Self-similar regimes of ultrarelativistic acceleration of particles trapped by an electrostatic wave in an inhomogeneous isotropic plasma

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The ultrarelativistic acceleration of charged particles trapped in the potential well of a plasma wave, which is in turn incident on a density slope in a weakly inhomogeneous, isotropic plasma, has been studied analytically and numerically. The dynamics of the long-term (in principle, permanent) confinement of the trapped particles in the accelerating phase of the field and that of the ultrarelativistic (theoretically unbounded) acceleration of these particles are analyzed for self-similar profiles of the phase velocity of the plasma wave.

INTRODUCTION

In connection with the technological progress made in the field of high-power lasers, active research has recently been undertaken on various possibilities for the ultrarelativistic acceleration of charges by intense electromagnetic waves (this research was stimulated primarily by Refs. 1–3). The basic idea here is that intense electrostatic waves (Langmuir waves) excited in a plasma, with fields $E \sim m_e c\omega_{pe}/e$ and with phase velocities $v_{ph} \equiv \omega/k$ approaching c (the velocity of light in vacuum), could accelerate charges to ultrarelativistic energies. The rate at which the particles would acquire energy would be significantly higher than that which has been achieved in conventional accelerators. Several mechanisms for acceleration of this type have been discussed; examples are plasma beat-wave acceleration (PBWA), i.e., an acceleration by a plasma wave excited in a plasma by beats stemming from a superposition of two laser beams with approximately equal frequencies; plasma wakefield acceleration (PWFA), i.e., acceleration by the wake fields of charged bunches; plasma feeder acceleration (PFA);[4] and surfing in a magnetic field.[5] A question which arises in connection with all these schemes is that of an upper limit on the energy of the accelerated particles.

In the case of surfing of charges on a longitudinal wave in a transverse magnetic field,[5] there is no theoretical upper limit on the energy which the accelerated particles can acquire, but the plasma would have to be in a magnetic field, and the field strength H_0 would determine the acceleration rate. As the magnetic field is weakened, the acceleration rate decreases, and it vanishes at $H_0 = 0$.

Possibilities for unbounded acceleration of charges in an isotropic plasma ($H_0 = 0$) were pointed out by Faĭnberg.[3] One example was acceleration by a longitudinal wave with a phase velocity $v_{ph} = c\beta_{ph}$ equal to c in a homogeneous plasma. In this case, however, there is the question of the dynamics of the acceleration of the trapped particles by the plasma wave in an inhomogeneous isotropic plasma. For example, we would like to study the conditions for the confinement of trapped particles by a wave with a varying phase velocity; we would like to examine the phase stability, the rate at which the charges acquire energy, and how the phase velocity depends on the profile. In this connection we have carried out a detailed analysis of the dynamics of the unbounded acceleration of trapped particles by an electrostatic wave in a weakly inhomogeneous plasma for various profiles of the relativistic factor γ_{ph}, calculated in terms of the varying wave phase velocity: $\gamma_{ph} = 1/(1-\beta_{ph}^2)^{1/2}$.

The results show that two fundamentally different acceleration regimes are possible: acceleration with charge condensation at the bottom of the potential well and acceleration in which charges stick to the wall of the potential well, in a process like that described by Faĭnberg[3] for the case of a homogeneous plasma. In the former case the energy of the particles increases in proportion to t^s, where $s \leqslant 2/3$.

$$\gamma_{ph} = 1/(1-\beta_{ph}^2)^{1/2}.$$

In other words, the acceleration falls off as time elapses, but the energy spread of the accelerated particles shrinks. In the latter case, the energy of the charges increases linearly with time, i.e., the acceleration rate reaches a constant value. This constant value depends on the particular point on the wall of the potential well to which the trapped particle is stuck, in accordance with the results of Ref. 3. At the same time, the energy spread of the accelerated charges may shrink substantially in this regime because of the inhomogeneity of the plasma. Our analysis also shows that the acceleration is extremely sensitive to small regular deformations of the density profile, which determine the growth rate $\gamma_{ph}(x)$.

