Solution of the Derrida model for an arbitrary number of colors and an asymmetric distribution of the coupling constants

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The Derrida model of a spin glass, with symmetry group $Z(Q)$ with arbitrary Q, is analyzed. A solution is given for the case of a Potts interaction. The case in which there is an admixture of a ferromagnetic interaction is analyzed for the Potts interaction and also for a vector interaction. In the limit $T\to 0$ these models generate a coding method which is optimal from the standpoint of the Shannon theorems for channels with Gaussian noise.

1. INTRODUCTION

The Derrida model[1] is one of several spin-glass models which can be solved exactly. Gross and Mezard[2] have found a solution on the basis of the Parisi theory.

We assume that a system of N spins is strongly coupled; in other words, any p spins of the N can interact. We then write

$$H=\sum_{1\leqslant i_1\ldots i_p\leqslant N} j_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}, \tag{1.1}$$

where the $j_{i_1\ldots i_p}$ obey a Gaussian distribution, i.e.,

$$\rho(j)\propto\exp\left(-\frac{j^2N^{p-1}}{J^2p!}\right),\quad \sigma=\pm1,\quad z=\frac{N^p}{p!}. \tag{1.2}$$

The model in (1.1), (1.2) was solved in Refs. 1 and 2 for the case $p\to\infty$, $N\gg p$. It was found that the exact solution of this model describes a breaking of replica symmetry even in first order.

The model (1.1), (1.2) was subsequently generalized[3] to the case $\sigma_i=\exp(2\pi ik/Q)$, and the spin interaction was assumed to be a vector interaction. In Ref. 4, the limit $T\to 0$ with $\langle J\rangle\neq 0$ was linked with the problem of optimal coding for the transmission of information over noisy channels.

In the present paper we first consider the case of a model with a Potts interaction of spins in a magnetic field with $\langle j\rangle=0$. We then take up models with a Potts interaction and with a vector interaction with $h=0$, $\langle j\rangle\neq 0$ (h is the magnetic field).

2. SOLUTION OF THE MODEL WITH A POTTS INTERACTION FOR $\langle j\rangle=0$

The Hamiltonian of the system is

$$H=\sum_{1\leqslant i_1\ldots<i_p\leqslant N}\sum_{r=1}^{Q=1}(\tau_{i_1\ldots i_p})^r\sigma_{i_1}{}^r\ldots\sigma_i{}^r j_{i_1\ldots i_p}, \tag{2.1}$$

where $\sigma_i=\exp(2\pi ik/Q)$, $k=1, Q$, and the coupling constant τ is distributed uniformly over the Q values of σ. The quantity j obeys a Gaussian distribution:

$$\rho(j)\propto\exp\left[-\frac{j^2}{J^2}\left(\frac{p!}{N^{n-1}}\right)^{-1}\right]. \tag{2.2}$$

In the mean-field approximation we find

$$\langle Z^n\rangle=\exp\Bigg\{\frac{B^2J^2}{4}\Bigg[n(Q-1)+\sum_{\alpha\neq\beta}\sum_{r=1}^{Q-1}(Q_{\alpha\beta}^{r,Q-r})^{p}\Bigg] -\sum_{\alpha\neq\beta,r}\frac{Q_{\alpha\beta}^{r,Q-r}\lambda_{\alpha\beta}^{r,Q-r}}{2}+\ln\operatorname{Tr}\exp\Bigg[\frac{1}{2}\sum_{a\neq b,r}\lambda_{ab}^{r,Q-r}\sigma_a{}^r\sigma_b{}^r +h\sum_a(Q\delta_{\sigma_a,1}-1)\Bigg]\Bigg\}, \tag{2.3}$$

where B is the inverse temperature, and $\lambda_{\alpha,\beta}$ are Lagrange multipliers.

We first consider the case in which replica symmetry is conserved.

