Ignition of deuterium-tritium fusion in mixtures and chemical compounds

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The effect of admixtures on the ignition of inertially confined deuterium-tritium fusion is analyzed. The plasma is assumed to be transparent to its own emission and to the fusion neutrons. The α-particle component of the heat transfer is taken into account systematically on the basis of an equation derived analytically. Ignition conditions are found for the cases with and without electron thermal conductivity. Above the ignition threshold, the temperature rises so rapidly that the hydrodynamic removal of energy is inconsequential, and the Lawson criterion does not apply. Burning waves propagate at a supersonic velocity, but this conclusion does not contradict the general theory of detonation and combustion. As an application, the energy which a point microexplosion would have to have in order to ignite a thick spherical deuterium-tritium shell with inert admixtures is calculated.

1. INTRODUCTION

Burning waves in DT plasmas have been studied by many workers. For example, Koval'skiĭ[1] studied ignition in an ultradense plasma without a bulk radiative loss. The effect of electron thermal conductivity was analyzed in Refs. 2–4. Burning waves in optically nondense plasmas were studied in Ref. 5; the local release of energy in α particles, the thermal conductivity, and the radiative loss were all taken into account. Detonation waves in a DT mixture were studied in Refs. 6–9.

In the present paper we examine the conditions for the onset of plane burning waves in mixtures and chemical compounds containing chemical elements other than the deuterium and the tritium. We assume that the plasma is transparent to radiation.

We will see that even a small admixture has a marked effect on the nature of the burn:[1)] The emission of electromagnetic radiation and the nonlocal release of energy in α particles become the major competing effects, while the role played by the electron thermal conductivity and the motion of the plasma decreases. The presence of admixtures simplifies the theoretical description of the ignition, although it undoubtedly complicates the ignition itself.

There are reasons for the research interest in the ignition of DT plasmas containing impurities. Most of the inertial-fusion proposals which have been advanced use the assumption that only an insignificant part of the plasma volume (we will call it the "inner part") is heated to the fusion temperature by the driver energy, while most of the DT plasma (the "outer part") is ignited as burning propagates out of the inner volume. In this situation, admixtures in the outer volume might play a helpful role in the target compression dynamics, provided that they are not too great a hindrance to ignition.

We are thinking primarily of the suggestion that a mixture of DT with heavy elements might be used to convert driver energy into thermal radiation for indirect compression of a fusion target. Such conversion is presently regarded as an extremely promising direction in inertial-fusion physics. If the shell in which the conversion is to occur is made of deuterium and tritium with an admixture of a heavy element, which raises the conversion efficiency, then thermonuclear burn in this shell after a microexplosion of the main target would raise the energy yield and the efficiency.

In particular, the possibility of using the radiation emitted from a "cumulative" converging spherical shock wave in DT (with or without admixtures) for an ablative compression of an inner igniter target has been discussed.[10] The results of the present study can be used to determine the requirements which the energy yield of the inner target must meet in order to achieve ignition of the outer DT with admixtures.

Even in the more conventional schemes for inertial fusion, the presence of impurities in the outer part of the target would probably be capable of raising the value of ρR and the energy efficiency of the target. For example, an admixture of a heavy element in the outer part of the DT volume might help block the radiation and keep the compression symmetric, as a result of radiative heat transfer. In pinches[11] and during plasma compression by a converging shock wave,[12] admixtures would intensity the radiative cooling and thereby raise the plasma density.

Thermonuclear burning in chemical compounds of DT (e.g., $LiBD_4 + LiBT_4$) was discussed in Ref. 13. These compounds, in contrast with pure DT, would be in a condensed state at room temperature. We show below that under the conditions assumed here a fusion burn would not be possible in such compounds.

Finally, the results derived here may be of interest for research on ignition of pure DT in certain specific situations. For example, Panarella[14] has suggested that a converging shock wave be used to create pressure at the boundary of the region with the fusion temperature; this pressure would retard the expansion of the hot region and would prolong the inertial confinement. A leading role in limiting the burn would be played by the processes by which energy is removed from the hot region by α particles and electrons, rather than by an expansion of the plasma. The ignition conditions would be determined by the equations derived in the present study. In other cases, the use of these equations for pure DT mixtures would yield only a lower estimate on the ignition criteria, because the motion of the plasma is ignored.

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2. STATEMENT OF THE PROBLEM

We are interested in the processes which occur in an immobile, fully ionized plasma with an ion density $n = \text{const}$. The plasma contains equal numbers of deuterium and tritium ions ($n_d = n_t$); it contains other ions as well. We characterize the plasma by means of the relative DT density $a = 2n_d/n$, the average charge number $\langle z \rangle$, and the mean square charge state $\langle z^2 \rangle$.

In the one-temperature approximation for the plasma, the relaxation of thermonuclear α's and the radiative losses are taken into account. We assume that the neutrons and the electromagnetic radiation pass freely out of the plasma, and we ignore all nuclear reactions except D + T. We wish to determine the criteria for the appearance of one-dimensional burning waves.

3. ONE-DIMENSIONAL HEAT-CONDUCTION EQUATION WITH α-PARTICLE TRANSPORT

Various methods have been used in numerical calculations on α-particle transport: the one-group approximation,[9] the front-back approximation,[15] and the track method.[16] In the one-dimensional case, the equation for heat transfer by α particles can be solved analytically. Some of the expressions for the coefficients of α-particle transport and the method used for studying soliton solutions in the present study are close to those used in Refs. 17 and 18.

