Collision integral to terms linear in the curvature

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Kinetic equations are derived for a plasma in a gravitational field. In the Landau approximation a correct technique for separating the kernel of the collision integral is proposed. For vacuum spaces the resulting kernel is expanded to second order in the small parameter r_D/λ_g, where r_D is the Debye radius and λ_g is the characteristic length scale of the variation of the gravitational field. It is shown that in the zeroth approximation the kernel of the collision integral is the covariant generalization of the Belyaev–Budker kernel. In the second approximation in the nonrelativistic limit the correction to the Belyaev–Budker kernel is obtained in an explicit form that depends linearly on the Riemann tensor. The properties and possible applications of the kinetic equation obtained are discussed.

1. INTRODUCTION

In attempts to construct a kinetic theory for a plasma situated in a gravitational field a multiplicity of difficulties arise, many of which were noted and analyzed in the series of papers Refs. 1–3. A serious problem that arises in this situation is associated with Currie's theorem,[4] which was formulated and proved for the special theory of relativity (STR). The problem consists in the impossibility of preserving the covariance of the equations for interacting particles. The situation is further complicated when one goes over to the general theory of relativity (GTR). Three possible ways of overcoming this difficulty have been proposed. In the case of the STR Balescu[5,6] suggested that the field of the interaction between the particles be treated not dynamically (equations-of-motion of the field), but statistically (kinetic equations). This leads to consideration of the complete distribution function of the particles and field oscillators. In this approach, in the case of electromagnetic interaction between the particles the dynamical Maxwell equations are replaced by kinetic equations for the distribution function of the field oscillators. Systematic development of this method makes it possible to take radiative friction into account in the kinetic equations.[7]

In the case of the GTR, Israel and Kandrup obtained in Refs. 1–3 a covariant closed kinetic equation for the kinetic part of the distribution function of the particles, using the projection-operator method developed in Ref. 8. For a specific application of the equations obtained to conformally static gravitational fields, see Refs. 9 and 10. The third method, proposed in Ref. 11 (predicative relativistic mechanics) consists simply in postulating the solutions of the equations-of-motion.

Another problem arises when one attempts to construct a complete kinetic theory that takes into account the self-consistent gravitational field. The problem consists in the necessity of calculating the average "macrometric" from the "micrometric." In Ref. 12 it was shown that the average of the energy-momentum tensor is not equal to the Einstein tensor calculated on the average metric, this being a consequence of the nonlinearity of the Einstein equations. In the construction of a kinetic theory on the background of a gravitational field, which is what we shall study below, this problem drops out. There exist, however, situations in which it is necessary to take the self-consistent gravitational field into account. For example, for the case of a weak plane gravitational wave there exists a region of frequencies in which allowance for the self-consistent gravitational field substantially alters the dispersion law of the gravitational waves in the medium (see Ref. 13).

At temperatures $T<\{137L/\ln(137L)\}^{1/2}\cdot mc^2$ (Ref. 14), when the scattering cross section prevails over the bremsstrahlung cross section, it is possible to use with success Klimontovich's method[15,16] for constructing a kinetic theory for a plasma in an external gravitational field, since at such temperatures there is no longer any need to take the statistics of the electromagnetic field into account. In this method the freedom that remains in the choice of the averaging hypersurface corresponds to "gauge" invariance of the kinetic theory developed by Kandrup.[2] The gauge invariance in Ref. 2 has as its cause the freedom to impose constraints on the proper times of the particles, corresponding to the freedom to choose the observer (hypersurface) in the method of Klimontovich. Previously, this method has been used in the Landau approximation to obtain a kinetic equation with a collision term for a plasma situated in an external gravitational field, for two types of metric—the Friedmann metric,[17] and the metric of Bondi, Pirani, and Robinson for a plane gravitational wave.[18]

In the present paper we use this method in the Landau approximation to obtain for a plasma in a gravitational field a kinetic equation that is valid when the condition $r_D/\lambda_g \ll 1$ is fulfilled, where r_D is the Debye radius and λ_g is the characteristic length scale of the variation of the gravitational field. For vacuum spaces ($R_{ik}=0$) the collision integral obtained will be expanded to second order in the above-indicated parameter. The calculations will be performed in the nonrelativistic region and exclusively for vacuum spaces, for visualizability and compactness of the expressions obtained. All the notation used coincides with the notation in Ref. 19.

2. DERIVATION OF THE EQUATIONS

The method of Klimontovich is based on considering the microscopic phase density of particles of type a, which

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can be covariantly generalized in a unique manner[17,18] and has the following form:

$$N_a=\sum_{i=1}^{n_a}\int ds_{(i)}\delta^{(4)}(x^k-x^k_{(i)}(s_i))\delta^{(4)}(p_k-p_{k(i)})(s_i)). \quad (1)$$

This method is described most fully and in most detail in Ref. 20 for the case of the STR. In the following we shall use an 8N-dimensional space (see, e.g., Ref. 21), which is a fiber bundle in which the base is space-time and the fiber is the space of the covariant momenta.