We introduce

$$\lambda_{\alpha\beta}^{r,Q-r}=\lambda_r,\quad Q_{\alpha\beta}^{r,Q-r}=q_r. \tag{2.4}$$

From (2.3) we then find

$$\langle Z^n\rangle=\exp\Bigg\{\frac{B^2J^2}{4}\Bigg[n(Q-1)+n(n-1)\sum_r q_r{}^p\Bigg] -\frac{n(n-1)}{2}\sum_{r=1}^{Q-1}q_r\lambda_r+\ln\operatorname{Tr}\exp\Bigg[\frac{1}{2}\sum_{a,b}\lambda_r\sigma_a{}^r\sigma_b{}^{Q-r} +h\sum_a(Q\delta_{\sigma_a,1}-1)\Bigg]\Bigg\}. \tag{2.5}$$

From the condition for an extremum with respect to q_r we find

$$\lambda_r=pq_r^{p-1}B^2J^2/2. \tag{2.6}$$

For $q_r<1$, $p\to\infty$, we have

$$\lambda\to 0. \tag{2.7}$$

Expanding the exponential function in (2.5) in λ, we find

$$-BF=\frac{nB^2J^2(Q-1)}{4}+n\ln[e^{Bh(Q-1)}+(Q-1)e^{-Bh}] +\sum_r\left[\frac{\lambda_rq_r}{2}-\frac{\lambda_r}{2}\left(\frac{e^{BhQ}-1}{e^{BhQ}+Q-1}\right)^2\right] \tag{2.8}$$

(F is the free energy) and thus

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$$q_r=[\exp(Bh)-1]^2/[\exp(Bh)+(Q-1)]^2,$$

$$-BF={}^1/_4B^2J^p(Q-1)+\ln[\exp(Bh(Q-1))+(Q-1)\exp(-Bh)]. \quad (2.9)$$

We turn now to the case in which replica symmetry is broken.

The solution (2.9), (2.10) is valid up to the value B_c, at which the entropy vanishes. Let us assume that the replica symmetry group n is broken to subgroup m. We denote by q_0, λ_0 the quantities q_{ab}, λ_{ab} for a and b from different subgroups, while we use q_1, λ_1 for the case in which they are from the same subgroup. We can then write

$$\sum_{a\neq b} q_{ab}{}^p=n(m-1)q_1{}^p+(n-m)nq_0{}^p \to n(m-1)q_1{}^p-nmq_0{}^p. \quad (2.11)$$

We then perform the following transformation:

$$\mathrm{Tr}_\sigma \exp\Big(\frac{1}{2}\sum_{a\neq b,r}\lambda_{ab}{}^r\sigma_a{}^r\sigma_r{}^{Q-r}\Big)=\mathrm{Tr}_\sigma\Big[\sum\frac{\lambda_0{}^r}{2}\sum_{a,b}\sigma_a{}^r\sigma_b{}^{Q-r}$$

$$+\frac{1}{2}\sum_{k=1}^{n/m}\sum_{a,b=(k-1)m+1}^{km}(\lambda_1{}^r-\lambda_0{}^r)\sigma_a{}^r\sigma_b{}^{Q-r}\Big]\exp\Big(-\frac{n\lambda_1}{2}\Big)$$

$$=\exp\Big(-\frac{n\lambda_1}{2}\Big)\mathrm{Tr}_\sigma\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}\exp\Big(-\sum_r|z_{r,0}|^2\Big)dz_{r0}$$

$$\times dz_{r0}{}^*\exp\Big[\sum_r\Big(\frac{\lambda_{r0}}{2}\Big)^{1/2}2\,\mathrm{Re}\,z_{r0}\sigma_a{}^r\Big]\prod_{k=1}^{n/m}\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{rk}$$

$$\times dz_{rk}{}^*\exp\Big(-\sum_r|z_{rk}|^2\Big)\exp\Big[\sum_r(\lambda_{1r}-\lambda_{0r})^{1/2}$$

$$\times\Big(\sum 2\,\mathrm{Re}\,z_{rk}\sigma_a{}^r\Big)\Big]=\exp\Big(-\frac{n\lambda_1}{2}\Big)\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r0}\,dz_{r0}{}^*$$

$$\times\exp(-|z_{r0}|^2)\Big\{\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r1}\,dz_{r1}{}^*\exp(-|z_{r1}|^2)$$

$$\times\Big[\mathrm{Tr}_\sigma\exp\Big(\frac{\lambda_{r0}}{2}\Big)^{1/2}2\,\mathrm{Re}\,z_{r0}\sigma_a{}^r+(\lambda_{r1}-\lambda_{r0})^{1/2}2\,\mathrm{Re}\,z_{r1}\sigma_a{}^r\Big)\Big]^m\Big\}^{n/m}$$

