Static properties and small-amplitude-oscillation spectrum of 2π solitons of the discrete double sine-Gordon equation

V. V. Beloshapkin, G. P. Berman, A. G. Tret'yakov, and E. V. Shtukkert

Institute of Physics, Siberian Branch, Academy of Sciences of the USSR
(Submitted 7 December 1990)
Zh. Eksp. Teor. Fiz. **100**, 1238–1246 (October 1991)

The stability of 2π solitons of the discrete double sine-Gordon equation has been studied by numerical analysis. The width of the soliton decreases abruptly as the parameter H (the strength of the external field) is increased. The field dependence of the gap in the small-oscillation spectrum is found. The frequency of the local oscillation in the width of the 2π soliton is a nonmonotonic function of the strength of the external field. The small-oscillation spectrum contains a band not found in the continuum approximation. This additional band disappears in the transition to the continuum approximation. The properties of the localization of small oscillations in a chain with 2π structural solitons as the parameter H is varied were studied. It is possible to control the filtering properties of 2π solitons by means of an external field.

1. INTRODUCTION

Nonlinear effects manifested by self-localized states (static and dynamic solitons) which are solutions of the equations of motion of a many-particle system have been the subject of active research over the past decade in solid state physics. One-dimensional models are convenient for treating nonlinear interactions of arbitrary strength most comprehensively. In several cases, such models make it possible to derive analytic results with relatively little computational difficulty.

Depending on the nature of the interparticle interaction, a classical spin chain has both regular and stochastic properties. Solitons in such models have been the subject of several studies.[1–6] For example, the properties of a spin chain in the limit of strong in-plane anisotropy are described in the continuum limit by the sine-Gordon equation.[1] When the discrete nature of the system is taken into account, some new physical results, which do not follow from the sine-Gordon model, are found. In particular, the discrete nature leads to pinning of solitons, a change in the spectrum of excitations,[2] and the formation of an amorphous structure.[3] The local properties of dynamic nonlinear excitations were studied in Ref. 4 in a continuum model of a uniaxial ferromagnet. A numerical study of the dynamics of solitons of the discrete double sine-Gordon equation was carried out in Ref. 5 for various values of the parameters of the system. The thermodynamic properties of a magnetic chain describable by the double sine-Gordon equation were studied analytically (in the continuum approximation) and numerically in Ref. 6. The nonlinear theory of magnetic excitations in magnetically ordered media was reviewed in Ref. 7 for spaces of various dimensionalities and for various types of magnetic interaction.

The stability of solitons and the excitation spectrum in a biaxial magnetic material were studied in Ref. 8 in the long-wavelength approximation. The polarization of the polymer polyvinylidene fluoride (PVF_2) by an electric field was studied in Ref. 9. A phenomenological model whose dynamics is described by the double sine-Gordon equation was used to describe this process.

In the present paper we report a numerical analysis of the static and dynamic properties of the soliton states of the discrete double sine-Gordon equation and of the conditions for the localization of small oscillations in the soliton-pinning region.

2. DESCRIPTION OF THE MODEL

We consider the model Hamiltonian

$$\mathscr{H}=\sum_n \dot{\varphi}_n^2/2-J_0\sum_n \cos(\varphi_{n+1}-\varphi_n)$$
$$-A_0\sum_n \cos^2\varphi_n-H_0\sum_n \cos\varphi_n, \quad (1)$$

where J_0, A_0, and H_0 are parameters of the model. From the conditions for an equilibrium of the system, $\partial\mathscr{H}/\partial\varphi_n=0$, with the Hamiltonian (1), we find a discrete analog of the double sine-Gordon equation:

$$\sin(\varphi_{n+1}-\varphi_n)-\sin(\varphi_n-\varphi_{n-1})=A\sin 2\varphi_n+H\sin\varphi_n, \quad (2)$$

where we have introduced the dimensionless parameters

$$A=A_0/J_0, \quad H=H_0/J_0. \quad (3)$$

Equations (2) were used in the continuum approximation to describe structural solitons in anisotropic quasi-one-dimensional magnetic materials in Refs. 1–4, 7, and 8. In this case the J_0 represents the exchange constant, A_0 represents the magnitude of the anisotropy, and H_0 represents the external magnetic field. A one-dimensional model with the Hamiltonian (1) was used in Ref. 9 to describe the change in polarization due to the motion of solitons in the polymer PVF_2, in which a dipole moment **p** is associated with each CF_2–CF_2–CF_2 monomer, and the angle φ determines the orientation of this moment. In this case the constant A_0 is related to the interaction between chains, while H_0 is related to both the interaction between chains and the interaction with the external electric field. The magnitude of the constant H_0 may vary with the external electric field. In Ref. 9, and also in Ref. 5, a study was made of the dynamic properties of solitons in the discrete model (1).