To analyze the motion of an α particle with an energy $E = 3.5$ MeV, the plasma can be assumed to be stationary. Estimates show that at a plasma temperature $T \ll 100$ (here and below, temperatures are expressed numerically in kiloelectron volts, while other properties are in cgs units, unless otherwise stipulated) the only force acting on an α particle is the friction force exerted by electrons. At $T \gg 0.5$ the velocity of the electrons is large in comparison with that of the α particles, so we have a simplified equation of motion for the particles:

$$M_\alpha \frac{d\mathbf{v}_\alpha}{dt} = -M_\alpha \mathbf{v}_\alpha \nu(n, T). \tag{3.1}$$

Here M_α, v_α, and z_α, are respectively the mass, velocity, and charge of the α particle, and the collision rate μ is given by

$$\nu = \frac{4(2\pi m_e)^{1/2} z_\alpha^2 e^4 \langle z \rangle n \Lambda}{3T^{3/2} M_\alpha} \approx 5\cdot 10^8 \left(\frac{\langle z \rangle n}{4{,}5\cdot 10^{22}}\right)\left(\frac{T}{10}\right)^{-3/2} \tag{3.2}$$

with the Coulomb logarithm

$$\Lambda \approx 31 - \ln(n^{1/2}/T) \approx 7,$$

for $n \sim 10^{23}$ and $T \sim 10$. It follows from (3.1) that, after it appears, an α particle moves along a straight line. After traveling a distance s it has a velocity

$$v_\alpha = v_0 - \int_0^s \nu\, ds, \qquad v_0 = \left(\frac{2E}{M_\alpha}\right)^{1/2} \approx 1{,}3\cdot 10^9, \tag{3.3}$$

and it is suffering an energy loss per unit length

$$-\frac{d}{ds}\left(\frac{M_\alpha v_\alpha^2}{2}\right) = \nu M_\alpha \left(v_0 - \int_0^s \nu\, ds\right). \tag{3.4}$$

The volume element $d\mathbf{r}'$ emits an isotropic flux of α particles:

$$J(\mathbf{r}')d\mathbf{r}' = \frac{n^2 a^2}{4} \langle \sigma v \rangle d\mathbf{r}', \tag{3.5}$$

where

$$\langle \sigma v \rangle = 2{,}63\cdot 10^{-12} T^{-2/3} \left\{ (1+0{,}16T)\exp\left[-\frac{19{,}98}{T^{1/3}} - \left(\frac{T}{10{,}34}\right)^2\right] + 0{,}0108\exp\left(-\frac{45{,}07}{T}\right)\right\} \tag{3.6}$$

(Ref. 8). At a distance s from the point $\mathbf{r}'$, this flux is distributed over a surface area of $4\pi s^2$. It leads to a bulk energy evolution with a power density dQ which is equal to the particle flux density multiplied by the energy loss per particle per unit length:

$$dQ = M_\alpha \nu \left(v_0 - \int_0^s \nu\, ds\right) \frac{J(\mathbf{r}')d\mathbf{r}'}{4\pi s^2}. \tag{3.7}$$

The total power density of the energy evolution at the point r is found by integrating (3.7) over $d\mathbf{r}'$:

$$Q = M_\alpha \nu(\mathbf{r}) \int J(\mathbf{r}') \left(v_0 - \int_0^{|\mathbf{r}-\mathbf{r}'|} \nu\, ds\right) \theta\left(v_0 - \int_0^{|\mathbf{r}-\mathbf{r}'|} \nu\, ds\right) \times \frac{d\mathbf{r}'}{4\pi(\mathbf{r}'-\mathbf{r})^2}. \tag{3.8}$$

The inner integral is evaluated over a straight line segment between the points $\mathbf{r}$ and $\mathbf{r}'$, and $\theta(x)$ is the unit step function ($\theta = 1$ for $x > 0$; $\theta = 0$ for $x < 0$).

If all functions depend on only a single Cartesian coordinate, x, we can integrate (3.8) over the transverse coordinates:

$$Q(x) = \frac{M_\alpha v_0 \nu}{2} \int_{-\infty}^{+\infty} J(x')\theta(1-|y|)\left(|y| - 1 + \ln\frac{1}{|y|}\right) dx', \tag{3.9}$$

$$y = v_0^{-1} \int_x^{x'} \nu(s)\, ds. \tag{3.10}$$

Integrating (3.9) by parts twice, we find

$$Q(x) = EJ + \frac{\partial}{\partial x}\left\{E v_0^2 \int_{-1}^{1} \frac{1}{\nu(x')} \frac{\partial}{\partial x'}\left[\frac{J(x')}{\nu(x')}\right] \times \left(\frac{1}{12} - \frac{|y|}{2} + \frac{y^2}{4} + \frac{|y|^3}{6} + \frac{y^2}{2}\ln\frac{1}{|y|}\right) dy. \right. \tag{3.11}$$

Finally, if the function $\nu^{-1}\partial(J/\nu)/\partial x$ is assumed to vary only slightly when y changes by ± 1, we find

$$Q = EJ - \frac{\partial q}{\partial x}, \qquad q = -\frac{E v_0^2}{36\nu} \frac{d}{dx}\left(\frac{J}{\nu}\right), \tag{3.12}$$

in which the local energy evolution EJ and the thermal conductivity associated with α-particle transport have been singled out. If the density is homogeneous, the heat flux is proportional to grad T:

$$q=-C(T)\frac{\partial T}{\partial x}, \quad C=\frac{Ev_0^2}{36\nu}\frac{d}{dT}\left(\frac{J}{\nu}\right), \tag{3.13}$$

where $C(T)$ depends on neither n nor the electron thermal conductivity and is given by

$$C_e=1{,}4\cdot10^{19}\,T^{5/2}. \tag{3.14}$$

We have omitted a function $\gamma/\langle z\rangle$ from the expression for C_e; that function decreases slowly with increasing charge number. Introducing an effective index

$$\beta=\frac{\partial \ln J}{\partial \ln T}$$

in the standard fashion, we find from (3.13)

$$C=\left(\frac{a}{\langle z\rangle}\right)^2 T^2\frac{\langle\sigma v\rangle}{10^{-17}}(\beta+3/2)\cdot5{,}3\cdot10^{18}. \tag{3.15}$$

Using $\langle\sigma v\rangle\approx1.3\cdot10^{-17}$ and $\beta\approx3.5$ at $T=5$, we find, for a pure DT mixture (i.e., with $a=\langle z\rangle=\langle z^2\rangle=1$), $C>C_e$ at $T>5$. Comparing the energy production rate EJ with the bulk radiative-loss power Q_r,

$$EJ=\frac{\langle\sigma v\rangle}{10^{-17}}a^2n^2\cdot1{,}4\cdot10^{-23}, \tag{3.16}$$

$$Q_r=n^2T^{1/2}\langle z\rangle\langle z^2\rangle\cdot0{,}53\cdot10^{-23}, \tag{3.17}$$

we easily find that the relation $EJ>Q_r$ holds at these temperatures ($T>5$).