In the case $n\to 0$ we find

$$\ln\exp\Big(\frac{1}{2}\sum_{a,b,r}\lambda_{rab}\sigma_a{}^r\sigma_b{}^{Q-r}\Big)=-\frac{n}{2}\sum_r\lambda_{r1}$$

$$+\frac{n}{m}\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r0}\,dz_{r0}{}^*\exp(-|z_{r0}|^2)\ln\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r1}$$

$$\times dz_{r1}{}^*\exp\Big(-\sum_r|z_{r1}|^2\Big)\sum_k\exp\Big\{\mathrm{Re}\Big[\sum_r(2\lambda_{r0})^{1/2}z_{r0}\exp(ir\varphi_k)$$

$$+[2(\lambda_{r0}-\lambda_{r1})]^{1/2}z_{r1}\exp(ir\varphi_k)\Big]\Big\}. \quad (2.12)$$

Taking logarithms in (2.12), and using (2.11), we find

$$-\frac{BF}{nN}=(Q-1)\frac{B^2J^2}{4}+\sum_r[(m-1)q_{r1}{}^p-mq_{r0}{}^p]\frac{B^2J^2}{4}$$

$$-\frac{1}{2}\sum_r[(m-1)q_{r1}\lambda_{r1}-m\lambda_{r0}q_{r0}]-\sum_r\frac{\lambda_{r1}}{2}$$

$$+\frac{1}{m}\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r0}\,dz_{r0*}\exp(-|z_{r0}|^2)\ln\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r1}\,dz_{r1}{}^*$$

$$\times\exp(-|z_{r1}|^2)\sum_k\exp\{[2(\lambda_{r1}-\lambda_{r0})]^{1/2}\,\mathrm{Re}\,z_{r0}$$

$$\times\exp(ir\varphi_k)+(2\lambda_{r0})^{1/2}\,\mathrm{Re}\,z_{r0}\exp(ir\varphi_k)+Bh(Q\delta_{k,1}-1)\}. \quad (2.13)$$

At the point of the extremum

$$\lambda_{r0}={}^1/_2B^2J^2pq_{r0}^{p-1}, \quad (2.14)$$
$$\lambda_{r1}={}^1/_2B^2J^2pq_{r1}^{p-1}$$

are self-consistent, with

$$q_{r0}<1,\quad q_{r1}=1,\quad \lambda_{r0}\to 0,\quad \lambda_{r1}\to\infty. \quad (2.15)$$

We now consider

$$\frac{1}{m}\ln\prod_r\int dz_{r0}\,dz_{r1}\Big\{\sum_k\exp\Big[\sum_r(2\lambda_{r0})^{1/2}z_{r0}{}^*\exp(ir\varphi_k)$$

$$+[2(\lambda_{r1}-\lambda_{r0})]^{1/2}z_{r1}{}^*\exp(ir\varphi_k)+Bh(Q\delta_{k,1}-1)\Big]\Big\}^m. \quad (2.16)$$

In the limit $\lambda_{r1}\to\infty$ we find

$$\frac{1}{m}\ln\prod_r\int dz_{r0}\,dz_{r1}\sum_k\exp\Big\{m\sum_r(2\lambda_{r0})^{1/2}z_{r0}{}^*\exp(ir\varphi_k)$$

$$+[2(\lambda_{r1}-\lambda_{r0})]^{1/2}mz_{r1}{}^*\exp(ir\varphi_k)+mBh(Q\delta_{k,1}-1)\Big\}.$$

Evaluating the integral over z_{r1}, we find

$$\frac{m}{2}\sum_r(\lambda_{r1}-\lambda_{r0})+\frac{1}{m}\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r0}\,dz_{r0}{}^*\exp(-|z_{r0}|^2)$$

$$\times\frac{1}{m}\ln\sum_k\exp\Big[mBh(Q\delta_{k,1}-1)+m\sum_r(2\lambda_{r0})^{1/2}z_{r0}{}^*\exp(ir\varphi_k)\Big]. \quad (2.17)$$