The function (1) satisfies the following equation:

$$u^i\tilde{\nabla}_iN_a(q)+\frac{e_a}{c}F_{ik}(x)u^k\,\partial N_a/\partial p_i=0, \qquad (2)$$

where $\tilde{\nabla}_i$ is the Cartan derivative;[21] $q^A=(x^i;\ p_k)$ and $A=1,...,8$. The electromagnetic field acting on a particle and originating from the other particles obeys the Maxwell equations

$$F^{ik}{}_{;k}=-\frac{4\pi}{c}j^i=-\sum_a 4\pi e_a\int\frac{d^4p}{(-g)^{1/2}}u^iN_a(q),$$
$$F_{(ik;\,l)}=0. \qquad (3)$$

We shall define a function L^{ab}_{il} by the following relation:[15,16]

$$\frac{e_a}{c}F_{ik}(x)=\sum_b\int d^8q'L_{ik}{}^{ab}(x;q')N_b(q'). \qquad (4)$$

Then the function L^{ab}_{ik} can be expressed in terms of the retarded vector Green function $D_{jk'}(x;\ x')$:

$$L_{ij}{}^{ab}(x;q')=-\frac{4\pi e_ae_b}{c}\nabla_{[i}D_{j]k'}(x;x')u'^k, \qquad (5)$$

which satisfies the equation

$$g^{nm}\nabla_n\nabla_mD_{k'}{}^{l}-R^l_nD^{\bar n}_{k'}=\delta_k{}^l\frac{\delta^{(4)}(x;x')}{(-g(x))^{1/2}}. \qquad (6)$$

In relation to this definition of L^{ab}_{ik} it is necessary to note two important points. First, to construct a kinetic theory in an arbitrary gravitational field it is necessary to know exactly the retarded vector Green function in an arbitrary gravitational field. However, this function can be calculated in closed form only for restricted classes of spaces. In Ref. 18, the Green function obtained in Ref. 22 in the space of a plane gravitational wave was used for these purposes. Second, as pointed out in Ref. 23, the retarded Green function cannot be determined in the general case, in view of the effect of the inverse scattering on the curvature. Only the casual Green function can be correctly determined. However, in constructing the collision integral we shall use only local properties of the Green function. Locally, it is possible to determine the retarded Green function using the adiabatic-expansion method of DeWitt and Schwinger.[23]

In order to eliminate the self-action of the particles, we shall proceed as follows. We write the continuity equation (2) for one particle of type a:

$$\left\{u^i\tilde{\nabla}_i+\sum_c\int d^8q''L_{il}{}^{ac}(x;q'')u^lN_c'(q'')\frac{\partial}{\partial p_i}\right\}$$
$$\times\int\delta^{(8)}(q-q^a_{(j)}(s_j))ds_j=0, \qquad (7)$$

where

$$N_c'=\begin{cases}N_c, & c\neq a,\\ \sum\limits_{k\neq j}^{n_a}\int\delta^{(8)}(q-q_{(k)}(s_k))ds_k, & c=a.\end{cases} \qquad (8)$$

Next, summing (7) over j we obtain

$$u^i\tilde{\nabla}_iN_a+\sum_c\int d^8q''L_{il}{}^{ac}(x;q'')u^l\frac{\partial}{\partial p_i}(N_a(q)N_c'(q''))=0. \qquad (9)$$

We average this equation on a given hypersurface, then multiply by $N'_b(q')$, and again average.[17,18,20] Next, using the relations

$$\langle N_a(q)\rangle=n_aF_a(q),$$
$$\langle N_a(q)N_b'(q')\rangle=n_an_bF_{ab}(q;q'),$$
$$\langle N_a(q)N_b'(q')N_c(q'')\rangle=n_an_bn_cF_{abc}(q;q';q'')$$
$$+\delta_{bc}n_an_bF_{ac}(q;q'')\int ds''\delta^{(8)}(q''-q_{(b)}(s''|q')),$$

where $q_b(s''|q')$ is the trajectory of a particle of type b passing through the point q' at the moment of proper time $s''=0$, and using also the standard expansions of the distribution functions into a vacuum part and a correlation part

$$n_aF_a=f_a,$$
$$n_an_bF_{ab}=f_af_b+g_{ab}(q;q'),$$
$$n_an_bn_cF_{abc}=f_{(a}g_{bc)}+g_{abc}(q;q';q'')+f_af_bf_c,$$

we obtain a chain of equations, the first two of which have the form

$$u^i\tilde{\nabla}_if_a+\frac{e_a}{c}\langle F_{il}{}^a\rangle u^l\frac{\partial f_a}{\partial p_i}=-\sum_b\frac{\partial}{\partial p_s}J_s,$$
$$J_s=\int d^8q'L_{sl}{}^{ab}(x;q')u^lg_{ab}(q;q'), \qquad (10)$$

$$u^i\tilde{\nabla}_ig_{ab}+\frac{e_a}{c}F_{il}{}^{ab}u^l\frac{\partial g_{ab}}{\partial p_i}+\frac{e_a}{c}\langle F_{il}{}^a\rangle u^l\frac{\partial g_{ab}}{\partial p_i}$$
$$=-\int ds''L_{il}{}^{ab}(x;q_b(s''|q'))u^lf_b(q_b(s''|q'))\frac{\partial f_a}{\partial p_i}$$
$$-\frac{\partial f_a}{\partial p_i}\sum_c\int d^8q''L_{il}{}^{ac}(x;q'')u^lg_{bc}(q';q'')$$
$$-\sum_c\int d^8q''L_{il}{}^{ab}(x;q'')u^lg_{abc}(q;q';q''). \qquad (11)$$

It is obvious that, by making the replacements $a\leftrightarrow b$ and $q\leftrightarrow q'$ in the initial equation (7), we would have obtained an analogous equation, and it follows from this that the function g_{ab} satisfies Eq. (11) with the corresponding replacements. In Eqs. (10) and (11) we have used the following notation:

$$\langle F_{il}{}^a\rangle=\frac{c}{e_a}\sum_b\int d^8q'L_{il}{}^{ab}(x;q')f_b(q')$$

is the mean field acting on a particle of type a;

$$F_{il}{}^{ab}(x;q')=\frac{c}{e_a}\int ds''L_{il}{}^{ab}(x;q_b(s''|q'))$$

is the field created, at the position of a particle of type a, by a particle of type b moving along the trajectory $q_b(s''|q')$.