We arrive at an interesting conclusion, which holds quite accurately for the case of DT and also for its mixtures with small values of $\langle z^2\rangle$: heat is transferred out of the ignition regions, with $EJ>Q_r$, primarily by α particles, while it is transferred in cold regions ($EJ<Q_r$) by electrons. It follows that in analyzing the ignition of an optically transparent plasma we must treat α-particle transport as the dominant process.

4. CONDITION FOR THE EXCITATION OF A HEATING WAVE IN A LORENTZ PLASMA

Using the results found above, we can write the energy equation as

$$\frac{3}{2}n(1+\langle z\rangle)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(C_e(T)\frac{\partial T}{\partial x}\right)+T^{-3/2}\int_{-1}^{1}T^{3/2}EJ\left(|y|-1+\ln\frac{1}{|y|}\right)dy-Q_r. \tag{4.1}$$

Linear perturbations of the temperature superposed on the background of the cold ($T=0$) plasma decay. Perturbations which are localized along x and which are of finite amplitude, such that the condition $EJ>Q_r$ holds at the maximum, may grow if the length of the perturbation is large enough; otherwise they will decay. It is natural to assume that there exists a certain intermediate state, namely a localized steady-state solution of Eq. (4.1), which is the boundary between growing and decaying perturbations and which satisfies a steady-state integrodifferential equation

$$\frac{\partial}{\partial x}\left(C_e\frac{\partial T}{\partial x}\right)+T^{-3/2}\int_{-1}^{1}T^{3/2}EJ\left(|y|-1+\ln\frac{1}{|y|}\right)dy=Q_r. \tag{4.2}$$

Solutions of this equation are analyzed in Secs. 4 and 5 of this paper.

To determine whether the integral term in (4.2) can be replaced by the differential relation (3.12), and to estimate the error of this approximation, we consider a case of methodological interest, in which Eq. (4.2) can be solved with both the differential operator and the integral operator (approximately). This is the case of a Lorentz plasma ($\langle z\rangle^2\gg1$), but with $a\approx1$ and $\langle z\rangle\approx1$. This situation corresponds to the presence of a small amount of a large-z admixture in a DT plasma. In a Lorentz plasma, electron-electron collisions do not contribute to the thermal conductivity, and the latter can be calculated easily. It differs from (3.14) by a factor of $4.3\langle z\rangle/\langle z^2\rangle$. Replacing EJ by Q_r (which has the same order of magnitude) in (3.13) for an estimate, and comparing (3.14) with (3.13), we find that under the condition

$$\langle z^2\rangle\gg3 \tag{4.3}$$

the electron thermal conductivity is small in comparison with the heat transfer by α particles, and the corresponding term in (4.2) can be discarded. We introduce the temperature T_0, which is the temperature at which the equality $EJ=Q_r$ holds, i.e., at which we have

$$\langle\sigma v\rangle=\langle z^2\rangle T_0^{1/2}\cdot0{,}38\cdot10^{-17}. \tag{4.4}$$

According to (4.3), the condition $T_0>8$ must hold. In this temperature region we have $\beta\approx\text{const}\approx2$, and Eq. (4.2) becomes

$$\int_{-\infty}^{+\infty}\psi^{3,5}(u')K(u-u')du'=\psi^2(u), \tag{4.5}$$

$$K(u-u')=\left(|u-u'|-1+\ln\frac{1}{|u-u'|}\right)\theta(1-|u-u'|), \tag{4.6}$$

$$\psi=\frac{T}{T_0}, \quad \frac{T_0}{10}\approx\left(\frac{\langle z^2\rangle}{8}\right)^{2/3},$$
$$u=v_0^{-1}\int_0^x \nu\,dx, \quad u'=v_0^{-1}\int_0^{x'}\nu\,dx. \tag{4.7}$$

To find an approximate solution of the nonlinear integral equation (4.5), we approximate its kernel $K(u-u')$ by the Gaussian function

$$K_G=\frac{3}{\pi^{1/2}}\exp[-9(u-u')^2]. \tag{4.8}$$

We choose the parameters of this function in such a way that the zeroth and second moments of K and K_G are the same. The error of this approximation of the equation is $8\cdot10^{-5}\partial^4(\psi^{3.5})/\partial u^4$ in order of magnitude and is quite small for smooth functions ψ. After K is replaced by K_G, Eq. (4.5) has a solution

$$\psi=1{,}75^{1/3}\exp(-27u^2/14). \quad (4.9)$$

The x dependence of ψ is determined by (4.9) and by equation

$$\frac{du}{dx}=\nu(T_0)v_0^{-1}\psi^{-3/2}.$$

We will not reproduce it here. We restrict the discussion to two integral characteristics: the length Δx of the perturbation (this length is finite, although u varies from $-\infty$ to $+\infty$),

$$\Delta x=\frac{v_0}{\nu(T_0)}\int_{-\infty}^{\infty}\psi^{3/2}\,du=\frac{7\pi^{1/2}v_0}{9\nu(T_0)}, \quad (4.10)$$

and L, its energy divided by the cross-sectional area,

$$L=3nT_0\int_{-\Delta x/2}^{\Delta x/2}\psi\,dx=\left(\frac{7}{4}\right)^{5/6}\left(\frac{28\pi}{135}\right)^{1/2}\frac{3nv_0T_0}{\nu(T_0)}. \quad (4.11)$$

We now solve the equation ($\beta=$ const)

$$\frac{1}{36}\frac{\partial^2\psi^{3,5}}{\partial u^2}+\psi^{3,5}-\psi^2=0, \quad (4.12)$$

which is found from (4.2) by replacing the integral operator by a differential operator in accordance with (3.13). A finite solution of this equation is

$$\psi=\left(\frac{14}{11}\right)^{2/3}\cos^{4/3}\frac{9u}{7}. \quad (4.13)$$

From (4.13) we can calculate Δx, L, and the maximum temperature T_m:

$$T_m\approx1{,}2T_0,\ \Delta x\approx4{,}0(n/4{,}5\cdot10^{22})^{-1}(T_0/10)^{3/2},$$
$$L\approx8{,}4(T_0/10)^{5/2}\ \mathrm{MJ/mm^2}. \quad (4.14)$$

In the equations found from (4.8)–(4.11), the numerical coefficients are 1.2, 3.6, and 7.5, respectively. The replacement of the integral operator in (4.2) by a differential one simplifies the equation, at an inconsequential cost in calculation accuracy.