1. ACCELERATION WITH CONDENSATION OF TRAPPED PARTICLES AT THE BOTTOM OF THE POTENTIAL WELL

Let us consider the Cherenkov interaction of the trapped particles with a longitudinal wave

$$\mathbf{E} = \mathbf{e}_x E_m(x/L)\cos\Psi, \qquad \Psi = \Psi_0 + \omega t - \int_0^x k(x)\,dx.$$

The wave is propagating along the density gradient of a weakly inhomogeneous plasma, whose variations have a length scale L. In terms of the dimensionless variables $\tau = \omega t$, $X = \omega x/c$, and $\beta = v/c$, the relativistic equations of motion of a charge q with a rest mass m_0 can be written as follows:[6]

$$\frac{d}{d\tau}\gamma\beta_x = \alpha(\xi)\cos\Psi, \qquad \Psi = \Psi_0 + \tau - \rho\int_0^{\xi}\frac{d\xi}{\beta_{ph}(\xi)},$$
$$p_\perp = \text{const}, \quad \gamma = \gamma_\perp(1-\beta_x^2)^{-1/2}, \quad \xi = X/\rho, \tag{1}$$
$$\gamma_\perp = (1+(p_\perp/m_0c)^2)^{1/2}, \qquad \frac{\alpha(\xi)}{\alpha_0} = \left(\frac{\beta_{ph}(\xi)}{\beta_{ph}(0)}\right)^{1/2}.$$

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Here $\rho = \omega L/c$ is the semiclassical parameter which is a measure of the weak inhomogeneity of the plasma ($\rho \gg 1$), and $\alpha(\xi) = qE_m/m_0 c\omega$ is the dimensionless amplitude of the longitudinal wave. The following relations will be useful below:

$$\beta_x = \beta_{ph}(\xi)[1-\Psi_\tau], \quad \gamma_\tau = \alpha(\xi)\beta_x \cos\Psi.$$

To analyze the acceleration of the charges, we go over from (1) to a nonlinear equation for the phase of the wave on the trajectory of the trapped particle:

$$\frac{d^2\Psi}{d\tau^2} + \frac{\alpha(\xi)[1-\beta_x^2]^{1/2}}{\beta_{ph}(\xi)\gamma_\perp}\cos\Psi = (1-\Psi_\tau)^2\frac{d\beta_{ph}(\xi)}{dX}. \quad (2)$$

We begin with a brief outline of the WKB analysis of the acceleration. During adiabatic acceleration, the motion of a trapped particle in the field of the wave consists of fast phase oscillations with a slowly varying period and a slowly varying amplitude. Introducing the small parameter $\varepsilon = 1/\rho$ and the slow time $s = \varepsilon\tau$, we seek a solution of (2) through an asymptotic expansion:

$$\Psi(\tau, s) = W_0(\tau, s) + \varepsilon W_1(\tau, s) + \ldots,$$
$$\xi = \xi_0(s) + \varepsilon\xi_1(\tau, s) + \ldots.$$

The slow variables ξ_0 and s are related by

$$s(\xi_0) = \int_0^{\xi} d\xi/\beta_{ph}(\xi),$$

while the longitudinal velocity and the relativistic factor of the particle are given in the zeroth approximation by

$$\beta_0(\tau, s) = \beta_{ph}(\xi_0)(1-\partial W_0/\partial\tau), \quad \gamma_0(\tau, s) = \gamma_\perp(1-\beta_0^2(\tau, s))^{-1/2}.$$

The equation for $W_0(\tau,s)$

$$\frac{\partial^2}{\partial\tau^2}W_0 + \frac{\alpha(\xi_0)\gamma_\perp^2\cos W_0}{\beta_{ph}(\xi_0)\gamma_0^3(\tau, s)} = 0, \quad (3)$$

i.e., the bounce frequency of the phase oscillations of the trapped particles is

$$\Omega(\tau, s) = (\gamma_\perp/\gamma_0)(\alpha(\xi_0)/\beta_{ph}(\xi_0)\gamma_0)^{1/2}.$$