In the approximation of a rarefied plasma and pair collisions, it is possible to omit the second term on the left-hand side and the last term on the right-hand side of Eq. (11). The second term on the right-hand side of (11) takes into account the phenomenon of dynamical polarization of the plasma,[24] and leads in the nonrelativistic case to the Balescu–Lenard equation. We shall consider Eq. (11) in the Landau approximation, i.e., without allowance for the dynamical polarization of the plasma. The corresponding equation was obtained in the nonrelativistic case by Landau and in the case of the STR by Belyaev and Budker.[24] Thus, Eq. (11) takes the following form:

$$u^i\tilde{\nabla}_i g_{ab}(q;q')+\frac{e_a}{c}\langle F_{il}{}^a\rangle u^l\frac{\partial g_{ab}}{\partial p_i}$$
$$=-\int ds' L_{il}{}^{ab}(x;q_b(s'|q'))u^l f_b(q_b(s'|q'))\frac{\partial f_a}{\partial p_i}. \quad (12)$$

3. SOLUTION OF THE EQUATION FOR THE CORRELATION FUNCTION

In a plasma situated in a gravitational field, self-consistent electromagnetic fields can arise, and are taken into account by the term $\langle F^a_{il}\rangle$ in Eq. (12). In the case of a plane gravitational wave the self-consistent electromagnetic field was calculated in Ref. 25, while in the presence of a supplementary external magnetic field it was calculated in Ref. 26. We shall confine ourselves to the approximation in which the influence of the self-consistent electromagnetic field on the act of collision of the particles is neglected. This approximation will be valid for a homogeneous plasma in the field of a weak gravitational wave, since in this case, as shown in Ref. 25, the self-consistent field is of order h^2 (where h is the amplitude of the gravitational wave), and is therefore negligibly small.

Thus, the equation for the correlation function takes the following form:

$$u^i\tilde{\nabla}_i g_{ab}=-\int ds' L_{il}{}^{ab}(x;q_b(s'|q'))u^l f_b(q_b(s'|q'))\frac{\partial f_a}{\partial p_i}. \quad (13)$$

Equation (13) is easily solved by the method of characteristics. Taking into account that the function g_{ab} also satisfies Eq. (13) with the replacements $a\leftrightarrow b$ and $q\leftrightarrow q'$, the solution can be represented in the form

$$g_{ab}(q;q')=g_{ab}{}^0(q_a(s_0|q);\ q_b(s_0'|q'))+g_{ab}{}^+(q;q')+g_{ab}{}^-(q;q'),$$
$$g_{ab}{}^+(q;q')=-\int_{s_0}^{\bar{s}} ds\int_{-\infty}^{+\infty} ds' L_{il}{}^{ab}(x_a(s|q);q_b(s'|q'))$$
$$\times u^l(s|q)f_b(q_b(s'|q'))\frac{\partial f_a(q_a(s|q))}{\partial p_i(s|q)}, \quad (14)$$
$$g_{ab}{}^-(q;q')=g_{ab}{}^+(q;q')|_{a\leftrightarrow b,\ q\leftrightarrow q'},$$

where $q_a(s|q)$ is the solution of the geodesic equations

$$\frac{Du_a{}^i(s|q)}{ds}=0,\qquad u_a{}^k(s|q)=\frac{dx_a{}^k(s|q)}{ds},$$

which are characteristics of the kinetic operator $\hat{K}=u^i\tilde{\nabla}_i$. The quantity $\bar{s}$, which henceforth we shall set equal to zero, is determined by the relation $q_a(\bar{s}|q)=q$. When the self-consistent field is taken into account, the equations of a geodesic are replaced by equations of a world line

$$\frac{Du_a{}^i(s|q)}{ds}=\frac{e_a}{m_a c^2}\langle F_k{}^i\rangle u_a{}^k,\qquad u_a{}^k(s|q)=\frac{dx_a{}^k(s|q)}{ds}.$$

At this stage of the derivation of the collision integral on a flat background, the parameter s_0 in the solution (14) is usually allowed to go to minus infinity, and one uses Bogolyubov's principle of the weakening of correlations, i.e., one assumes that

$$g^0|_{s_0\to-\infty}\to 0.$$

The subsequent calculations of the collision integral can be performed in two ways. In the collision integral it is possible to cut off the limits of integration over r by the Debye radius at large r and by $r_{\min}$ at small r, but leave the interaction potential unscreened. This leads to the appearance of a Coulomb logarithm L (Ref. 24). In the opinion of Balescu,[27] it is more correct to replace the Coulomb potential $1/r$ by a screened potential $\exp(-r/r_D)/r$ both in the collision integral and in the correlation function, and to retain the cutoff only for small r. This leads to a small change of the Coulomb logarithm: L is replaced by $L-1/2$. One can also propose a third way of calculating, which we shall demonstrate here for the example of the derivation of the Belyaev–Budker collision integral.