Eqs. (4.14), along with the definition of T_0 in (4.7), lead to the following dependence on $\langle z^2\rangle$:

$$T_m\sim\langle z^2\rangle^{2/3},\quad \Delta x\sim\langle z^2\rangle,\quad L\sim\langle z^2\rangle^{5/3}.$$

The condition for the appearance of plane ignition waves in a Lorentz plasma containing deuterium and tritium is that the parameter values in (4.14) be exceeded.

5. PARAMETERS OF A STEADY-STATE THRESHOLD PERTURBATION INCLUDING ELECTRON THERMAL CONDUCTIVITY

If the mixture has small values of $\langle z\rangle$ and $\langle z^2\rangle$, there is no basis for ignoring the thermal conductivity in comparison with the energy transport by α particles. Let us compare the ratio of coefficients $C(T)/C_e(T)$ with the ratio EJ/Q_r. Using (3.14)–(3.17), and including the $\langle z\rangle$ dependence of $C_e(T)$, which was omitted in (3.14) $[C_e(T)\sim\gamma(z')/\langle z\rangle]$, we find

$$b=\frac{C/C_e}{EJ/Q_r}\approx\frac{(\beta+3/2)\langle z^2\rangle}{7{,}0\gamma}. \quad (5.1)$$

We see that the ratio in (5.1) is of order unity in the case $\langle z^2\rangle\sim\langle z\rangle\sim1$. This result means that, under conditions such that the relation $EJ\sim Q_r$ holds, the electron thermal conductivity is close in order of magnitude to the α-particle thermal conductivity and must in general be considered as important as the latter.

It follows from (5.1) that the simultaneous description of the two thermal-conductivity mechanisms can be simplified somewhat. The reason is that the quantity b turns out to depend on neither n nor a. It depends only very weakly on the temperature T (only through β) and on the charge composition of the ions. Furthermore, the dependence on the temperature and the dependence on the charge composition cancel out to a large extent, since an increase in z is accompanied by a slight increase in the ratio $\langle z^2\rangle/\gamma$, while there are simultaneous increases in the characteristic temperatures, and the value of β decreases. We can thus assume that b is a constant; for simplicity we set it equal to one[2)] ($\beta\approx3.5$–2, $\langle z\rangle\sim1$–2). We thus find

$$C_e\approx\frac{CQ_r}{EJ}. \quad (5.2)$$

It might appear that switching to (5.2) cost us dearly in calculation accuracy ($\approx30\%$ for $\langle z^2\rangle=\langle z\rangle=1$). However, the exact expression for the electron thermal conductivity of a plasma with a complex ion charge composition is difficult to find. Furthermore, it depends on not only $\langle z^2\rangle$ and $\langle z\rangle$ but also the concentration of each ion species. Any approximate expression for C_e will unavoidably be simply a crude estimate, so an error of 20–30% is not excessive.

Using (5.2), and replacing the integral operator in (4.2) by a differential relation, we can put Eq. (4.2) in the form

$$\frac{\partial}{\partial x}\left\{C\left[1+\frac{Q_r}{JE}\right]\frac{\partial T}{\partial x}\right\}+JE-Q_r=0. \quad (5.3)$$

For a solution which is localized along x, the values of T and G grad T tend toward zero at the boundary, so the integral of motion of (5.3) gives us

$$\left\{C\left[1+\frac{Q_r}{JE}\right]\frac{\partial T}{\partial x}\right\}^2=2I, \quad (5.4)$$

$$I(T)=\int_0^T C(\tau)EJ(\tau)\left[\frac{Q_r^2(\tau)}{E^2J^2(\tau)}-1\right]d\tau. \quad (5.5)$$

From (5.4) we find the condition for the existence of a localized solution of Eq. (5.3): There must exist a maximum temperature T_m such that

$$I(T_m)=0, \quad (5.6)$$

$$I(T)>0,\ 0<T<T_m. \quad (5.7)$$

The sign of $I(T)$ is determined only by the value of T and by the parameter

$$S=\langle z\rangle\langle z^2\rangle a^{-2}, \quad (5.8)$$

since the part of the integrand in square brackets in (5.5) is a function of only τ and S. It follows that (5.6) gives T_m as a function of the one parameter S.

It is convenient to replace S by the temperature $T_0(S)$

at which the equality $EJ = Q_r$ holds, by analogy with (4.4). This temperature is found from the equation

$$\langle\sigma v\rangle = ST_0^{1/2}\cdot 0{,}38\cdot 10^{-17}, \tag{5.9}$$

which follows from (3.16), (3.17), and (5.8). Figure 1 shows a curve of $T_0(S)$. This curve was found without considering the relativistic temperature dependence of the bremsstrahlung;[20] that dependence is also ignored in (3.17) and (5.9). A solution of Eq. (5.9) exists only in the case $S < S_0 \approx 33$ [for the extreme value $T_0(S_0) \approx 40$]. This solution is triple-valued. We understand $T_0(S)$ here to be the central branch of the multivalued function, on which we have $0 < T_0 < T_0(S_0)$.

The absence of a solution for $S > S_0$ has a simple physical meaning: For $S > S_0$, at any temperature above absolute zero, the bremsstrahlung loss exceeds the heat evolution in α particles.

