On the Zhdanov-Trubnikov equation for premixed flame instability

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(Submitted 21 December 1990)
Zh. Eksp. Teor. Fiz. **100**, 428–432 (August 1991)

We exhibit exact solutions to a generalized version of the equation recently proposed by Zhdanov and Trubnikov[1] to describe the hydrodynamical instability of premixed flames. In a limiting case we show how steady solutions to the nonlinear ZT-equation can be deduced from those of a linear integral problem. We next present arguments suggesting that the original ZT equation possesses a continuum of solutions, which would be broken into a discrete set by the slightest stabilizing curvature effect. Finally, related open mathematical problems are discussed.

1. Using expansions in the wrinkling amplitude, Zhdanov and Trubnikov[1] recently obtained an equation, denoted by ⟨⟨ZT⟩⟩ in what follows, to describe how two-dimensional wrinkles of an initially planar premixed flame grow as a consequence of the well-known Landau[2]-Darrieus[3] hydrodynamic instability. When written in terms of suitably scaled time (t) and space (x) variables, the ZT equation reads as follows:

$$\dot{\psi}+{}^1/_2(\psi^2+cH^2(\psi))_x+H(\psi_x)=0, \tag{1}$$

where the subscript x denotes the partial derivative $\partial/\partial x$, $\dot{\psi}\equiv\partial\psi/\partial t$; $\psi(t,x)$ is a scaled function representing the local angle made by the flame front to its mean location. $H(\psi)$ stands for the Hilbert transform of ψ:

$$H(\psi)=\pi^{-1}\int(z-x)^{-1}\psi(t,z)\,dz \tag{2}$$

and $1>c>0$ is a constant parameter which approaches zero (unity) when the density change across the flame is very weak (very large). Our present purpose is to show a way to obtain solutions to (1).

2. We first note that the model leading to (1) assumed a constant local burning speed; as an unrealistic consequence, wrinkles of arbitrarily short wavelengths are allowed to grow, so that the dynamical problem is ill-posed. A more complete description would acknowledge that actual burning speeds do depend on local flame shape and flow structure.[4–6] It therefore makes sense to formally augment (1) into the following modified equation (mZT):

$$\dot{\psi}+{}^1/_2(\psi^2+cH^2(\psi))_x+H(\psi_x)=\nu\psi_{xx}, \tag{3}$$

in which $\nu\psi_{xx}$ accounts for the combined stabilizing ($\nu\geqslant 0$ is assumed) influences of front curvature and flame stretching by flow nonuniformity.[6] We next note that (3) is a differentiated form of

$$\dot{\varphi}+{}^1/_2(\varphi_x^2+cI^2(\varphi))=\nu\varphi_{xx}+I(\varphi), \tag{4}$$

where $\varphi_x\equiv\psi$, and $I(\cdot)=-(d/dx)H(\cdot)$ is the Landau-Darrieus operator (multiplication by $|k|$ in the Fourier-space conjugate to x). We shall concentrate on (4); then the limit $\nu\to+0$ will lead us back to (1).

3. For $c=0$ (small density change) (4) reduces to a well-known equation due to Sivashinsky,[7] for which exact, relevant solutions can be found, e.g. by the method of pole-decomposition.[8–10] We now show that the method also works when $c\neq 0$.

Let us first introduce the real functions

$$\varphi^{(\alpha)}=\ln\left[\sin\frac{k}{2}(x-x_\alpha)\sin\frac{k}{2}(x-\bar{x}_\alpha)\right], \tag{5}$$

where $x_\alpha=\mathrm{Re}(x_\alpha)+i\mathrm{Im}(x_\alpha)$ is a complex constant, $\bar{x}_\alpha$ is its conjugate and k is any positive real constant. We next note the identities[11]

$$\varphi_x^{(\alpha)}=2k\sum_{n=1}^{\infty}e^{-nk|\mathrm{Im}(x_\alpha)|}\sin[nk(x-\mathrm{Re}(x_\alpha))], \tag{6}$$

$$\mathrm{sh}\,z/(\mathrm{ch}\,z-\cos x)=1+2\sum_{n=1}^{\infty}e^{-nz}\cos nx,\qquad z>0. \tag{7}$$