In the case of a flat space-time the function L^{ab}_{il} has the following form:

$$L_{il}{}^{ab}(x;q')=-\frac{e_a e_b}{c}\partial_{[i}\delta(\sigma)\theta(\Delta x^4)u'_{l]}, \quad (15)$$

where $\sigma(x;x')=\frac{1}{2}(x-x')^2$ is the Synge "world" function[23] for a flat space-time. In order to integrate in (14) over s' we go over from the derivative with respect to $x^i(s|q)$ in (15) to the derivative with respect to s' and integrate by parts. Then,

$$g_{ab}{}^+=-\int_{s_0}^{0} ds\int_{-\infty}^{+\infty} ds'\frac{e_a e_b}{-c}\theta(\Delta x_T{}^4)\frac{d}{ds'}\left\{\frac{\sigma_{[i}u'_{l]}u^k}{(\sigma_{k'}u'^k)}\right\}$$
$$\times\delta(\sigma)f_b(q')\frac{\partial f_a(q)}{\partial p_i}. \quad (16)$$

Here, we have used the standard assumption that the change of the distribution function during the collision time is small. In order to integrate over s' in (16) it is necessary to solve the system of equations $\sigma=0$, $\Delta x^4_T\geqslant 0$. Since, in zeroth order in the interaction, the trajectories are straight lines, this system can be written in the following form:

$$\sigma(s;s')={}^1/_2(x^i-x'^i+u^i s-u'^i s')^2=0,$$
$$\Delta x_T{}^4=x^4-x'^4+u^4 s-u'^4 s'>0. \quad (17)$$

The system (17) has the unique solution

$$s'=(yu')+\omega s-[(\omega^2-1)s^2-2(y\xi)s+R_0'^2]^{1/2}, \quad (18)$$

where

$$y^i=x^i-x'^i,\quad \omega=(uu'),\quad \xi^i=u^i-\omega u'^i,\quad R_0'^2=(yu')^2-y^2.$$

The quantities ξ^i and R'_0 reduce, in the rest frame of the particle, which has velocity u_i', to the relative velocity and relative distance, respectively. Under the condition (17), the quantity $-\sigma_{k'}u'^k$ takes the form

$$-\sigma_{k'}u'^k=\{(\omega^2-1)s^2-2(y\xi)s+R_0'^2\}^{1/2}\equiv\rho.$$

It is not difficult to convince oneself that the quantity ρ has the meaning of the relative distance between particles moving along straight lines when the retardation of the interaction is taken into account ($\sigma=0$). In the nonrelativistic limit,

$$\rho=|\mathbf{x}+\mathbf{v}t-(\mathbf{x}'+\mathbf{v}'t)|.$$

Now, using the principle of weakening of correlations, we can find the parameter s_0 that appears in g_{ab}^+ by requiring that correlation be absent when the particles fly apart to a distance equal to the correlation length l (of the order of the Debye radius). Thus, now the parameter s_0 becomes a function of q and q', and is found from the relation

$$\rho=[(\omega^2-1)s_0^2-2(y\xi)s_0+R_0'^2]^{1/2}=l.$$

The solution satisfying the condition $s_0(1\to\infty)\to-\infty$ is easily found:

$$s_0=s_0(q;q';l)=\{(y\xi)-[(y\xi)^2-(\omega^2-1)(R_0'^2-l^2)]^{1/2}\}/(\omega^2-1),$$
$$\bar{s}_0=s_0(q\leftrightarrow q').$$

Here, it is important that, although s_0 is now a function of q and q', the correlation function g_{ab} again satisfies Eq. (13). Using the quantity R_0' defined above, we can specify the region of integration Ω in the collision integral in a relativistically invariant manner:

$$\Omega: r_{min}\leqslant R_0'=\rho|_{s=0}\leqslant l=r_D.$$

After substitution of the correlation function into (10) we obtain the Belyaev–Budker collision integral, in which, in place of the Coulomb logarithm L, we have $L+1/2-\ln 2$.

Now it is not difficult to formulate a method for determining s_0 in the general case. This quantity should satisfy the system

$$\begin{gathered}\sigma(x_a(s_0|q),\quad x_b(s_*'|q'))=0,\\ x_a^4(s_0|q)-x_b^4(s_*'|q')>0,\\ -\sigma_{k'}(x_a(s_0|q),\quad x_b(s_*'|q'))u'^k(s_*'|q')=l^2,\end{gathered}\tag{19a}$$

where $\sigma(x;x')$ is the world function for the specified gravitational field. In analogy with the plane case, it is possible to show that the function g_{ab} with s_0 from (19a) satisfies Eq. (13), since $u^i\tilde{\nabla}_i s_0=0$. The region of integration Ω' in the collision integral is specified by the relation

$$\Omega': r_{min}\leqslant-\sigma_{k'}u'^k|_{s=0,\,\sigma=0}\leqslant l.\tag{19b}$$

Since the conditions (19) fix a certain region (having size of the order of r_D) in which the particles interact, it is possible to use for the Green function the DeWitt–Schwinger expansion.[23] The adiabaticity condition used in the DeWitt–Schwinger method implies in the given case that the characteristic length scale of the variation of the gravitational field is much greater than r_D.

4. DETERMINATION OF THE KERNEL OF THE COLLISION INTEGRAL

In order to determine the kernel of the collision integral correctly, it is necessary to change the order of the integration in J_s (10). But, as was noted in Ref. 21, this cannot be done, because the phase space is a fiber bundle and it is necessary, therefore, to integrate first over a fiber $[p_i'(x')]$, and only then over the base (x'^i). This difficulty can be overcome if we construct some correspondence between different fibers. To this end, we parallel-transport the vectors $p_i'(x^i)$ from the fiber above x'^i along the geodesic to the point x^i. We also go over from the coordinates x'^i to Riemannian coordinates with origin at the point x^i (which are the covariant generalization of the difference $x^i-x'^i$). Thus, we obtain the following change of coordinates in the phase space:

$$\begin{gathered}x'^i\to y^i=\sigma^i(x;x'),\\ p_i'(x')=g_{i'}{}^k(x';x)p_k'(x),\end{gathered}\tag{20}$$

where $g_{i'}^k(x';x)$ is the bivector of the parallel transport along the geodesic; $\sigma_i=\nabla_i\sigma$. Using the relations given in Ref. 23 for the world function and parallel-transport bivector, it is not difficult to calculate the Jacobian of the transformations (20):

$$J=\frac{1}{\Delta(x;x')},$$

where $\Delta(x;x')$ is the van Vleck–Morette determinant. After these transformations, it is then possible to change the order of the integration in (13).