Again in the case $S < S_0$, however, the condition for the existence of a steady-state solution of Eq. (5.3) does not always hold. We can find the extreme value of the parameter at which a solution still exists: $S = S_c$. Analysis of the function $I(T)$ shows that it is zero at $T = 0$. As T increases, it goes through a maximum (at $T = T_0$), then goes through a minimum (at $T = T_0'$; Fig. 1), and then increases again. The existence of a value T_m which satisfies conditions (5.6) and (5.7) depends on whether the minimum of $I(T)$ lies above or below the $I = 0$ axis. In the limiting case $S = S_c$, this minimum must lie exactly on the axis (Fig. 2). At $S = S_c$, the value of T_m thus satisfies the following two equations simultaneously:

$$I(T_m) = 0, \quad \frac{dI(T_m)}{dT} = 0. \tag{5.10}$$

For $T > 20$, as we see from (3.6), the expression for β becomes

$$\beta = \frac{45{,}07}{T} - \frac{2}{3}. \tag{5.11}$$

Using (3.13) and (5.11), we find from (5.5)

$$I(T) = \frac{v_0^2}{36v^2}\left[\left(\frac{45{,}07}{T} + \frac{5}{24}\right)Q_r^2 - \frac{1}{2}(JE)^2\right], \tag{5.12}$$

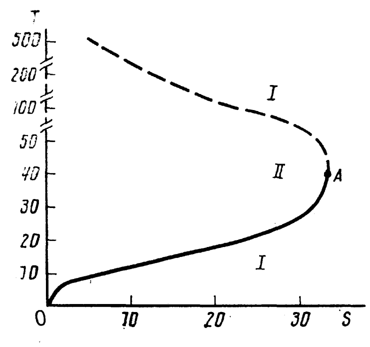

FIG. 1. Curves of $T_0(S)$ (from point 0 to point A) and of $T_0'(S)$ (above point A), on which the equality $Q_r = EJ$ holds. In region I we have $Q_r > EJ$; in region II we have $Q_r < EJ$.

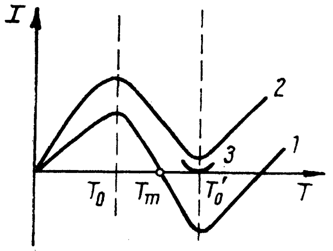

FIG. 2. Schematic plot of $I(T)$ for (curve 1) $1 < S < S_c$ and (curve 2) $S_c < S < S_0$. At $S = S_c$, curve 3 is tangent to the $I = 0$ axis at the point $T = T_m = T_0'$.

which holds for $T > 20$. It follows that Eqs. (5.10) have a solution $T_m \approx 51$ (i.e., $I = 0$ at $Q_r = EJ$) at $S = S_c$, where S_c satisfies (5.9) with $T \approx 51$ and $S_c \approx 32$. When the relativistic temperature correction to Q_r is taken into account approximately,[20] we find $S_c \approx 27$. A localized steady-state solution of Eq. (5.3) thus exists for $S \lesssim 27$.

The absence of a steady-state solution in the interval $27 < S < 33$ of this parameter might seem strange, since at these values there is a temperature interval $T_0 < T < T_0'$ in which the relation $EJ > Q_r$ holds, so that ignition would be possible in principle. As it turns out, however, wave propagation of burning is not possible at these values of S (Sec. 7).

Finally, we can express T_m, Δx, and L in terms of T_0. As $S \to S_c - 0$, the integral parameters Δx and L go off to infinity; in addition, the conditions for the applicability of our model are violated. According to (5.4), Δx and L should be found from

$$\Delta x = 2\int_0^{T_m} \frac{C[1 + Q_r/JE]\,dT}{(2I)^{1/2}}, \tag{5.13}$$

$$L = 3(1 + \langle z\rangle)n \int_0^{T_m} \frac{TC[1 + Q_r/JE]\,dT}{(2I)^{1/2}}. \tag{5.14}$$

Under the condition (5.10), these integrals diverge at their upper limit. The tendency of Δx to diverge requires consideration of the Compton effect and the slowing of the fusion neutrons. For this reason we will restrict the discussion to the case $T_0 < 20$ ($S < 24$).

We can assume that β varies only slightly in this region, having a value of about 3 in the interval $5 < T_0 < 10$ and a value of about 2 in the interval $10 < T_0 < 20$. Setting $\beta = \text{const}$ in (5.12)–(5.14), we find

$$I(T) = \frac{v_0^2 Q_r^2(T_0)}{36v^2(T_0)}\left[\frac{\beta + {}^3/_2}{4}\left(\frac{T}{T_0}\right)^4 - \frac{1}{2}\left(\frac{T}{T_0}\right)^{3+2\beta}\right]$$

$$T_m = T_0\left(\frac{\beta}{2} + \frac{3}{4}\right), \tag{5.15}$$

$$\Delta x = \frac{v_0(\beta + {}^3/_2)}{3v(T_0)} \int_0^{T_m/T_0} \frac{y^{1/2} + y^\beta}{[{}^1/_4(3 + 2\beta)y - y^{2\beta}]^{1/2}}\, y^{1/2}\, dy, \tag{5.16}$$

$$L=\left(\frac{1+\langle z\rangle}{2}\right)\frac{nv_0T_0(\beta+{}^3/_2)}{\nu(T_0)}\int\limits_0^{T_m/T_0}\frac{y^{1/2}+y^\beta}{[{}^1/_4(3+2\beta)y-y^{2\beta}]^{1/2}}y^{3/2}\,dy. \tag{5.17}$$

Evaluating the integrals for $\beta=2$ and $\beta=3$, and substituting in expression (3.2) for ν and expression (3.4) for v_0, we find

$$T_m\approx1{,}2T_0, \tag{5.18}$$

$$\Delta x=2{,}0\langle z\rangle^{-1}\left(\frac{T_0}{5}\right)^{3/2}\left(\frac{4{,}5\cdot10^{22}}{n}\right), \tag{5.19}$$

$$L\approx1{,}8\left(\frac{1+\langle z\rangle}{2\langle z\rangle}\right)\left(\frac{T_0}{5}\right)^{5/2}\quad \text{MJ/mm}^2. \tag{5.20}$$

In these expressions we can set $T_0/5\approx S^{0.4}$ at $S\lesssim8$; at larger values of S we can determine $T_0(S)$ from the curves in Fig. 1.