 0038-5646/91/100683-05$03.00

The dynamic and static properties of solitons of the double sine-Gordon equation and the small-oscillation spectrum were studied in detail in the continuum approximation in Refs. 1, 2, 6, and 10. In the discrete case, however, several new properties arise, in both the structure of the solitons and the small-oscillation spectrum of soliton structures.

The values given for the constants A_0 and J_0 in Ref. 9 indicate that the effects of discretization must be taken into account in describing soliton configurations of model (1) when that model is applied to the polymer PVF_2. It follows from Ref. 9 that the value of the dimensionless anisotropy constant is $A = 0.6$ in this case. It follows from Ref. 3 that this is a region where the solitons are strongly pinned and the discretization of the sysem has a substantial effect on the properties of the solitons. We will be discussing these points in the following sections of this paper.

3. STABILITY OF 2π SOLITONS

In the continuum approximation, Eqs. (2) become the double sine-Gordon equation:

$$\frac{d^2\varphi}{dz^2} - A\sin 2\varphi - H\sin\varphi = 0. \quad (4)$$

The solutions of Eq. (4) which describe 2π-soliton configurations are[4–6,8,10]

$$\varphi(z) = 2\,\mathrm{arctg}\{(1-2\eta)^{1/2}\operatorname{cosech}[(1-2\eta)^{1/2}s]\},$$
$$\eta = -\frac{A}{H}, \qquad s = H^{1/2}z. \quad (5)$$

The stability of the solutions (5) in the continuum approximation was studied in Refs. 6 and 10. The following numerical simulation was carried out to learn about the stability of 2π solitons in the discrete model (2) as a function of the parameter H (the external magnetic or electric field). A relaxation method[11–13] was used to determine the configurations of the system (1) which are stable in the static case. According to that method, the stable equilibrium structures are found from the solutions of the equations

$$\dot{\varphi}_i = -\frac{\partial \mathscr{H}}{\partial \varphi_i}, \qquad i = 1 - N. \quad (6)$$

As $t \to \infty$, the functions $\varphi_i(t)$ describe a stable configuration of Eqs. (1). The integration of Eqs. (6) is terminated when the quantity

$$T = (1/N)\Sigma_i|\partial\mathscr{H}/\partial\varphi_i|,$$

becomes sufficiently small (in most cases, 10^{-14}–10^{-15}). The specification of various sets of initial conditions $\{\varphi_i(0)\}$ determines various stable configurations.

Equations (6) have been used to find a stable 2π soliton in a chain of 40 particles with the parameter values $A = 0.5$ and $H = 0.05$. The resulting structure is shown by the solid line in Fig. 1. The quantity plotted along the y axis is $I(n) = \sin(\varphi_{n+1} - \varphi_n)$. The small-oscillation spectrum is found from the equation

$$|\mathscr{H}_{nn'} - W^2\delta_{nn'}| = 0, \quad (7)$$

where

$$\mathscr{H}_{nn'} = \frac{\partial^2\mathscr{H}}{\partial\varphi_n\,\partial\varphi_{n'}}.$$

As the field H is increased to a certain $H^{(1)}$, we observe "softening" (at $H = H^{(1)}$) of the frequencies of the natural modes which are localized at the soliton and which belong to the lower band of the small-oscillation spectrum (7) (these are oscillations of the soliton as a whole and of its width). The structure of the small-oscillation spectrum is described in more detail in the following section of this paper.

For $W^2 \simeq 0$, the soliton undergoes restructuring, in the direction of a decrease in its width. The result is the formation of a stable 2π soliton of smaller width and with a nonzero gap in the small-oscillation spectrum. A further increase in the field leads to more softening of the oscillations which are localized at the soliton. At a certain $H = H^{(2)}$, the width of the soliton again changes abruptly, etc. In our numerical simulation we observed several such jumps in the width of the 2π soliton.