From these and the definition of $I(\cdot)$ it follows that

$$I(\varphi^{(\alpha)})=1+i\frac{d}{dx}\left[\varepsilon_\alpha\ln\sin\frac{k}{2}(x-x_\alpha)+\varepsilon_{\bar{\alpha}}\ln\sin\frac{k}{2}(x-\bar{x}_\alpha)\right] \tag{8}$$

with $\varepsilon_\alpha=\mathrm{sign}[\mathrm{Im}(x_\alpha)]$ and $\varepsilon_{\bar{\alpha}}=-\varepsilon_\alpha$. The term pole-decomposition refers to the property that (4) admits exact, spatially $2\pi/k$-periodic solutions of the form

$$\varphi(t,x)=-h(t)+\sum_{n=1}^{2N}a_\alpha\ln\sin\left(\frac{k}{2}(x-x_\alpha(t))\right), \tag{9}$$

for which the x_α's are the complex poles of φ_x. As φ is real when x is, these must come in conjugate pairs in (9); their total number $2N$ is fixed, but arbitrary. That (9) solves (4) is shown by direct substitution. First the a_α's must be chosen so as to ensure that the most divergent terms—coming from φ_x^2, $I^2(\varphi)$ and φ_{xx} once (5), (6) are taken into account—cancel each other, which implies $a_\alpha=-2\nu(1-c)^{-1}$. One next uses the trigonometric formula:

$$\mathrm{ctg}\,a\,\mathrm{ctg}\,b=-1+\mathrm{ctg}(a-b)\,[\mathrm{ctg}\,b-\mathrm{ctg}\,a] \tag{10}$$

with $2a=k(x-x_\alpha)$ and $2b=k(x-x_\beta)$ to convert the remaining products of unlike cotangents ($\alpha\neq\beta$) into sums. The x-dependent terms still present in (4) then disappear if the poles move according to the $2N$ complex ODEs:

$$\dot{x}_\alpha=\frac{\nu k}{1-c}\sum_{\beta\neq\alpha}(1-\varepsilon_\alpha\varepsilon_\beta c)\,\mathrm{ctg}\left[\frac{k}{2}(x_\beta-x_\alpha)\right]-i\varepsilon_\alpha\left(1+\frac{2N\nu kc}{1-c}\right). \tag{11}$$

For $c=0$, the $2N$-body dynamics found in Ref. 8 is recovered and again, nonperiodic patterns—representing collections of elementary crests—and their evolution are readily obtained from (8), (11) by allowing k to go to zero, x and the x_α's being kept fixed. Finally, the x-independent terms

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disappear from (4) provided that $h(t)$ satisfies

$$(1-c)\dot{h}=2\nu kN(1-\nu kN). \tag{12}$$

For steady patterns ($\dot{x}_\alpha=0$) $\dot{h}$ measures the increase in flame speed due to instability and is a convenient norm of the wrinkling, as

$$\dot{h}=-\langle\dot{\varphi}\rangle=\langle\varphi_x{}^2\rangle+c\langle I^2(\varphi)\rangle$$

then follows from (4) upon transverse averaging (denoted by brackets).

4. The simplest nontrivial solution of (4) obtains when $N=1$. More general patterns correspond to superpositions of this simple one, possibly once shifted; in this sense, such an elementary pattern is the basic soliton for (4), as the poles are indestructible. Due to translational invariance one may take $x_1=iB$, $x_2=-iB$. The evolution of $B(t)$ follows from (11), viz.

$$(1-c)\dot{B}=\nu k(1+c)\,\mathrm{cth}(kB)-(1-c+2c\nu k) \tag{13}$$

and its fate as $t\to\infty$ mainly depends on νk. B may tend to a constant B_∞, easily found from (13), in which case the final expression of φ reads:

$$(c-1)\varphi(t,\ x)=\nu k(1-\nu k)t+2\nu\ \ln[1-\mathrm{sech}(kB_\infty)\cos(kx)] \tag{14}$$

up to arbitrary, constant shifts in x or φ. Large values of kB_∞ lead to a nearly sinusoidal wave; B may also ultimately move to $+\infty$. The bifurcation ($B_\infty=\infty$) of course corresponds to $\nu k=1$, i.e., to a wavenumber which is marginal for the linearized version of (4) or (3). Vanishing values of k lead to a cell amplitude that grows logarithmically, i.e., less rapidly than k^{-1}; $-\langle\dot{\varphi}\rangle$ then vanishes again (even though the crests sharpen, the troughs get flatter and flatter as k decreases); further bifurcations may occur [see (16)].