We have thus found the conditions for the existence of, and integral characteristics of, a steady-state solution of the energy transport equation. These solutions determine the conditions for ignition in a mixture of deuterium and tritium with admixtures. As the admixture concentration approaches zero we have $\langle z\rangle\to1$, $T_0\to5$, and the quantities in (5.18)–(5.20) are approximately the same as the conditions found through numerical calculations for pure DT. However, they are inaccurate because the motion of the plasma is ignored.

6. INSTABILITY OF THE STEADY-STATE SOLUTIONS; GROWTH RATE OF A PERTURBATION ABOVE THE IGNITION THRESHOLD

We repeat that the steady-state solutions found here are meaningful only when they are unstable, in contrast with the usual situation.

The reason is that our assumption that the plasma is homogeneous and immobile is justified if the time scale Δt is short in comparison with the time taken by sound waves to propagate out of the localization region:

$$\Delta t\ll\frac{\Delta x}{2c_s(T_0)}, \tag{6.1}$$

where c_s is the sound velocity. For a steady-state solution we have $\Delta t\to\infty$ and condition (6.1) does not hold. If the steady-state solution is unstable, however, and the growth rate λ satisfies

$$\frac{\lambda\Delta x}{2c_s}\gg1, \tag{6.2}$$

the model which we are using here can be used to calculate the evolution of the temperature distribution. The temperature either exceeds the perturbation threshold (and therefore rises rapidly) or remains below the threshold everywhere and therefore decays rapidly. The steady-state solution of the energy equation, in contrast, has the sole physical meaning of being the threshold between growing and decaying perturbations.

We can prove that steady-state localized solutions of the equation

$$A\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[C\left(1+\frac{Q_r}{JE}\right)\frac{\partial T}{\partial x}\right]+JE-Q_r,\quad A={}^3/_2n(1+\langle z\rangle) \tag{6.3}$$

[this equation is found from (4.1) by replacing the integral operator by a differential operator], are indeed unstable at all values of S for which they exist. We designate a steady-state solution by $\Theta(x)$, and we designate a small perturbation superposed on the steady-state solution by δT [$\delta T\sim\exp(\lambda t)$]. Linearizing (6.3), we find

$$A\lambda\,\delta T=\frac{\partial^2}{\partial x^2}[G(\Theta)\delta T]+\delta T\left[\frac{d}{dT}(JE-Q_r)\,|_{T=\Theta(x)}\right], \tag{6.4}$$

where

$$G(T)=C(T)\left[1+\frac{Q_r}{JE}\right].$$

Equation (6.4) for the function $G\delta T$ is self-adjoint. The Sturm oscillation theorem[21] holds for the eigenfunctions of this equation on the line segment $-\Delta x/2<x<\Delta x/2$ [we put the origin, $x=0$, at the maximum of the function $\Theta(x)$]. In particular, the Sturm theorem asserts that the eigenvalues of the quantity $\mathscr{E}=-A\lambda$ can be numbered in order of increasing value ($\mathscr{E}_0<\mathscr{E}_1<\mathscr{E}_2<\dots$) and that the eigenfunction corresponding to an eigenvalue $\mathscr{E}_m$ has precisely m internal zeros on the line segment $-\Delta x/2<x<\Delta x/2$. This theorem is formulated for the Schrödinger equation in Ref. 22.

Points of importance to the discussion below are that the function $\Theta(x)$ is positive, is even, has a single maximum at $x=0$, and is equal to zero at $x=\pm\Delta x/2$.

We now note that the function $\delta T_1=\varepsilon\partial\Theta/\partial x$, where $\varepsilon\ll\Delta x$ holds, satisfies Eq. (6.4) with $\lambda=0$ and vanishes at the ends of the segment. In other words, it is a neutral mode of the perturbation.

Furthermore, since we have $\delta T_1=0$ at a single internal point, $x=0$, it is clear that δT_1 is an eigenfunction corresponding to the eigenvalue $\mathscr{E}_1$. It follows from the oscillation theorem that we have $\mathscr{E}_0<\mathscr{E}_1=0$ for the fundamental perturbation mode, so we have $\lambda_0>0$.

The instability of the steady-state solution has been proved. The fundamental perturbation mode turns out to be unstable, the first mode has neutral stability, and the higher-order modes are therefore stable. Since the unstable mode is of positive sign, it follows rigorously that an initial temperature distribution $W(x)$ such that $W>\Theta$ and $W-\Theta\ll\Theta$ hold begins to grow with time, while any initial distribution $w(x)$ for which $w<\Theta$ and $\Theta-w\ll\Theta$ hold will decay.

We now need to find an estimate of the instability growth rate λ_0. From (5.4) we find a relation between dx and $d\Theta$:

$$G\,d\Theta=(2I)^{1/2}dx. \tag{6.5}$$

This relation holds for $-\Delta x/2<x<0$. For $0<x<\Delta x/2$ we need to insert a minus sign in front of the radical in (6.5). Now introducing the functions ξ and φ by means of

$$\varphi=G^{3/4}\delta T,\quad d\xi=G^{-1/2}dx, \tag{6.6}$$

we can put Eq. (6.4) in the form

$$\lambda A\varphi=\varphi_{\xi\xi}-\frac{\varphi\Phi_{\xi\xi}}{\Phi}, \tag{6.7}$$

$$\Phi=G^{-1/4}(2I)^{1/2}. \tag{6.8}$$

The sign of $(2I)^{1/2}$ in (6.8) is chosen in the same way as earlier.

Equation (6.7) is a steady-state Schrödinger equation with an energy $-A\lambda$ and a potential $\Phi_{\xi\xi}/\Phi$. The potential is negative and reaches a minimum at the point $x=0$.