The expansion of the integrand in (14) in powers of s, s', and y^i is completely equivalent to the convolution of the expansions of the two quantities

$$Z_i=L_{jl}{}^{ab}(x_a(s|q);\ q_b(s'|q'))u^l(s|q)g_{\cdot i}{}^{j}(x_a(s|q);\ x),\tag{21a}$$

$$V^i=f_b(q_b(s'|q'))\frac{\partial f_a(q_a(s|q))}{\partial p_j(s|q)}g_j{}^{\cdot i}(x_a(s|q);x).\tag{21b}$$

This is a consequence of the covariant constancy of $g_i^j[x;\,x_a(s|q)]$ along the geodesic, and of the relation $g_i^{j'}g_{j'}^k=\delta_i^k$. This representation is convenient in that the quantities Z_i and V^i are vectors at the point x and scalars at the point $x_a(s|q)$, and therefore the integration of these quantities separately is correct (only the scalars can be integrated). Analogously, for g_{ab}^-,

$$\begin{gathered}\bar{Z}_i=L_{jl}{}^{ba}(x_b(s|q'),q_a(s'|q))u'^l(s|q')\\ \times(g_{\cdot k'}^{j}(x_b(s|q');x')g_{\cdot i}^{k'}(x';x)),\end{gathered}\tag{22a}$$

$$\begin{gathered}\bar{V}^i=f_a(q_a(s'|q))\frac{\partial f_b(q_b(s|q'))}{\partial p_j'(s|q')}\\ \times(g_j{}^{\cdot k'}(x_b(s|q');x')g_{k'}{}^{\cdot i}(x';x)).\end{gathered}\tag{22b}$$

The expansions of V^i and $\bar{V}^i$ in powers of s, s', and y^i are the covariant generalizations of the expansions in the STR:

$$\begin{gathered}V^i=\{f_b(x;p'(x))-y^l\tilde{\nabla}_l'f_b+s'u'^l(x)\tilde{\nabla}_l'f_b+\dots\}\\ \times\left\{\frac{\partial f_a(q)}{\partial p_i}+su^l\tilde{\nabla}_l\frac{\partial f_a}{\partial p_i}+\dots\right\},\end{gathered}\tag{23}$$

$$\bar{V}^i=\{f_a(q)+u'^l\tilde{\nabla}_l f_a s+\ldots\}$$
$$\times\left\{\frac{\partial f_b(x;p'(x))}{\partial p_i'(x)}-y^l\tilde{\nabla}_l'\frac{\partial f_b}{\partial p_i'}+u^l\tilde{\nabla}_l'\frac{\partial f_b}{\partial p_i'}s+\ldots\right\}.$$

In taking the self-consistent field into account it is necessary everywhere to replace the bivector of parallel transport along the geodesic by the bivector of parallel transport along the world line, and, in (23), to replace

$$\tilde{\nabla}_l\to\tilde{\nabla}_l+\frac{e_a}{c}\langle F_{il}{}^a\rangle\frac{\partial}{\partial p_i}.$$

The contribution of the first-order terms in the expansion (23) in the plane nonrelativistic case was taken into account by Klimontovich.[28] It was shown that these terms lead to allowance for fluctuations in the energy-conservation law. We shall take into account only the zeroth-order terms in (23), and shall expand the quantities Z_i and $\bar{Z}_i$. For the subsequent calculations it is convenient to go over to orthogonal-basis components of the vectors y^i, p_i', and p_i. In order not to encumber the notation, henceforth, by convention, we shall take Latin indices to be basis indices.

Thus, the collision integral takes the form

$$J_s=-\int d^4p'(x)\left\{M_{is}{}^{ab}\frac{\partial f_a}{\partial p_i}f_b(x;p'(x))\right.$$
$$\left.+M_{is}{}^{ba}\frac{\partial f_b(x;p'(x))}{\partial p_i'(x)}f_a(q)\right\},\qquad(24)$$

where the kernels are

$$M_{is}{}^{ab}=\int_{\Omega'}\frac{d^4y}{\Delta}L_{sl}{}^{ab}(x;q')u^l\int_{s_0}^{0}ds\int_{-\infty}^{+\infty}ds'Z_i,\qquad(25a)$$

$$M_{is}{}^{ba}=\int_{\Omega'}\frac{d^4y}{\Delta}L_{sl}{}^{ab}(x;q')u^l\int_{\bar{s}_0}^{0}ds\int_{-\infty}^{+\infty}ds'\bar{Z}_i.\qquad(25b)$$