5. Following Ref. 8, we next note that $(x_\beta-x_\alpha)(\dot{x}_\alpha-\dot{x}_\beta)\approx4\nu k$, when two poles x_α and x_β get very close to each other in the same half plane. As a consequence, in much the same way as for $c=0$, the poles tend to form alignments parallel to the imaginary axis. Let $\pm B_n$ ($n=1,\dots,N$; $B_n>0$) denote the pole imaginary parts, when perfect alignment is reached in each cell. Equations (11) then degenerate into N algebraic conditions:

$$\nu k\sum_{m\neq n}(f_+^{(n,m)}+f_-^{(n,m)})+\nu kf_+^{(n,n)}=(1-c+2\nu Nck),$$
$$n=1,\dots N,\qquad f_\pm^{(n,m)}=(1\pm c)\,\mathrm{cth}\left(\frac{k}{2}(B_n\pm B_m)\right). \tag{15}$$

The B_n's may be ordered to have $B_N=\max(B_n)<\infty$, so that for $n=N$, all the above hyperbolic contangents exceed unity and the following constraint on N obtains:

$$2N\leqslant1+(\nu k)^{-1}, \tag{16}$$

which does not contain c. If $(\nu k)^{-1}$ is an odd integer, equality holds in (16) and a pair of poles branches off from infinity; hence an infinite sequence of bifurcations exists. It is natural to conjecture that

$$N=N_{\max}(\nu k)=\mathrm{Int}[(1+(\nu k)^{-1})/2]$$

[$\mathrm{Int}(\cdot)$ = Integer Part of $(\cdot)$] generically corresponds to steady patterns obtained from sufficiently irregular initial conditions[12] ("many available poles"). If $N\geqslant N_{\max}$ holds at $t=0$, $N_{\max}-N$ pairs of poles ultimately escape to $\pm i\infty$ and no longer contribute to $\langle\dot{\varphi}\rangle$ and φ in the long-time limit.

6. If $N=N_{\max}(\nu k)$ is indeed the relevant value for steady patterns, then $2N\approx1/\nu k\to\infty$ as $\nu k\to0$. As for $c=0$ one may then replace the discrete sum over the B_n's involved in (14) by an integral, with weight $p(B')/\nu$, over the current pole location B': $p(B')dB'/\nu$ denotes the number of poles whose imaginary parts lie in the interval $[B',B'+dB']$; obviously $p(B)$ may be continued as an even function. When evaluated at $B=B_n$, the continuous version of (15) leads to

$$k\int_{-\infty}^{\infty}p(B')K(B,B')dB'-\mathrm{sign}\,B=0 \tag{17}$$

with a kernel $K(B,B')$ defined by

$$K(B,\ B')=(1-c\ \mathrm{sign}\,B/\mathrm{sign}\,B')\,\mathrm{cth}((B-B')k/2). \tag{18}$$

Once the linear problem (17) is solved for $p(B)$ with the normalization

$$\int_{-\infty}^{\infty}\frac{p(B)}{\nu}dB=2N\approx\frac{1}{\nu k},\quad\text{hence}\quad k\int_{-\infty}^{\infty}p\,dB=1, \tag{19}$$

the function $\varphi(t,x)$ is available:

$$(c-1)\varphi=t/2+2\int_{-\infty}^{\infty}p(B)\ln\left[\sin\frac{k}{2}(x-iB)\right]dB. \tag{20}$$

By construction, this expression solves the mZT equation in the limit $\nu k\to+0$. Hence its derivative $\psi=\varphi_x$ should solve the original ZT equation...but we have so far been unable to show this directly, upon substitution of (19) into (1) (due to the difficulty encountered when handling singular double integrals). Because p depends on k only through the grouping kB, the wrinkling amplitude given by (20) is proportional to the wavelength, as it should be, since no intrinsic length appears in (1).

An interesting extension would be to determine how (17) generalizes to unsteady situations. Within the continuous approximation this amounts to writing a continuity equation for $p(B,t)$ which, once the convection in B-space is evaluated from (11), reads

$$(1-c)\dot{p}+\frac{\partial}{\partial B}\left[p\left(k\int_{-\infty}^{\infty}p(B')K(B,B')dB'-\mathrm{sign}\,B\right)\right]=0. \tag{17'}$$

Obviously, this is compatible with (17) if $p(B=\pm\infty)=0$. Unfortunately, finding a non-trivial solution to (17′) seems to be even more difficult than solving (1) itself!