To find an estimate of the zeroth eigenvalue, it is natural to assume that in calculating the first two eigenvalues we can use a quadratic approximation of the potential well. For a quadratic potential we have[22]

$$\mathscr{E}_n=U_0+\hbar\omega(n+1/2)$$

where U_0 is the minimum value of the potential. We thus find

$$\mathscr{E}_1-U_0=3(\mathscr{E}_0-U_0).$$

In our case we have $\mathscr{E}_1=0$ and $\mathscr{E}_0=2U_0/3$. We then find an estimate of the growth rate:

$$\lambda_0=-\frac{2}{3A}\left[\frac{\Phi_{\xi\xi}}{\Phi}\right]_{x=0}$$

$$=\frac{2}{3A}\left[\frac{d}{dT}[EJ-Q_r]-\frac{1}{4G}\frac{dG}{dT}(EJ-Q_r)\right]_{T=T_m}. \tag{6.9}$$

Numerically, λ_0 is given approximately by

$$\lambda_0\approx\frac{2}{3}\beta\frac{EJ(T_m)}{AT_m}\approx\frac{4Q_r(T_0)}{AT_m}. \tag{6.10}$$

Let us substitute expressions (6.10) and (5.19) for λ_0 and Δx into condition (6.2). We estimate the sound velocity from the formula for a pure DT mixture:

$$c_s(T)=0{,}8\cdot10^8(T/5)^{1/2}. \tag{6.11}$$

The condition under which we can ignore the motion of the plasma becomes

$$\frac{\langle z^2\rangle}{a^2}\left(\frac{2}{1+\langle z\rangle}\right)\left(\frac{T_0}{5}\right)^{1/2}\gg1. \tag{6.12}$$

Whether this condition holds depends only on the DT content and the ion charge composition; it does not depend on the plasma density.

For a pure DT plasma, the left side of this inequality is of order unity. As the amount of impurity in the plasma increases, condition (6.12) becomes satisfied by a progressively larger margin. Correspondingly, there is a decrease in the effect of the plasma motion on the ignition process.

We reach an important conclusion: In studying the ignition of an optically transparent plasma with admixtures in an inertial-ignition situation, we cannot use one of the fundamental criteria of CTR physics, the Lawson criterion, which is based on the inertial-expansion time. The minimum size of the ignition region is determined not by the condition for inertial confinement but by the diffusion length of the fusion α particles or, more precisely, by the parameter Δx. We might add that the Lawson criterion is of limited applicability again in the case of a pure DT plasma confined by an external pressure. If a plasma of minimal size can be confined for a substantial time, the ignition region will become independent of the expansion time, being determined again by the quantity Δx which we have calculated here.

7. TIME-VARYING TEMPERATURE RISE

After the ignition threshold is exceeded, a temperature perturbation begins to grow and expand. For $S\ll S_c$, we can assume that the value of EJ above the threshold increases much more rapidly than Q_r with the temperature. When the condition $EJ\gg Q_r$ holds, Eq. (6.3) can be put in the following form, where we are using (3.13):

$$A\frac{\partial T}{\partial t}=\frac{v_0^2}{36}\frac{\partial}{\partial x}\left[\frac{1}{\nu}\frac{\partial}{\partial x}\left(\frac{JE}{\nu}\right)\right]+JE. \tag{7.1}$$

In contrast with the equation studied in Ref. 4, the thermal conductivity is due to α particles, rather than electrons, so we are dealing with the HS regime (in the terminology of Ref. 4) of the temperature growth, in the course of which the length Δx of the perturbation increases. Comparing terms on the right side of (7.1), we find[3)]

$$\Delta x\sim v_0/6\nu\sim T^{3/2}.$$

The expansion velocity is given in order of magnitude by

$$V\approx\frac{1}{2}\frac{d\Delta x}{dt}\sim\frac{3}{4}\frac{\Delta x}{AT}JE\sim\frac{1}{8}\frac{v_0}{\nu}\frac{JE}{AT}\sim T^{\beta+1/2}. \tag{7.2}$$

The ratio of V to the sound velocity c_s near the threshold is equal in order of magnitude to expression (6.12); under the conditions $1\ll S\ll S_c$ we have $V/c_s>1$. Above the threshold we have $V/c_s\sim T^\beta\gg1$.

Strictly speaking, the wave propagation of the temperature at a supersonic velocity would make it wrong to label this process "combustion," since the velocity of a classical combustion wave is always below the velocity of sound.[23] A more legitimate term would be "detonation" or, more precisely, "weak detonation."[24]

These comments may sound strange since detonation usually stems from the propagation of a shock wave, and there is no shock wave in our case. However, in a pioneering study of detonation we find the following point emphasized:[25] "...A detonation may propagate at a velocity greater than that calculated from the Jouguet rule...when the igniting agent...propagater more rapidly than a shock wave... ."

In the case of weak detonation in a mixture of DT with admixtures, the igniting agent is the α-particle thermal conductivity, which differs from the classical gaseous thermal conductivity[23] in that it can support the propagation of a burn at a velocity $V\gg c_s$ (in which case we have $n\approx$const).

In the course of the time-varying growth, when the temperature reaches a value $T_0'\sim50$–200, above which the condition $Q_r>EJ$ holds, the temperature rise comes to a halt. During the subsequent evolution of the solution of Eq. (6.3), steady-state traveling waves $T(x-Vt)$ such that

$$-AV\frac{\partial T}{\partial x}=\frac{\partial}{\partial x}\left(G\frac{\partial T}{\partial x}\right)+JE-Q_r \tag{7.3}$$

and

$$T(-\infty)=0, \quad T(+\infty)=T_0'$$

play an important role. From (7.3) we find

$$EJ(T_0')=Q_r(T_0');$$

i.e., T_0' satisfies Eq. (5.9) (the condition $S<S_c$ must hold here). In contrast with T_0, the value of T_0' belongs to the upper branch of our triple-valued function (see Fig. 1 and Sec. 5). In this case the wave is a transition between two linearly stable states.