Figure 1 shows a sample sequence of structures observed in this simulation. The particular sequence of equilibrium stable structures shown in Fig. 1 was found under the assumption that there is fast relaxation to the equilibrium position in the system. In the continuum approximation we do not find the soft modes, the nonmonotonic dependence of the frequency of the internal oscillation of a 2π soliton on the field H, and the abrupt change in the width of solitons.[10] These differences from the continuum approximation stem directly from the pinning of solitons in the discrete model. As $A \to 0$, the pinning of the solitons disappears, and along

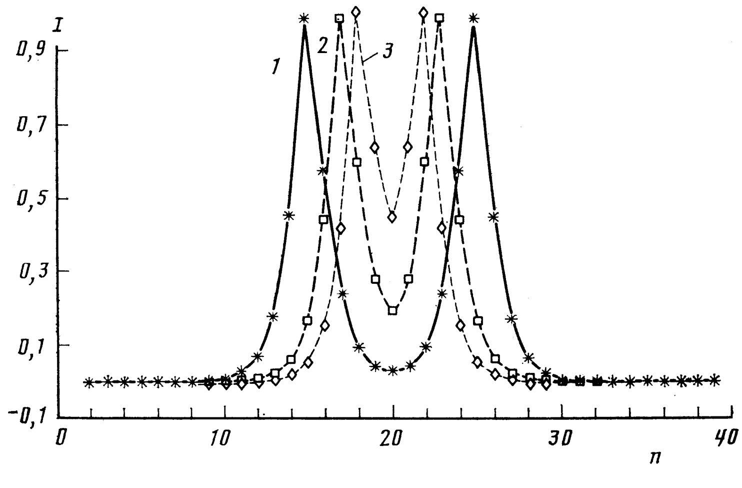

FIG. 1. Sequence of soliton structures found in a numerical simulation with $A = 0.5$ as the field H was raised. 1—$H = 0.5$; 2—0.7; 3—0.12.

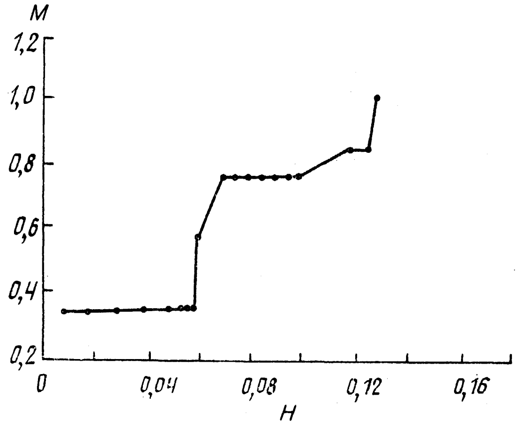

FIG. 2. The "magnetization" M versus the field strength H at $A = 0.5$.

with it all the differences from the results of the continuum description disappear. We found that the field dependence of the gap g in the small-oscillation spectrum as $H \to H^{(1)}$ has the form of a power law:

$$g = [H^{(1)} - H]^{\alpha}, \quad \alpha = 0{,}18. \tag{8}$$

We define the parameter M by

$$M = \frac{1}{N} \sum_{n=1}^{N} \cos \varphi_n. \tag{9}$$

This parameter serves as an analog of the magnetization in the case of a magnetic chain or of the polarization in polymer chains. Figure 2 shows M as a function of the applied field, H. We see that the behavior of the "magnetization" of the chain as a function of the field is not monotonic; the change is abrupt and is determined by the abrupt nature of the change in the width of the soliton with increasing H. The 2π solitons and 2π antisolitons consist of two π solitons and two π antisolitons, respectively. The distinction between a soliton and an antisoliton is that for solitons we have $I(n) > 0$ for all n, while for antisolitons we have $I(n) < 0$. In a discrete medium, the 2π solitons and 2π antisolitons may be accompanied by bound pairs of a π soliton with a π antisoliton. We have shown numerically that there can exist a stable random sequence consisting of 2π solitons and 2π antisolitons.

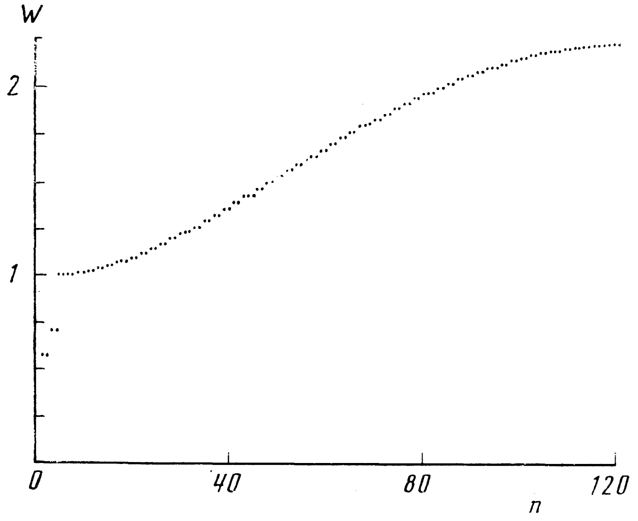

FIG. 4. Spectrum of small oscillations in the chain with the 2π soliton shown in Fig. 3a.