It is convenient to introduce the functions g and r:

$$g(\theta, s) = \varkappa(s) + \cos\theta - \cos\theta_m, \quad r(\theta, s) = (g^2(\theta, s) - \varkappa^2(\theta, s))^{1/2},$$

where $\theta = W_0 + \pi/2$, θ_m is the right-hand turning point for the trapped particle in the symmetric potential well, and the parameter $\varkappa$ is

$$\varkappa(s) = \gamma_\perp/\alpha(\xi_0)\beta_{ph}(\xi_0)\gamma_{ph}(\xi_0).$$

From (3) we then easily find the following expression for $G(\tau,s) \equiv \partial\theta/\partial\tau$:

$$G^+ = \frac{r/\beta_{ph}\gamma_{ph}^2}{g - r\beta_{ph}}, \quad G^- = -\frac{r/\beta_{ph}\gamma_{ph}^2}{g + r\beta_{ph}}. \quad (4)$$

When (4) is used, the relativistic factors for the forward and retrograde motions of the trapped particle and the oscillation period $T(\theta_m,s)$ can be found from

$$\gamma_0^+ = (\gamma_\perp\gamma_{ph}/\varkappa)(g - \beta_{ph}r), \quad \gamma_0^- = (\gamma_\perp\gamma_{ph}/\varkappa)(g + \beta_{ph}r),$$
$$T(\theta_m, s) = (2\beta_{ph}\gamma_{ph}^2/\omega)\langle g(\theta, s)/r(\theta, s)\rangle. \quad (5)$$

Here the angle brackets mean the averaging operation

$$\langle R(\theta, s)\rangle \equiv \int_{-\theta_m}^{\theta_m} d\theta R(\theta, s).$$

Expressions (4) can be used to write a solution for $\theta(\tau,s)$ in quadrature with the unknown function $\theta_m(s)$; the latter function is found from the condition for the soluability of the equation for $W_1(\tau,s)$. In the limit $\gamma_{ph} \gg 1$, we find the following result for the oscillations of the trapped particles near the bottom of the potential well, i.e., for the case $\theta_m \ll 1$:

$$\theta_m(s) \approx \theta_m(0)(\gamma_{ph}(0)/\gamma_{ph}(\xi_0))^{3/4}. \quad (6)$$

According to (6), the trapped particles condense at the bottom of the potential well with increasing γ_{ph}, forming a bunch; i.e., permanent confinement of the trapped particles in the potential well of the wave, with a γ_{ph} which increases without bound, is achieved (under the condition that the situation is adiabatic). As a result [as is easily seen from (5)], the energy of the particles increases in proportion to the increase in γ_{ph}:

$$\gamma_0 \approx \gamma_\perp\gamma_{ph}(\xi_0).$$

Consider this condition for an adiabatic situation. Using the expression for the bounce frequency, we write this condition as follows:

$$d(\Omega^{-1})/d\tau \approx (\gamma_\perp/\alpha\rho^2)^{1/2}[d(\gamma_{ph}^{3/2})/d\xi] \ll 1.$$

It follows that although the bounce frequency tends toward zero with increasing γ_{ph}, the situation will remain adiabatic for the oscillations of the trapped particles for an arbitrarily long time, provided that $\gamma_{ph}(\xi)$ increases no more rapidly than the power function $\xi^{2/3}$ as $\xi \to \infty$.

Let us examine in more detail the self-similar asymptotic behavior of the solution of Eq. (2) for the power-law profile $\gamma_{ph}(\xi) = \gamma_*\xi^{2/3}$ under the condition $\xi \gg 1$. Switching to the new variables

$$R(\xi) = \gamma/\gamma_\perp\gamma_{ph}(\xi), \quad \zeta = \ln\xi,$$

we find from (2) the following equation for a nonlinear oscillator with a positive friction for the self-similar function $R(\zeta)$:

$$\frac{d^2R}{d\zeta^2} + \frac{dR}{3d\zeta} - \frac{2R}{9} + \chi\left(1 - \frac{1}{R^2}\right) = 0, \quad (7)$$

