepsilon = \varepsilon(\Omega)$ are branch points, since the points $v = \omega_{\pm}/k$ of the respective logarithms in the integrand for ε are branch points. If for these logarithms we cut the complex plane upward along the straight line Rev = const, as noted earlier, the cuts originating at the singular points of the function $\varepsilon(\Omega)$ must be made downward along the straight line ReΩ; see Fig. 4.)] Thus, the singular points of function ε find themselves within an arbitrarily small neighborhood of the threshold point

$$\Omega(K_1) = K_1 + \frac{K_1^2}{4M}$$

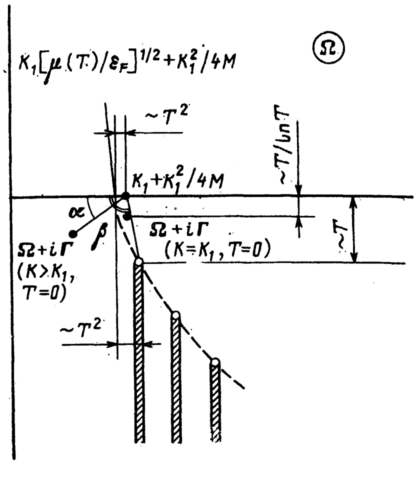

FIG. 4.

as $T\to +0$. Hence, we can rigorously define the damping constant Γ of the wave at $K=K_1$ and $T=0$ as

$$\lim_{T\to+0} \Gamma(T)\,|_{K=K_1}.$$

This limit, as could be expected, is zero since simple estimates suggest that at point $K=K_1$ asymptotically

$$\operatorname{Im}\varepsilon\sim T, \qquad \frac{\partial\,\operatorname{Re}\varepsilon}{\partial\Omega}\sim \ln T, \qquad \Gamma(T)\sim\frac{T}{\ln T}\to 0.$$

as $T\to +0$. Representation (31) is admissible since at low temperatures $\Gamma(T)\sim T/\ln T \ll T$, and the distance between the real axis and the closest singular point is proportional to T; see Fig. 4 (at low temperatures the chemical potential of a Fermi gas,

$$\mu(T)\approx\mathscr{E}_F-\frac{\pi^2}{12}\frac{T^2}{\mathscr{E}_F}$$

see Ref. 17).

It is also easy to see that point $\Omega+i\Gamma$ at a certain $K>K_1$ corresponding to a wave propagating at absolute zero cannot coincide with a singular point of the function ε at $T\neq 0$ (when the hyperbola moves to the left). Indeed (see Fig. 4), as noted above, for small positive differences $K-K_1$ we have

$$\operatorname{tg}\alpha\sim \ln^{-1}(|\xi|^{-1})\ll 1,$$

but at the same time

$$|\operatorname{tg}\beta|\sim T^{-1}\gg 1,$$

Hence, $\alpha\neq\beta$ (actually, angle β is even greater than $\pi/2$, since the first singular point lies to the right of point $K_1+K_1^2/4M$), with the result that the singular point of ε closest to the real axis (and, the more so, the other singular points) cannot coincide with the point $\Omega+i\Gamma$ (for $K>K_1$ and $T=0$).

When one formulates an initial-value problem in a plasma, e.g., that of the excitation of a longitudinal wave by nonlinear processes of some sort (e.g., decay processes), the "end products of the evolution of the system" include not only the waves corresponding to the roots $\Omega=\Omega(K)$ of the equation $\varepsilon(\Omega,K)=0$ but also some excitations $\Omega=\overline{\Omega}(K)$ associated with singular points of the function ε. At very low temperatures T, the contribution from the singular point of ε nearest the real axis decays more slowly than a longitudinal wave, because this singular point is closer than the point $\Omega+i\Gamma$ to the real axis in this case. (In the limit $T=0$, at which singularities appear on the real axis itself, at the points $\Omega=K\pm K^2/4M$, in the Ω plane, the associated perturbations decay more slowly than exponentially (by a power law) over time. In the limit $t\to+\infty$, the wave contribution is predominant (i.e., decays more slowly than the perturbations stemming from the singular points of ε) if T satisfies the inequality

$$(K-K_1)/M\ll T/\mathscr{E}_F.$$

6. CONCLUSION

The expression (32) obtained in Sec. 4 is valid for a weakly nonideal Fermi gas. However, the general remarks made in Sec. 4 that allowing for the purely kinematic effect, quantum recoil under emission (absorption), leads to nonzero Landau damping are valid for a Fermi liquid where at $T=0$ the difference between the phase velocity of zero sound and the Fermi velocity is also exponentially small.[7]