4. PROPERTIES OF THE SMALL-AMPLITUDE SPECTRUM IN A CHAIN WITH 2π SOLITONS

To study the small-amplitude spectrum, we consider a chain of 120 particles with a single 2π soliton. This structure was found by means of the relaxation equations (6) with $A = 0.5$ and $H = 0.01$; it is shown at the top in Fig. 3.

Figure 4 shows the small-oscillation spectrum found for this structure from expression (7). This spectrum consists nominally of three bands. The frequencies of the lower band correspond to localized symmetric and antisymmetric oscillations of π solitons forming a 2π soliton. The eigenvectors (discrete eigenfunctions) of the oscillations of this band are shown at the top in Fig. 5 a and b. These are oscillations of the 2π soliton as a whole and of its width. The central band is

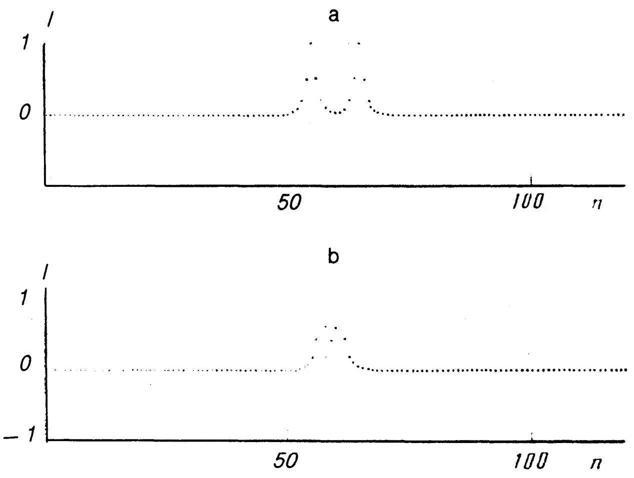

FIG. 3. Chain of 120 particles with a 2π soliton. a—$A = 0.5$, $H = 0.01$; b—$A = 0.5$, $H = 0.12$.

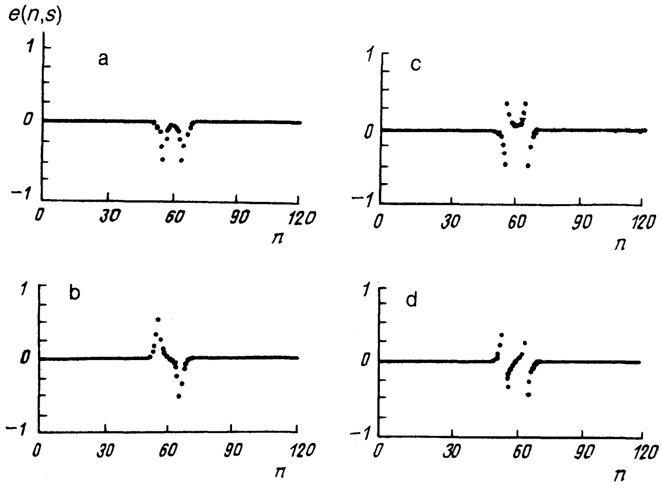

FIG. 5. The eigenvectors $e(n,s)$ corresponding to localized oscillations of the two lower bands of the spectrum (Fig. 4). a—$s = 1$; b—2; c—3; d—4.

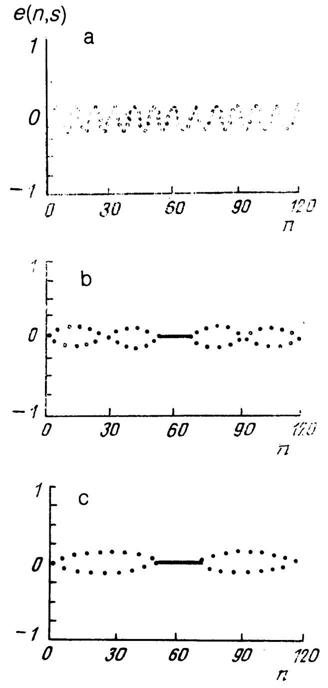

FIG. 6. Eigenvectors $e(n,s)$ of normal oscillations of the upper band of the spectrum (Fig. 4). a—$s = 30$; b—118; c—120.