where $\chi = (\alpha\rho^2/2\gamma_\perp\gamma_*^3)$ is a large parameter. Analysis of (7) shows that, for the initial data which would be natural for the problem of ultrarelativistic acceleration, all motions of the nonlinear oscillator R decay with increasing ζ, in proportion to $\exp(-\zeta/6)$. The trajectories in the phase plane approach a focus singular point with $R \approx 1 + (1/9\chi)$, $R_\zeta = 0$. The energy of the trapped particles increases in accordance with

$$\gamma(\tau) \approx \gamma_\perp\gamma_*(\tau/\rho)^{2/3}.$$

Simultaneously, the phase of a wave decreases monotonically on the trajectory of a trapped particle:

$$\theta(\tau) \approx (2\gamma_\perp\gamma_*/3\alpha\rho)(\rho/\tau)^{1/3}. \quad (8)$$

With increasing ξ, the small oscillatory increments in γ and θ decay in proportion to $\xi^{-1/6}$.

There is a fundamental point worth noting here. It is associated with the difference between (6) and (8). The derivation of (6) used an expansion in the small parameter ε; in this expansion, the inertial force on the right side of Eq. (2) was automatically assumed small. However, analysis of the self-similar case corresponding to (8) leads to the conclusion that this assumption is incorrect in the sense that after a sufficiently long time the inertial force becomes comparable to the electrical force. For this reason, and despite the satisfaction of the adiabatic condition $d\Omega^{-1}/d\tau \ll 1$, the motion of the trapped particles basically corresponds to a slow creep along the rear wall of the electric potential $\varphi \sim \cos\theta$ at the bottom of the well. In other words, an ultrarelativistic trapped particle automatically goes into the accelerating phase of the field of the plasma wave and stays there permanently, undergoing an unbounded acceleration. A corresponding conclusion follows for other self-similar cases in which the $\gamma_{\rm ph}$ profile is a power law $\gamma_{\rm ph}(\xi) = \gamma_* \xi^n$ with $n < 2/3$. To demonstrate this point, we note that the change of variables

$$\gamma(\tau) = \gamma_\perp \gamma_{\rm ph}(\xi) R(\eta), \quad \eta = \xi^\nu, \quad \xi = \tau/\rho,$$

where $\nu = 1 - 1.5n < 0$, leads us to the following equation for R:

$$\frac{d^2R}{d\eta^2} + \frac{n}{2\nu\eta}\frac{dR}{d\eta} = \frac{\chi}{\nu^2}\left(\frac{1}{R^2} - 1\right) + \frac{n}{\nu}\left(1 + \frac{n}{2\nu}\right)\frac{R}{\eta^2}. \tag{9}$$

This equation describes the motion R of a nonlinear oscillator with an energy

$$\mathscr{E} = 0{,}5R_\eta^2 + \tilde{U}(R, \eta)$$

in a potential well

$$\tilde{U}(R, \eta) = \frac{\chi}{\nu^2}(R + R^{-1}) - \frac{n}{2\nu}\left(1 + \frac{n}{2\nu}\right)\frac{R^2}{\eta^2}$$

with a positive friction. The action (an adiabatic invariant) of the oscillator

$$J = \oint dR\, R_\eta = \oint dR (\mathscr{E} - \tilde{U})^{1/2},$$

can be shown to fall off with increasing η in accordance with

$$J(\eta)/J(\eta_0) = (\xi_0/\xi)^{n/2}.$$

In other words, the trapped particles condense at the bottom of the potential well, $\theta(\infty) = 0$. The energy and phase of the particles have the asymptotic behavior

$$\gamma(\tau) \approx \gamma_\perp \gamma_* (\tau/\rho)^n, \quad \theta(\tau) \approx (\gamma_\perp \gamma_* n/\alpha\rho)(\rho/\tau)^{1-n}.$$

The acceleration rate thus falls off to zero as time elapses.