Expanding in powers in λ_{r0}, we find

$$\frac{m}{2}\sum_k(\lambda_{r1}-\lambda_{r0})+\frac{m}{2}\sum_r\lambda_{r0}\Big\{1-\frac{[\exp(mBhQ)-1]^2}{[\exp(mBhQ)+Q-1]^2}\Big\}$$

$$+\frac{1}{m}\ln\{\exp[mBh(Q-1)]+(Q-1)\exp(-mBh)\}. \quad (2.18)$$

The condition for an extremum yields the relations

$$q_{r0}=\Big[\frac{\exp(mBhQ)-1}{\exp(mBhQ)+Q-1}\Big]^2, \quad (2.19)$$

$$q_{r1}=1, \quad (2.20)$$

$$(Q-1)\frac{B^2J^2}{4}-\frac{1}{m^2}\ln\{\exp[mBh(Q-1)]+\exp(-mBh)\}$$

$$+(Q-1)\frac{Bh}{m}\frac{\exp(mBhQ)-1}{\exp(mBhQ)+Q-1}=0. \quad (2.21)$$

From (2.21) we find an equation for $B_c=mB$:

$$-F=(Q-1)\frac{J^2B_c}{4}+\frac{1}{B_c}\ln\{\exp[hB_c(Q-1)]+\exp(-hB_c)\}$$

$$=(Q-1)\frac{J^2B_c}{2}-h(Q-1)\frac{\exp(hQB_c)-1}{\exp(hQB_c)+Q-1}. \qquad (2.22)$$

3. SOLUTION WITH A POTTS INTERACTION FOR $\langle j\rangle\neq 0$

We consider the Hamiltonian (2.1) with the coupling constants

$$j_{i_1\ldots i_p}=J_0p!/N^{p-1}+j_1,$$
$$\rho(j_1)\propto\exp\left[-\frac{j_1^2}{J^2}\left(\frac{p!}{N^{p-1}}\right)^{-1}\right]. \qquad (3.1)$$

In place of (2.3) we then find

$$\langle Z^n\rangle=\exp\Big\{BJ_0\sum_{a;r}S_a^rt_a^r+\frac{B^2J^2}{4}\Big[(Q-1)n+\sum_{a\neq b,r}(Q_{ab}^{r,Q-r})^p\Big]$$
$$-\frac{1}{2}\sum_{a\neq b,r}Q_{ab}^{r,Q-r}\lambda_{ab}^r-\sum_{a,r}-S_a^rt_a^r$$
$$+\ln\mathrm{Tr}_\sigma\exp\Big(\frac{1}{2}\sum_{a\neq b}\lambda_{ab}^{r,Q-r}\sigma_a^r\sigma_b^{Q-r}+\sum_{a,r}t_a^r\sigma_a^r\Big)\Big\}, \qquad (3.2)$$

where S and t are Lagrange multipliers.

In the case in which replica symmetry is conserved we find

$$S_a^r=S_r,\quad Q_{ab}^{r,Q-r}=q_r,$$
$$-BF=BJ_0\sum_r S_r^p+\frac{B^2J^2}{4}\Big[(Q-1)-\sum_r q_r^p\Big]$$
$$+\sum_r\frac{q_r\lambda_r}{2}-\sum_r S_rt_r$$
$$+\ln\mathrm{Tr}_\sigma\exp\Big[\frac{1}{2}\sum_{a\neq b,r}\lambda_r\sigma_a^r\sigma_b^{Q-r}+\sum_{a,r}t_r\sigma_a^r\Big]. \qquad (3.3)$$

From the condition for an extremum we find

$$t_r=pBJ_0S_r^{p-1}, \qquad (3.4)$$
$$\lambda_r={}^1/_2pB^2J^2q_r^{p-1}. \qquad (3.5)$$

For $S_r<1$ we have the self-consistent expressions

$$t_r=0,\ S_r=0,\ \lambda_r=0,\ q_r=0, \qquad (3.6)$$
$$-BF={}^1/_4B^2J^2(Q-1)+\ln Q. \qquad (3.7)$$

In the case $S_r=1$ we have

$$-BF=BJ_0+\frac{B^2J^2}{4}\Big[(Q-1)-\sum_{r=1}^{Q-1}q_r^p\Big]+\sum_r\frac{q_r\lambda_r}{2}$$
$$-\sum_r t_r-\sum_r\frac{\lambda_r}{2}+n\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}\exp(-|z_r|^2)dz_r\,dz_r^*$$
$$\times\ln\sum_k\exp\Big[\sum_r(2\lambda_r)^{1/2}z_r\sigma_k^r+t_r\sigma_k^r\Big]. \qquad (3.8)$$