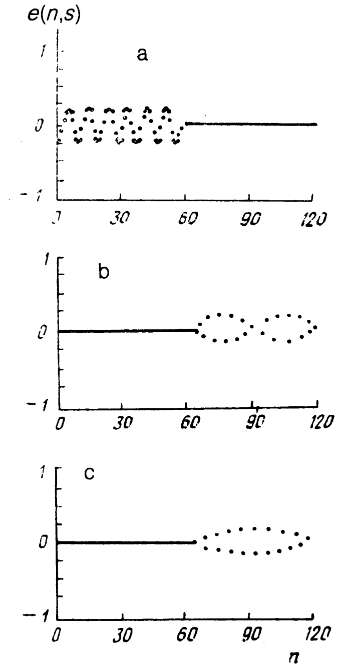

FIG. 7. Eigenvectors $e(n,s)$ of normal oscillations for a chain with 2π solitons with $A = 0.5$ and $H = 0.12$. a—$s = 30$; b—118; c—120.

a band of antisymmetric localized oscillations of π solitons forming a 2π soliton. The eigenvectors of these oscillations are shown in the lower part of Fig. 5 c and d. These are localized oscillations in the width of π solitons forming a 2π soliton. This oscillation band is present if the parameter A is sufficiently large, and effects of the discrete nature are important. As we go to the limit of a continuous medium ($A \to 0$), the oscillations of this band become collectivized, and the band coalesces with the upper band. At small values of H, the frequencies of the upper band correspond to a band of collective oscillations. The low-frequency oscillations of this spectral band constitute coupled oscillations of the soliton and of the rest of the chain (the 30th eigenvector in Fig. 6), while the high-frequency oscillations (the upper part of this band) propagate throughout the chain without interacting with the soliton. The transmission coefficient for the transmission of such oscillations through the soliton is close to unity.

We also observed the interesting result that it is possible to control the coefficient for the transmission of linear excitations through a 2π soliton by means of an external field. As the field H is raised, for example, the 2π soliton becomes narrower (see the preceding section of this paper). This narrowing leads to a change in the transmission of the soliton. As the field H is raised to $H = 0.12$, we obtain the structure shown at the bottom in Fig. 3. The transmission of the soliton has changed radically. While at $H = 0.01$ all oscillations of the upper part of the spectrum passed freely through the soliton, at $H = 0.12$ none of these oscillations passes through the soliton.

Figure 7 shows the same eigenvectors as in Fig. 6. The oscillations corresponding to the lower bands of the spectrum remain localized. In contrast with Fig. 6a, however, none of the oscillations of the upper band passes through the