2. UNBOUNDED ACCELERATION OF TRAPPED PARTICLES IN THE STICKING REGIME

We now consider the solution of Eq. (2) for a power-law profile of the relativistic factor, $\gamma_{\rm ph}(\xi) = \gamma_* \xi^n$, with a power n in the interval $2/3 < n < 1$. As in the preceding section of this paper, the substitution $R(\eta) = \gamma/\gamma_\perp \gamma_{\rm ph}(\xi)$, where $\eta = \xi^\nu$, leads us to Eq. (9) for the function $R(\xi)$, but in this case we have $\nu < 0$ and $\eta \to 0$ as $\xi \to +\infty$. With these differences in mind, we can show that in this case there is no potential well at all as $\xi \to +\infty$, i.e., as $\eta \to 0$, and R becomes infinite in accordance with the power law. Further analysis leads to the following scenario. At large values of ρ the trapped particles first condense near the bottom of the potential well $\tilde{U}$, but this condensation subsequently comes to a halt. As $\tau \to \infty$, the particles stick to the rear wall of the electric potential with a certain phase θ_∞, $0 < \theta_\infty < \pi$. As a result, the asymptotic behavior of γ and θ is

$$\gamma(\tau) \approx \gamma(\tau_c) + \alpha(\tau - \tau_c)\sin\theta_\infty,$$
$$\theta(\tau) = \theta_\infty + [\rho/2\gamma_*(2n-1)](\rho/\tau)^{2n-1},$$

where τ_c is a constant. As time elapses, the energy of the trapped particles thus increases linearly, as in the case of a homogeneous plasma, discussed by Faĭnberg.[3] In contrast with Ref. 3, however, all the trapped particles may localize in phase near the bottom of the well, if the parameter values are chosen correctly.

Let us examine the acceleration of particles for powers $n > 1$. Again in this case there is acceleration in the sticking regime, and there is no condensation of particles at the bottom of the potential well. For γ and θ we find the asymptotic behavior

$$\gamma(\tau) \approx \alpha\tau\sin\theta_\infty + \text{const},$$
$$\theta(\tau) = \theta_\infty - [\gamma_\perp^2/2\gamma(\tau)\alpha\sin\theta_\infty].$$

Numerical solutions have been carried out of Eqs. (1) for various profiles of the phase velocity of the plasma wave, $\beta_{\rm ph}(\xi)$, including some power-law profiles,

$$\beta_{\rm ph}(\xi) = \left[1 + \frac{1 - \beta_{\rm ph}(0)}{\beta_{\rm ph}(0)(1+\xi)^{2n}}\right]^{-1}, \quad \xi \geqslant 0,$$

with $n > 0$, and an exponential profile,

$$\beta_{\rm ph}(\xi) = \beta_{\rm ph}(0) + [1 - \beta_{\rm ph}(0)]\,\text{th}\,\xi.$$

The results of these calculations confirm the analysis above. To illustrate the results, we show in Fig. 1 a plot of the wave phase $\Psi(\tau)$ on the trajectory of a trapped particle and the relativistic factor $\gamma(\tau)$ for a power-law profile of $\beta_{\rm ph}(\xi)$ with the parameter values $n = 2$, $\beta_{\rm ph}(0) = 0.9$, $\rho = 10^3$, $\alpha = 0.05$, $\gamma_\perp = 1$, and $\Psi_0 = -0.75\pi$. We see from Fig. 1a that the rate at which the trapped particle is accelerated reaches a constant value fairly quickly. According to Fig. 1b, and in agreement with the discussion above, a sticking regime occurs in the course of the acceleration: After several oscillations, the trapped particle sticks to the rear wall of the electric potential.