Even for $q_r=1$, we have $t_r\gg\lambda_r^{1/2}$. Taking the leading term, corresponding to $k=1$, in (3.8), we find

$$q_r=1,\ S_r=1,\ -BF=BJ_0. \qquad (3.9)$$

We turn now to the case in which replica symmetry is broken. Proceeding as in the derivation of (2.13), we find

$$-\frac{BF}{nN}=(Q-1)\frac{B^2J^2}{4}+BJ_0S^p+\sum_r[(m-1)q_{r1}^p-mq_{r0}^p]$$
$$-\frac{1}{2}\sum_r[(m-1)\lambda_{r1}q_{r1}-m\lambda_{r0}q_{r0}]-\sum_r\frac{\lambda_{r1}}{2}$$
$$-\sum_r S_rt_r+\frac{1}{m}\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r0}\,dz_{r0}^*\exp(-|z_{r0}|^2)$$
$$\times\ln\prod_r\frac{1}{\pi}\int_{-\infty}^{\infty}dz_{r1}\,dz_{r1}^*\exp(-|z_{r1}|^2)$$
$$\times\sum_k\exp\Big\{\sum_r[[2(\lambda_{r1}-\lambda_{r0})]^{1/2}$$
$$\times\mathrm{Re}\,z_{r1}\exp(ir\varphi_k)+\mathrm{Re}(2\lambda_{r0})^{1/2}z_{r0}\exp(ir\varphi_k)+t_r\exp(ir\varphi_k)]\Big\}. \qquad (3.10)$$

At the point of the extremum we find

$$\lambda_{r0}={}^1/_2B^2J^2pq_{r0}^{p-1},$$
$$\lambda_{r1}={}^1/_2B^2J^2pq_{r1}^{p-1}, \qquad (3.11)$$
$$t_r=BJ_0pS^{p-1}.$$

For $S_r<1$ we find $t_r=0$; we then find the situation discussed in Sec. 2. For $h=0$ we have

$$q_{r0}=0,\quad q_{r1}=1,$$
$$B_c^2=4\ln Q/[(Q-1)J^2], \qquad (3.12)$$
$$-F={}^1/_2(Q-1)J^2B_c=[(Q-1)\ln Q]^{1/2}J.$$

We turn now to the case of absolute zero, i.e., $B\to\infty$. Obviously, under the condition

$$[(Q-1)\ln Q]^{1/2}J=J_0 \qquad (3.13)$$

a transition occurs from (3.12) to (3.9). At values of the ratio J_0/J large in comparison with (3.13), complete magnetization arises. At smaller values, there is zero magnetization. This result agrees with the results of Ref. 4. Bear in mind here that we are using $Q-1$ numbers for each bond.

Let us consider a vector interaction of spins. We are interested in the case

$$H=\sum_{1\leq i_1\ldots<i_p\leq N}\mathrm{Re}\,j_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p},$$
$$j_{i_1\ldots i_p}=J_0p!/N^{p-1}+j_{1,i_1\ldots i_p},$$
$$\rho(j_1)\propto\exp\left[-\left(\frac{j_1}{J}\right)^2\left(\frac{p!}{N^{p-1}}\right)^{-1}\right]. \qquad (3.14)$$

The calculation is similar to that in the case of a Potts interaction, but in this case there is no summation over the index r

$(r = 1)$, and $B^2J^2(Q-1)/4$ is replaced by $B^2J^2/4$.

We thus have the following phases:

$$S=0,\ q=0,\ -BF={}^1/_4B^2J^2+\ln Q, \tag{3.15}$$

$$S=1,\ q=1,\ -BF=BJ_0, \tag{3.16}$$

$$S=0,\ q_0=0,\ q_1=1,\ -BF=B(\ln Q)^{1/2}J. \tag{3.17}$$

In the limit $B\to\infty$, a transition occurs from ferromagnetism phase (3.16) to spin-glass phase (3.17) at

$$(\ln Q)^{1/2}J=J_0. \tag{3.18}$$

This result agrees with the ideas of Ref. 4.

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