Zero sound has not been observed in experiments. The reason may be that the very region where such type of vibrations exists is fairly narrow, ranging from $k\sim\omega_p/v_F$ to $k\sim(\omega_p/v_F)\ln^{1/2}(\mathscr{E}_F/\hbar\omega_p)$.

Here we have not considered collisional damping of waves, γ_{st}, which in the conditions of weak nonideality under consideration have a higher order of smallness in parameter $q^2/\hbar v_F$ (namely, $\sim q^4$) than collisionless Landau damping.

Termination of the longitudinal vibration spectrum also occurs in classical plasma, where qualitatively the k-dependence of ω is the same as the one depicted in Fig. 1, and $k_3\simeq 0.53 r_D^{-1}$, with r_D the electron Debye radius. In both classical and degenerate electron plasma the spectrum terminates when the wave number k reaches a value of the order of the ratio of the plasma frequency to the characteristic velocity in the particle distribution (thermal or Fermi, respectively). The situation is different in a weakly nonideal Bose gas of charged particles (the interaction of particles in a Bose gas with repulsion can be considered in the perturbation-theory setting) whose net charge is assumed, as above, compensated for by the infinitely heavy fixed "ions." In a Bose gas at $T=0$ all particles are in a condensate, with the result that there can be no characteristic velocity in the particle distribution; the dispersion law is

$$\omega^2(\mathbf{k})=\omega_p{}^2+\frac{\hbar^2\mathbf{k}^4}{4m^2}$$

for all $\mathbf{k}$'s, and there is no point where the spectrum terminates, while Landau damping $\gamma\equiv 0$ in the first order in q^2 (at finite temperatures this is not so).

[1] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Pergamon, Oxford (1981).

[2] V. P. Silin and A. A. Rukhadze, *Electromagnetic Properties of Plasma and Plasma-like Media* [in Russian], Atomizdat, Moscow (1961).

[3] A. F. Aleksandrov, L. S. Bogdankevich, and A. A. Rukhadze, *Fundamentals of Plasma Electrodynamics* [in Russian], Vysshaya shkola, Moscow (1988).

[4] A. A. Vlasov, Zh. Eksp. Teor. Fiz. **8**, 291 (1938).

[5] I. I. Gol'dman, Zh. Eksp. Teor. Fiz. **17**, 681 (1947).

[6] Yu. L. Klimontovich and V. P. Silin, Zh. Eksp. Teor. Fiz. **23**, 151 (1952).

[7] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2, Pergamon, Oxford (1980).

[8] V. P. Silin and V. N. Ursov, Kratk. Soobshch. Fiz., No. 1, 34 (1982).

[9] V. P. Silin and V. N. Ursov, Kratk. Soobshch. Fiz., No. 12, 53 (1982).

[10] V. P. Silin [Sov. Phys. JETP **6**, 387 (1958)].

[11] V. P. Silin [Sov. Phys. JETP **7**, 538 (1958)].

[12] V. P. Silin [Sov. Phys. JETP **8**, 870 (1958)].

[13] V. N. Tsytovich [Sov. Phys. JETP **13**, 1249 (1962)].

[14] V. V. Kosachev and B. A. Trubnikov, Zh. Eksp. Teor. Fiz. **60**, 595 (1971).

[15] D. G. Lominadze, G. I. Melikidze, V.P. Silin, and V. N. Ursov, Fiz. Plazmy **12**, 814 (1986).

[16] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, Oxford (1984).

[17] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part 1, Pergamon, Oxford (1980).

Translated by E. Yankovskii