The case $n = 0.75$ is illustrated in Fig. 2a by a plot of the phase $\Psi(\tau)$ for the parameter values $\rho = 500$ and $\Psi_0 = -0.5\pi$. The values of the parameters α, $\gamma_\perp$, and $\beta_{\rm ph}(0)$ are the same as for Fig. 1. Because of the relatively small value of ρ, the trapped particle first acquires a substantial momentum, and the peak-to-peak amplitude $\delta\Psi$ of the oscillations reaches a value on the order of unity. The oscillations then decay. We clearly see an increase in the oscillation period, which is a consequence of a substantial increase in the relativistic factor γ. After a sufficiently long time the trapped particle sticks to the rear wall of the potential. This event is demonstrated in Fig. 2b, which shows a plot of the self-similar function $R(\eta)$ in the case $n = 5/6$, in which we have $\xi = 1/\eta^4$. The unbounded increase in $R(\eta)$ with de-

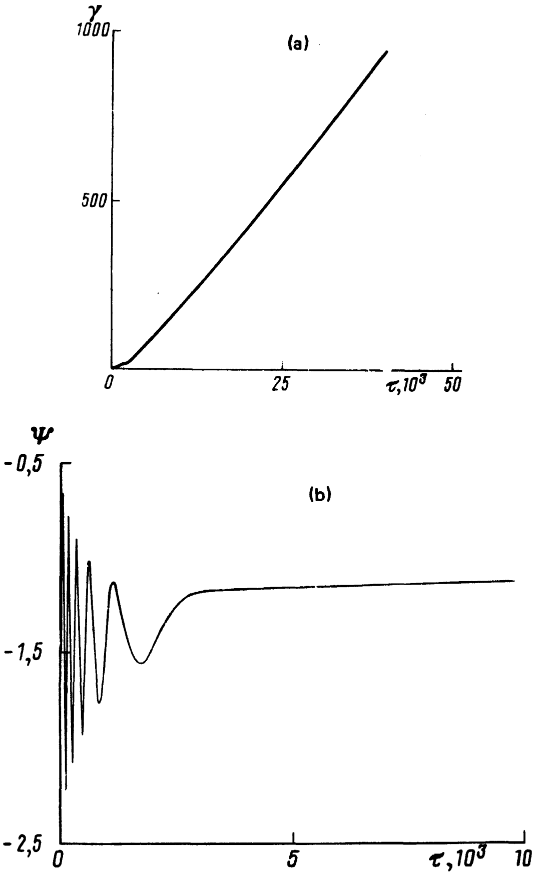

FIG. 1. Unbounded acceleration of a charge in the case of a power-law profile of the phase velocity of a plasma wave with exponent $n = 2$, which corresponds to the sticking regime. a—Relativistic factor of the charge; b—phase of the wave at the trajectory of the charged particle ($\rho = 10^3$, $\beta = 0.9$, $\alpha = 0.05$, $\Psi_0 = -0.75\pi$).