5. EXPANSION OF THE COLLISION INTEGRAL FOR VACUUM SPACES

For vacuum spaces ($R_{ik}=0$), to within the first power of the Riemann tensor, we have for the Green function the expression

$$D_{ik'}(x;x')=-\frac{1}{4\pi}\theta(\Delta x^4)\delta(\sigma)g_{ik'}(x;x'),\qquad(26)$$

since $\Delta=1+o^2$ (Ref. 23). To shorten the expressions, we take it henceforth that the free indices r and t are being displaced with velocities u^r and u^t, n and m are being displaced with velocities u'^n and $u'^m(x)$, and p and q are being displaced with velocities y^p and y^q. Using the known relations for the limits of coincidence of derivatives of σ (Ref. 23), we can obtain an expansion of the quantities appearing in (25):

$$\beta_i\equiv\sigma_j(x(s|q);x'(s'|q'))g_{.i}{}^j(x(s|q);x)$$
$$=\{y_i+u_is-{}^1/_3R_{iprq}s+{}^1/_6R_{irpt}s^2\}$$
$$-\{u_i'(x)+{}^1/_6R_{ipnq}-{}^1/_3(R_{inrp}+R_{iprn})s$$
$$+{}^1/_6R_{irnt}s^2\}s'+\{-{}^1/_3R_{inpm}(y^p+u^ps)\}s'^2,\qquad(27a)$$

$$\nu_i\equiv g_{jk'}(x(s|q);x'(s'|q'))u'^k(s'|q')$$
$$\times g_i{}^j(x(s|q);x)=\{u_i'(x)-{}^1/_2R_{inrp}s\}$$
$$+\{{}^1/_2R_{inpm}(y^p+u^ps)s'\},\qquad(27b)$$

$$C_i=\nabla_{[j}g_{t]k'}(x(s|q);x'(s'|q'))u^t(s|q)$$
$$\times u'^k(s'|q')g_{.i}{}^j(x(s|q);x)={}^1/_2R_{irnp}(y^p+u^ps),\qquad(27c)$$
$$\sigma(x(s|q);x(s'|q'))={}^1/_2\beta_i\beta^i,$$
$$s_0=s_0{}^0+\delta s_0,$$
$$s_0{}^0=\{(y\xi)-[(y\xi)^2-(\omega^2-1)(R_0'^2-l^2)]^{1/2}\}/(\omega^2-1),$$
$$\delta s_0=\{(yu'+\omega s)(R_{rpnq}s_0{}^0-2R_{rntp}s_0{}^{02})\qquad(27d)$$
$$+(y^2+2(yu)s_0{}^0+s_0{}^{02})R_{npmq}(y^p+u^ps_0{}^0)$$
$$\times(y^q+u^qs_0{}^0)+R_{rptq}s_0{}^{02}\}/6[(y\xi)^2-(\omega^2-1)(R_0'^2-l^2)]^{1/2}.$$

The region of integration Ω' is specified by the relation

$$\Omega':r_{min}\leqslant[(yu')^2-y^2+{}^1/_3R_{npmq}y^2]^{1/2}\leqslant r_D.\qquad(27e)$$

There exists a change of coordinates that reduces the region Ω' to the region Ω:

$$y^i\Rightarrow y^i+{}^1/_6y^2R^i_{.npm}.\qquad(27f)$$

Using (26), we can show that the kernels M_{is} satisfy the relations

$$M_{is}{}^{ab}u^i=0,\quad M_{is}{}^{ab}u^s=0,$$
$$M_{is}{}^{ba}u'^i(x)=0,\quad M_{is}{}^{ba}u^s=0.\qquad(28)$$

Since the expansions (27) contain only the Riemann tensor, we can immediately write out a possible structure for M_{is}:

$$M_{is}{}^{ab}=\frac{e_a{}^2e_b{}^2}{c^2}\left\{\frac{2\pi\omega^2}{(\omega^2-1)^{1/2}}\mathscr{E}_{is}+W_{is}{}^{ab}\right\},\qquad(29a)$$

$$M_{is}{}^{ba}=\frac{e_a{}^2e_b{}^2}{c^2}\left\{-\frac{2\pi\omega^2}{(\omega^2-1)^{1/2}}\mathscr{E}_{is}+W_{is}{}^{ba}\right\},\qquad(29b)$$

where the first term is the Belyaev–Budker kernel:

$$\mathscr{E}_{is}=P_{is}+\xi_i'\xi_s'/(\omega^2-1),\quad P_{is}=g_{is}-u_iu_s,$$
$$\xi_i'=u_i'(x)-\omega u_i,\quad \omega=(u(x)u'(x)),\qquad(30)$$
$$W_{is}{}^{ab}=\Phi_1R_{irst}+\Phi_2R_{insm}+\Phi_3R_{irsn}+\Phi_4R_{insr}+R_{irnt}(\Phi_5{}^1u_s+\Phi_5{}^2u_s')$$
$$+R_{inrm}(\Phi_6{}^1u_s+\Phi_6{}^2u_s')+R_{srnt}(\Phi_7{}^1u_i+\Phi_7{}^2u_i')$$
$$+R_{snrm}(\Phi_8{}^1u_i+\Phi_8{}^2u_i')+R_{nrmt}(\Phi_9{}^1g_{is}$$
$$+\Phi_9{}^2u_iu_s+\Phi_9{}^3u_i'(x)u_s'(x)+\Phi_9{}^4u_iu_s'+\Phi_9{}^5u_i'u_s).\qquad(31)$$