By analogy with the Kolmogorov-Petrovskiĭ-Piskunov theory,[26] the velocity V is found as an eigenvalue of a boundary-value problem. The equilibrium positions ($T=0$, $T_x=0$) and ($T=T_0'$, $T_x=0$) are saddle points and are connected by a common separatrix if there is a unique value of V for a given S. Multiplying Eq. (7.3) by $G\partial T/\partial x$, and integrating the result from $-\infty$ to $+\infty$, we find

$$VA\int G\left(\frac{\partial T}{\partial x}\right)^2 dx=I(T_0'). \tag{7.4}$$

We have already learned that under the conditions $1\leqslant S<S_c$ Eq. (6.3) has a steady-state threshold solution with $T\sim T_0$. Above this threshold ($T_0\ll T\ll T_0'$) a perturbation grows and expands. Finally, at $T\sim T_0'$, we have shown that steady-state traveling waves exist. The velocity of these waves is negative according to (7.4): The value of I at the point T_0' is negative under specifically the condition $S<S_c$ (Fig. 2). The propagation of such waves sends a $T=0$ state into a $T=T_0'$ state (Fig. 3a).

Traveling waves in the case $S_c<S<S_0$ are of a fundamentally different nature. In this case we have $I(T_0')>0$ (Fig. 2) and $V>0$. The wave thus sends a $T=T_0'$ state into a $T=0$ state and causes quenching, rather than ignition, of the fusion reaction (Fig. 3b). Such waves lead to a decay of the original temperature distribution, even if it has an amplitude $\sim T_0'$ and an arbitrarily great length.

In precisely this S interval we find the value $S=28.5$, which corresponds to the solid chemical compound $LiBD_4+LiBT_4$. Under the conditions which we have assumed here, a thermonuclear burn could not propagate in this compound (and the case against propagation would be even stronger in any other solid compounds of deuterium and tritium with higher values of S).

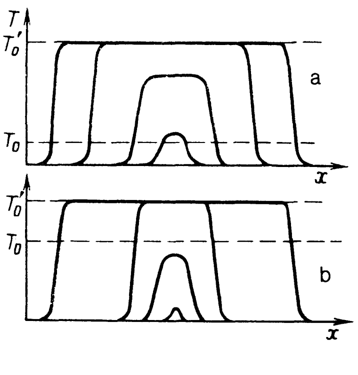

FIG. 3. Time evolution of the temperature distribution. a—$1<S<S_c$; b—$S_c<S<S_0$. The curves are in chronological order.

8. RANGE OF APPLICABILITY OF THESE RESULTS

The temperature rise leads to a violation of the conditions for the applicability of the model which we have adopted here. Let us calculate the optical thickness with respect to Compton scattering (the cross section is $\sigma_T=6.65\cdot10^{-25}$) for a perturbation with a length Δx, which we express in terms of the temperature with the help of (5.19) (or, equivalently, $\Delta x\sim v_0/\nu$):

$$\tau=\Delta x\sigma_T\langle z\rangle n\approx0{,}06\,(T/5)^{7/4}. \tag{8.1}$$

Starting at temperatures $T\sim$ 40–50, with $\tau\sim$ 1–2, we can no longer ignore the Compton effect in connection with the radiation,[27] even if the inverse bremsstrahlung ($\sim n^2$) is negligible. The length of the perturbation at such values of T is such that the heating of the plasma by thermonuclear neutrons must be taken into account. At even higher temperatures we need to take into consideration the burnup of the fuel ($S\neq$const) and the departure of the electron temperature from the ion temperature.

By virtue of the condition $T<$ 40–50, the range of applicability of the results on the time-varying stage, in which the temperature increases rapidly to $T\sim T_0'$, is very limited. In particular, for $1\leqslant S<S_c$ we have $T_0'(S)>50$. On the other hand, the ignition criteria are completely reliable, particularly for $1\ll S\ll S_c$ (in practice, we would have $S\sim$ 3–10). In this case we have $T\ll50$, and, on the other hand, the condition (6.14) holds.

As an application, let us find the condition for the ignition of weak detonation in the wall of a spherical cavity with a radius $R_0\gg\Delta x$ in connection with a point thermonuclear microexplosion inside the cavity. We assume that the energy of the microexplosion is released in the form of α particles and that this energy heats the wall to a depth $\sim\Delta x/2$. After the microexplosion, the total flux of α particles at the boundary of the cavity is zero (as it is at the point $x=0$ of the steady-state profile). Consequently, the threshold temperature profile near the cavity boundary is determined by the function $\Theta(R-R_0)$ for values of the radial coordinate R between R_0 and $R_0+\Delta x/2$. The threshold energy is half of (5.20), multiplied by $4\pi R_0^2$:

$$M\approx10\ [R_0/(1\text{ mm})]^2\cdot(T_0/5)^{5/2}\text{ MJ}. \tag{8.2}$$

9. CONCLUSION

1. The dominant effects during the appearance and propagation of a thermonuclear burn in an optically transparent DT plasma with impurities are the nonlocal release of energy in α particles and the radiative losses.

2. The motion of the plasma is inconsequential. The Lawson criterion based on the inertial-expansion time is no longer applicable.

3. The burn propagates in an unusual weak-detonation regime.

4. Ignition criteria have been derived with the help of an equation derived analytically for the heat transfer by fusion α particles.

5. For a pure DT plasma, these criteria can serve as lower estimates or as estimates for the case in which the plasma expansion is retarded by an external pressure.

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discussion of these results and to P. V. Sasorov for useful consultations.

[1)] It is not completely legitimate to call this process "combustion," as we will see below.

[2)] Essentially the only case in which b can differ greatly from unity (specifically, it would be much greater than unity) is the case of a Lorentz plasma, which we discussed above. In that case, we can use the criteria derived in Sec. 4. A comparison of those criteria with Eqs. (5.18)–(5.20) shows that incorporating the electron thermal conductivity ($b = 1$) changes T_m and L only slightly from their values for a Lorentz plasma ($b \gg 1$) and causes a slight increase in Δx. These results show us once more that the electron thermal conductivity is important only at low temperatures (it smooths the edge of the distribution and increases Δx). It is not important at high temperatures, where we find most of the energy of the distribution.

[3)] Under the condition $\beta = 1$ ($EJ \sim T$) we can derive an exact self-similar solution of Eq. (7.1), with an x dependence $(1 - x^2/l^2(t))^{2/3}$. This result confirms the estimates of Δx and V given above. In the case $\beta = \text{const} > 1$, self-similar separation of variables is possible, but it is difficult to derive an explicit expression for the x dependence and to accurately determine the constants in the expressions for Δx and V.

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