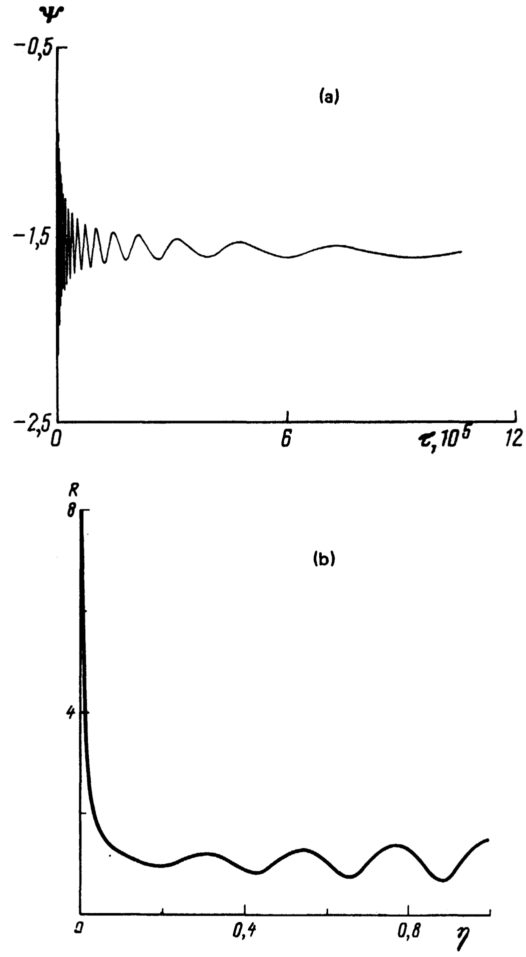

FIG. 2. Acceleration of a charge for values $2/3 < n < 1$ of the power in the expression for the profile of the wave velocity. a—Phase of the wave on the trajectory of the charge in the case $n = 3/4$ ($\nu = -1/4$, $\xi = 1/\eta$); b—plot of the self-similar function $R(\eta)$ for the case $n = 5/6$ ($\rho = 500$, $\beta = 0.9$, $\alpha = 0.05$, $\Psi_0 = \pi/2$).

creasing η signifies sticking of the particle, and the acceleration rate becomes constant.

An important conclusion follows from the discussion above. In this problem, the $\gamma_{\rm ph}$ profile depends on the distribution of the plasma density. This density increases monotonically up to a certain value along the propagation direction of the plasma wave. It is not difficult to see that small regular deformations in the plasma density distribution with $\beta_{\rm ph} \approx 1$ will lead to a pronounced change in the $\gamma_{\rm ph}$ profile, including a change in the exponent in its growth law, $n \equiv d \ln\gamma_{\rm ph}/d \ln\xi$. The growth law determines the type of acceleration regime. Consequently, the acceleration mechanism is sensitive to the plasma density distribution. In the general case of a variable n, there can be a mixed regime for the acceleration of trapped particles by a plasma wave in an inhomogeneous plasma.

3. DAMPING OF THE LONGITUDINAL WAVE BY THE TRAPPED PARTICLES

Let us briefly examine the effect of the damping of the longitudinal wave by the trapped particles. Since we are interested in ultrarelativistic acceleration, we will be assuming that the wave is loaded with a fairly low-density flux of trapped particles. The specific condition will be given below. We furthermore restrict the discussion to acceleration when trapped particles condense at the bottom of the well.

It is convenient to write the kinetic equation for the distribution function of the trapped particles in terms of the variables Ψ, ξ, and $q_x \equiv p_x/m_0c$, introduced above:

$$\left(1-\frac{\beta_x}{\beta_{\rm ph}}\right)\frac{\partial f}{\partial\Psi}+\frac{1}{\beta_{\rm ph}}\left(\frac{\partial\Phi}{\partial\Psi}\right)\frac{\partial f}{\partial q_x}=\varepsilon\left[\left(\frac{\partial\Phi}{\partial\xi}\right)\frac{\partial f}{\partial q_x}-\beta_x\frac{\partial f}{\partial\xi}\right]. \tag{10}$$

Here $\Phi(\xi,\Psi) \equiv q\varphi(\xi,\Psi)/m_0c^2$ is a dimensionless electric potential, and ε is the small parameter which was defined back in Sec. 1. Expanding $f(\Psi,q_x,\xi)$ in powers of the parameter ε, and we find from (10) that the distribution function f depends in the zeroth approximation on $\mathscr{E}$ and ξ, where $\mathscr{E}$ is the Hamiltonian

$$\mathscr{E}=\Phi(\xi,\Psi)+(1+q_x^2)^{1/2}-\beta_{\rm ph}q_x.$$

We assume for simplicity that the transverse momenta are small and $f\sim\delta(p_\perp)$. The condition under which the next approximation has a solution imposes the requirement that $f_0(\mathscr{E},\xi)$ be a function of the adiabatic invariant $J(\mathscr{E},\xi)$ alone. For trapped particles, this invariant can be calculated from

$$J(\mathscr{E},\xi)=2\beta_{ph}\gamma_{ph}^2\int_{\Psi_1}^{\Psi_2} d\Psi\,[(\mathscr{E}-\Phi)^2+\beta_{ph}^2-1]^{1/2}\equiv\oint d\Psi\, p_\Psi, \qquad (11)$$

where p_Ψ is the generalized momentum corresponding to the generalized coordinate Ψ, and $\Psi_{1,2}$ are the turning points; i.e.,

$$\mathscr{E}=\Phi(\xi,\Psi_{1,2})+\gamma_{ph}^{-1}.$$

For the untrapped particles we have, in place of (11),

$$J_{(\pm)}=\gamma_{ph}^2\int_{-\pi}^{\pi} d\Psi\,\{\beta_{ph}^2\Phi-\mathscr{E}\pm\beta_{ph}[(\mathscr{E}-\Phi)^2+\beta_{ph}^2-1]^{1/2}\}. \qquad (12)$$