The quantity W_{is}^{ba} will have an analogous structure. Using the relations (28), we can reduce the number of unknown coefficients from 17 to 10. Then W_{is} can be represented in the form

$$W_{is}{}^{ab}=\{\Phi_1R_{lrkt}+\Phi_2R_{lnkm}+\Phi_3R_{lrkn}+\Phi_4R_{lnkr}\}P_i{}^lP_s{}^k$$
$$+\{\Phi_5{}^2R_{lrnt}+\Phi_6{}^2R_{lnrm}\}\xi_0{}^sP_i{}^l+\{\Phi_7{}^2R_{lrnt}+\Phi_8{}^2R_{lnrm}\}\xi_i'P_s{}^l$$
$$+R_{nrmt}\left\{\Phi_9{}^1\mathscr{E}_{is}+\frac{1}{\omega^2}\left(\Phi_9{}^2-\frac{1}{\omega^2-1}\Phi_9{}^1-\omega\Phi_6{}^2\right.\right.$$
$$\left.\left.-\omega\Phi_8{}^2-\Phi_2\right)\xi_i'\xi_s'\right\},\qquad(32a)$$

$$W_{is}{}^{ba}=\{\Phi_1R_{lrkt}+\Phi_2R_{lnkm}+\Phi_3R_{lrkn}+\Phi_4R_{lnkr}\}P_i'{}^lP_s{}^k$$
$$+\{\Phi_5{}^2R_{lrnt}+\Phi_6{}^2R_{lnrm}\}\xi_s'P_i'{}^l+\{\Phi_7{}^1R_{lrnt}+\Phi_8{}^1R_{lnrm}\}\xi_iP_s$$
$$+R_{nrmt}\left\{\Phi_9{}^1\mathscr{E}_{is}-\frac{1}{\omega}\left(\Phi_9{}^2-\frac{1}{\omega^2-1}\Phi_9{}^1\right)\xi_i\xi_s'\right\},\qquad(32b)$$

where

$$P_{ks}'=g_{ks}-u_k'u_s',\quad \xi_i=u_i-\omega u_i'(x).$$

In order to calculate the coefficients that appear in (32), it is necessary to substitute the expansions (27) into

(25) and integrate. In the integration, the need arises to calculate the tensors

$$\mu_{i_1\ldots i_n}=\int_\Omega d^4y\frac{d\delta(^1/_2y^2)}{d(^1/_2y^2)}fq'_{i_1}\ldots q'_{i_n}, \tag{33}$$

where $q'_i=\omega y_i-u'_i(x)(yu)$, to fourth order; in these tensors, f depends on g_{ik}, u_i, $u'_i(x)$, and y^i. The tensors (33) can be calculated simply to arbitrary order, using the following procedure. From the vectors y^i, u^i, and u'^i we construct a vector θ_i that satisfies the relations $\theta_iu^i=\theta_iu'^i=0$:

$$\theta_i=y_i+u_i.\frac{(y\xi)}{a}+u_i'\frac{(y\xi')}{a},\qquad a=\omega^2-1. \tag{34}$$

From this, we express y^i as

$$y_i=\theta_i-u_i\frac{(y\xi)}{a}-u_i'\frac{(y\xi')}{a}$$

and substitute this, in q'_i, into (33). Then the problem of calculating the quantities (33) reduces to the problem of calculating the tensors

$$\mu'_{i_1\ldots i_n}=\int_\Omega d^4yF\theta_{i_1}\ldots\theta_{i_n},$$

where

$$F=\frac{f\,d\delta(^1/_2y^2)}{d(^1/_2y^2)}.$$

Taking into account that the tensor $\mu'_{i_1\ldots i_n}$ gives zero when contracted with any velocity and depends only on g_{ik}, u_i, and u'_i, we can obtain

$$\begin{aligned}&\mu'_{i_1\ldots i_{2k+1}}=0,\\&\mu_{ik}'=^1/_2\mathscr{E}_{ik}\hat\alpha(F),\\&\mu'_{iklj}=^1/_8\mathscr{E}_{i(k}\mathscr{E}_{lj)}\hat\beta(F),\end{aligned} \tag{35}$$

where the operators $\hat\alpha$, $\hat\beta$, and $\hat\gamma$ are given by

$$\begin{aligned}&\hat\alpha(F)=\int_\Omega d^4yF\theta^2,\\&\hat\beta(F)=\int_\Omega d^4yF\theta^4,\\&\gamma(F)=\int_\Omega d^4yF.\end{aligned} \tag{36}$$

The tensors $\mu_{i_1\ldots i_n}$ are then easily calculated:

$$\mu_i=\xi_i'\hat\gamma\left(F\frac{y\xi}{a}\right),$$

$$\mu_{ik}=\frac{\omega^2}{2}\hat\alpha(F)\mathscr{E}_{ik}+\xi_i'\xi_k'\hat\gamma\left(F\frac{y\xi^2}{a^2}\right),$$

$$\mu_{ikl}=\frac{\omega^2}{2}\hat\alpha\left(F\frac{y\xi}{a}\right)\mathscr{E}_{(ik}\xi_{l)}'+\gamma\left(F\frac{y\xi^3}{a^3}\right)\xi_i'\xi_k'\xi_l',$$

$$\mu_{iklj}=\frac{\omega^4}{8}\hat\beta(F)\mathscr{E}_{i(k}\mathscr{E}_{lj)}+\frac{\omega^2}{2}\hat\alpha\left(F\frac{y\xi^2}{a^2}\right)\{\mathscr{E}_{i(k}\xi_l'\xi_{j)}'$$
$$+\xi_i'\mathscr{E}_{(kl}\xi_{j)}'\}$$
$$+\hat\gamma\left(F\frac{y\xi^4}{a^4}\right)\xi_i'\xi_k'\xi_l'\xi_j'.$$

The calculations of the coefficients in W_{is} are conveniently performed in the rest frame of the particle, which has velocity u'^i (a boost along a basis index). In this coordinate frame we have $\theta^2=r^2(\cos^2\theta'-1)$, where θ' is the polar angle.