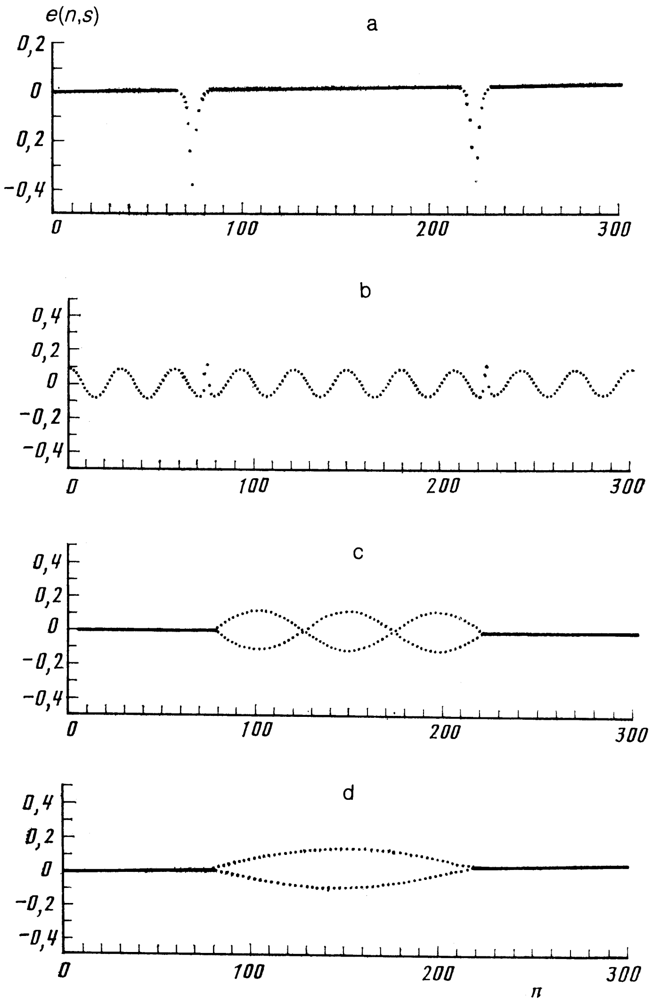

FIG. 8. Characteristic eigenvectors $e(n,s)$ of a resonator consisting of two solitons with $A = 0.1$ and $H = 0.25$. a—$s = 1$, b—25; c—296; d—300.

soliton; instead they remain "localized." Effects of localization of normal modes in the Frenkel'-Kontorova model associated with a change in the parameter of the one-particle potential were studied numerically in Ref. 14. In particular, localization of the high-frequency vibrations of atoms was observed when the soliton pinning parameter decreased.

It has thus been shown that it is possible to control the filtering properties of 2π solitons with respect to linear excitations by means of an external field H. As an example we have formed the structure in a chain of 300 particles containing two 2π solitons with $A = 0.1$ and $H = 0.25$. For this structure, some of the high-frequency vibrations are localized in the region between solitons. For these frequencies, the solitons form a resonator. Other frequencies (from the low-frequency part of the upper band), in contrast, pass through the entire chain (i.e., are not localized). Figure 8 shows characteristic eigenvectors of the natural modes of a resonator of two solitons.

5. CONCLUSION

1. It has been shown that the stability of 2π solitons in a discrete chain depends on the strength of the applied field, H. As this field is raised, the width of a 2π soliton changes abruptly. This change in width is preceded by the appearance of a soft mode in the small-oscillation spectrum of the system. It has been found that the gap in the small-oscillation spectrum is a power-law function of the external field. This functional dependence has been found.

2. The nonmonotonic dependence of the frequency of a local oscillation of the width of a 2π soliton on the external field has been found. The spectrum of small oscillations has a band not found in the continuum approximation. This new band disappears when the transition is made to the continuous-medium approximation.

3. It is possible to control the transmission coefficient of structural 2π solitons for high-frequency oscillations in the spectrum of natural excitations by means of an external field.

[1] H. J. Mikeska, J. Phys. C **11**, L29 (1978).
[2] C. Etrich, H. J. Mikeska, S. Magyari, H. Thomas, and R. Weder, Z. Phys. B **62**, 97 (1985).
[3] P. I. Belobrov, V. V. Beloshapkin, G. M. Zaslavskiĭ, and A. G. Tret'yakov, Zh. Eksp. Teor. Fiz. **87**, 310 (1984) [Sov. Phys. JETP **60**, 180 (1984)].
[4] G. E. Khodenkov, Fiz. Met. Metalloved. **61**, 850 (1986).
[5] S. Homma and S. Takeno, Prog. Theor. Phys. **74**, 618 (1985).
[6] C. A. Condat, R. A. Guyer, and M. D. Miller, Phys. Rev. B **27**, 474 (1983).
[7] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Magnetization Waves: Dynamic and Topological Solitons*, Naukovo Dumka, Kiev, 1983.
[8] M. M. Bogdan and A. M. Kosevich, in *Proceedings, IV International Workshop on Nonlinear and Turbulent Processes in Physics*, Vol. 1, Kiev, 1989, p. 50.
[9] H. Dvey-Aharon, T. J. Sluckin, P. L. Taylor, and A. J. Hopfinger, Phys. Rev. B **21**, 3700 (1980).
[10] D. K. Campbell, M. Peyrard, and P. Solano, Physica D **19**, 165 (1986).
[11] M. Peyrard and S. Aubry, J. Phys. C **16**, 1593 (1983).
[12] V. V. Beloshapkin, G. M. Zaslavskiĭ, and A. G. Tret'yakov, Fiz. Nizk. Temp. **12**, 733 (1986) [Sov. J. Low Temp. Phys. **12**, 415 (1986)].
[13] V. V. Beloshapkin, G. P. Berman, and A. G. Tret'yakov, Zh. Eksp. Teor. Fiz. **95**, 723 (1989) [Sov. Phys. JETP **68**, 410 (1989)].
[14] S. R. Sharma and B. Bergersen, Phys. Rev. B **30**, 6586 (1984).

Translated by D. Parsons