The $\pm$ in (12) correspond to untrapped particles with velocities $\beta_x>\beta_{ph}$ and $\beta_x<\beta_{ph}$, respectively. By analogy with the preceding sections of this paper, we choose the following expression for the electric potential and the Hamiltonian:

$$\Phi(\Psi,\xi)=-\alpha(\xi)\beta_{ph}(\xi)\cos\theta,\quad \Psi=-\pi/2+\theta,$$
$$\mathscr{E}=\gamma_{ph}^{-1}(\xi)-\alpha(\xi)\beta_{ph}(\xi)\cos\theta_m,$$

where $\theta_m(\xi)$ are the turning points for the trapped particles. Here we have $0\leqslant\theta_m(\xi)\leqslant\theta_*(\xi)$, and $\theta_*(\xi)$ is found from (11); it is determined by the maximum value J_* of the adiabatic invariant for the trapped particles. Equation (11) can be rewritten as

$$J\equiv J_c\int_{-\theta_m}^{\theta_m} d\theta\,[(\cos\theta-\cos\theta_m)(2\varkappa+\cos\theta-\cos\theta_m)]^{1/2},$$
$$J_c\equiv 2/\alpha(\xi)\varkappa^2(\xi).$$

It follows that the trapped particles first condense near the bottom of the potential well, but if the damping of the longitudinal wave is substantial, the levels rise, and the particles are gradually spilled out. The spatial distribution of the density of trapped particles has "holes." In particular, over the period $0\leqslant|\theta|\leqslant\pi$ this distribution is described by

$$\frac{n_{tr}(\theta,\xi)}{\langle n_{tr}\rangle}=\frac{4\beta_{ph}(0)}{\theta_*^2(\xi)\beta_{ph}(\xi)}\begin{cases}(\theta_*^2(\xi)-\theta^2)^{1/2}, & 0<|\theta|\leqslant\theta_*,\\ 0, & \theta_*\leqslant|\theta|\leqslant\pi,\end{cases}$$

where $\langle n_{tr}\rangle$ is the density averaged over the wave period.

Let us examine the energy flux density of the trapped particles, averaged over the wave period, during acceleration in the condensation regime. For simplicity we set $f_0(J)=(2\pi/J_*)=\text{const}$. Calculations lead to

$$\langle S_x\rangle=m_0c^3\langle n_{tr}\rangle\gamma_{ph}(\xi).$$

We write $q=ze$, and we denote by v_g the group velocity of the longitudinal wave. In the problem of the ultrarelativistic acceleration of charges, the condition for conservation of the total energy flux in the wave-plus-particle system determines the law describing the damping of the wave by the trapped particles:

$$\frac{\alpha(\xi)}{\alpha(\xi_0)}=\left[1-\frac{2Z^2c\langle n_{tr}\rangle}{\alpha^2(\xi_0)v_gn_p}\gamma_{ph}(\xi)\frac{m_e}{m_0}\right]^{1/2}, \qquad (13)$$

where n_p is the plasma density.

According to (13), the maximum energy of the bulk of the accelerated trapped particles is

$$\gamma_{max}\approx\frac{v_g}{c}\frac{E_m^2(\xi)}{8\pi m_0c^2\langle n_{tr}\rangle}.$$

Strong acceleration is achieved when the flux density of the trapped particles is low:

$$\langle n_{tr}\rangle\ll[v_gE_m^2(\xi)/8\pi m_0c^3\gamma_{ph}(\xi_0)].$$

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