7. Further remarks are due. First, that the integral equation (17) is an outer asymptotic one, valid for $B\leqslant B_N$ (and not too small) in the limit $\nu k\to+0$. Next, that the choice $N=N_{\max}(\nu k)$ is rather arbitrary isofar as only steady patterns are looked for: any value $N\approx\sigma N_{\max}$, $0\leqslant\sigma\leqslant1$, could in principle have been considered, provided that the RHS of (17) is modified into $((1-(1-\sigma)c)\mathrm{sign}(B)$. Moreover preliminary numerical solutions of (15) seem to indicate that B_N tends to a well defined limit as N grows, σ being kept fixed.

These remarks led us to the question: does the ZT equation admit a continuum of steady solutions parametrized by σ? We have no rigorous, definitive answer up to now, but we suggest below what could happen. For the sake of argument only $c=0$ is considered, in which case (17) is of convolution type and can be solved exactly by Fourier transformation,[8] to give

$$p(B)=\pi^{-2}\ln\left[\operatorname{cth}\left|\frac{kB}{4}\right|\right]+d, \tag{21}$$

where d is an integration constant. For $\sigma=1$, the normalization (18) yields $d=0$ and a pole density which vanishes at $|B|=\infty$ only. A smaller σ would lead to a negative d, and to a $p(B)$ vanishing at $|B|=B_*(\sigma)$. As p cannot become negative (it is a density), and (17) is no larger valid when $p(B)=0$, it is natural[8] to truncate p when $|B|\geqslant B_*$, thereby leading to

$$p(B)=\frac{1}{\pi^2}\ln\left[\frac{\operatorname{cth}|kB/4|}{\operatorname{cth}(kB_*/4)}\right], \tag{22}$$

for $|B|\leqslant B_*$ and to $p\equiv 0$ otherwise. A way to relate B_* to σ would then be to use the normalization condition:

$$1\geqslant\sigma=k\int_{-B_*}^{B_*}p(B)\,dB\left(\equiv\pi^{-2}\int_0^{kB_*}\frac{dx}{\operatorname{sh}(x/2)}\right), \tag{23}$$

in which case equations (9), (12) would yield

$$\varphi(t,x)=\sigma(\sigma/2-1)t-2\int_{-B_*}^{B_*}p(B)\ln\sin\left[\frac{k}{2}(x-iB)\right]dB \tag{24}$$

as another "steady" solution to the "inviscid" Sivashinsky equation [(4) with $\nu=c=0$].

If true, even for $c\neq 0$, this reasoning would mean that the ZT equation ($\nu k=0$) has a continuum of steady solutions (parametrized by σ), whereas the mZT equation only admits a countably infinite number of them (parametrized by N) as $\nu k\to +0$, in a way which is quite reminiscent of what recently happened in theories of viscous fingering,[13] bubble rise,[14] crystal growth,[15] etc. To the best of our knowledge the possibility of a continuum of solutions for infinitely-thin flames steadily propagating in tubes was first alluded to by Pelcé.[16] As (15) is a discretized version of (17), in a sense, this would also suggest that discretization-induced diffusion may play a quite nontrivial role in numerical studies of pattern selection!

8. One must finally acknowledge that the above methods all rely on a sort of miracle, namely that (4) admits exact solutions of the form given by (9). It would be nice to use these as a basis to incorporate extra effects as perturbations. However, the slightest well-chosen structural change in the evolution equation (4) may preclude any simple generalization. For example adding a term $-\psi_{xxxx}$ to the RHS of (4), so as to account for higher-order stabilizing ($\varepsilon>0$) curvature effects,[7] prevents one from using the pole-decomposition method, whatever the smallness of ε, even though the modification of φ clearly is small when $\varepsilon\to +0$ and x is real. Problems of the same kind arise when the extra term is $\varepsilon(\langle\varphi\rangle-\varphi)$ (influence of a small gravity field[17]) or $-\varepsilon\varphi$ (remote flame held around a point-source of reactants[18]) or is a small known forcing. It would be quite interesting mathematically to set up singular-perturbation analyses to determine how the former poles of φ_x get "unfolded" into more complicated, as yet unknown, sets of singularities.

ACKNOWLEDGMENT

Prof. G. I. Sivashinsky recently drew Ref. 1 to my attention and gave me a copy; for this and for many enjoyable discussions on flames, it is a real pleasure to thank him here.

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Edited by David L. Book