After tedious calculations, in the nonrelativistic limit we obtain

$$W_{is}{}^{ab}=(2\Phi_5{}^2)\xi_s'R_{irnt}u^ru'^nu^t+(2\Phi_7{}^2)\xi_i'R_{srnt}u^ru'^nu^t$$
$$+R_{rltk}u^ru^t\xi^l\xi^k\left(-\Phi_7{}^2\mathscr{E}_{is}+\frac{1}{a}\cdot2\Phi_5{}^2\xi_i'\xi_s'\right), \tag{37a}$$

$$W_{is}{}^{ba}=-(2\Phi_5{}^2)\xi_s'R_{imnt}u'^mu'^nu^t+(2\Phi_7{}^2)\xi_iR_{srnt}u^ru'^nu^t$$
$$+R_{rltk}u^ru^t\xi^l\xi^k\left(\Phi_7{}^2\mathscr{E}_{is}+\frac{1}{a}\cdot2\Phi_5{}^2\xi_i\xi_s'\right), \tag{37b}$$

where

$$\Phi_5{}^2=\frac{\pi r_D{}^2\omega^2}{12a^{5/2}}(1-8\ln2),\qquad\Phi_7{}^2=\frac{\pi r_D{}^2\omega^2}{24a^{5/2}}(1+8\ln2), \tag{38}$$
$$a=\omega^2-1.$$

It is not difficult to check that the kernels (37) satisfy the relations (28). Going over, finally, from the basis components of the vectors $p'_i(x)$ and p_i to the vectors $p'_i(x)$ and p_i, we obtain the following expression for the collision term of the kinetic equation:

$$J_s=\frac{e_a{}^2e_b{}^2}{c^2}\int\frac{d^4p'(x)}{(-g(x))^{1/2}}\left\{2\pi L\frac{\omega^2}{(\omega^2-1)^{1/2}}\mathscr{E}_{is}\left(\frac{\partial f_a}{\partial p_i'}f_b(x;p'(x))\right.\right.$$
$$\left.\left.-\frac{\partial f_b}{\partial p_i'}f_a(q)\right)+W_{is}{}^{ab}\frac{\partial f_a}{\partial p_i}f_b+W_{is}{}^{ba}\frac{\partial f_b}{\partial p_i'}f_a\right\}. \tag{39}$$

In this expression the kernels W_{is} are given by Eqs. (37) and (38), in which we have introduced the notation

$$\xi_i=u_i-\omega u_i'(x),\quad\xi_i'=u_i'-\omega u_i,\quad\omega=(u(x)u'(x)).$$

6. CONCLUSION

In conclusion, we shall summarize the results of the paper. We have obtained a collisional kinetic equation on the background of vacuum spaces to within terms linear in the Riemann tensor. The appearance of terms with curvature in the collision integral[39] is due, on the one hand, to allowance for the influence of the gravitational field on the elementary act of collision of two particles (deviation of the geodesics), and, on the other, to the influence of the gravitational field on the dynamics of the electromagnetic interaction of the particles with each other. As is well known,[29] the Coulomb potential of a charged particle in the neighborhood of its world line in an external gravitational field contains extra terms that depend on the curvature.

It should be noted that, so far as the author is aware, in the literature there is no correct derivation of the collision integral in a gravitational field, while in papers on general-relativistic kinetics (see, e.g., Ref. 26 and the references therein) a naive covariant generalization of the special-relativistic Belyaev–Budker collision integral is used. A similar generalization, which fails completely to take into account the specifics of the general-relativistic phase space, has also been used in Kandrup's paper.[2] The technique that we propose (Sec. 4) for identifying the kernel makes it possible to construct the collision integral in a gravitational field in a systematic way. The expression obtained for the collision integral reduces to the correctly generalized Belyaev–Budker integral with neglect of the variation of the gravitational field in the 4-dimensional collision region. It should also be noted that the collision integral (39) that we have constructed, like the Belyaev–Budker integral, has been obtained in the Landau approximation, i.e., without allowance for the dynamical polarization of the plasma. The next stage in complexity would be the derivation of the collision equation analogous to the Balescu–Lenard equation.

Comparison of the kernels linear in the curvature with the Belyaev–Lenard kernel shows that the condition for validity of the expansion used in the derivation of the collision integral has the following form:

$$h\frac{r_D^2}{\lambda_g^2} \ll \frac{v_T^2}{c^2} \equiv h\frac{\nu_g^2}{\nu_p^2} \ll 1, \quad (40)$$

where h, λ_g, and $\nu_g = c/\lambda_g$ are the characteristic amplitude, characteristic length scale, and characteristic frequency of the gravitational field; v_T and ν_p are the thermal velocity and plasma frequency. The condition (40) coincides exactly with the condition for smallness of the tidal deviation in the 4-dimensional collision region.

The kernels W_{is} linear in the Riemann tensor do not satisfy all the relations that are satisfied by the Belyaev-Budker kernel, but only the relations (28). An analogous situation arises in the construction of the theory of a nonideal plasma[28] and in allowance for the influence of an external electric field on the act of collision of particles.[27] The hydrodynamic equations obtained from the kinetic equation with the collision integral (39) will differ, because of the presence of the terms linear in the curvature, from the standard hydrodynamic equations. At the same time, it is known that calculations for gravitational-wave detectors are, as a rule, performed in the hydrodynamic approximation. Therefore, as one of the most important applications of the kinetic equation obtained in the present paper we may point to its possible use in the description of gravitational-wave experiments. Of course, this problem requires a separate investigation.

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