Mechanisms for suppressing induced sound scattering in liquids and gases

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An attempt is made to explain why there are no experimental data on stimulated scattering of sound by inelastic modes of a medium. Equations derived here in the geometric-acoustics approximation describe the propagation of a weak sound wave superposed on an intense pump wave. The excitation of relaxation modes of the medium—quasisteady entropy and vorticity waves—is taken into account. A detailed study is made of the influence of the acoustic streaming which is set up by the pump beam. Since this streaming is nonuniform, low-frequency beats of the sound vibrations can reach resonance with relaxation modes only in a narrow part of the overall interaction region. As a result there is a sharp decrease in the overall growth rate, and the steady-state scattering cannot compete with the excitation of harmonics of the pump wave. The possibility of observing scattering of this type under unsteady conditions is assessed. There are several stringent requirements to be met if an experiment is to be carried out.

1. INTRODUCTION

The propagation of intense sound could hypothetically be accompanied by an effect like stimulated thermal scattering of light.[1] The role of the slow scattering wave might be played by vortex modes,[2] thermal modes,[3,4] compressional modes,[5,6] and other inelastic modes of the medium.

The scattering can be described as follows. The incident sound wave and the scattered sound wave (of approximately the same frequency) interfere, exciting a quasisteady wave. The latter in turn interacts with the incident wave and pumps the scattered wave. This mechanism is effective if the interference wave is approximately the same as one of the relaxation modes of the medium, i.e., if the difference between the frequencies of the incident and scattered waves is on the order of the reciprocal of the decay time of the corresponding mode. The ratio of the oscillation period of the sound wave to the relaxation time determines a threshold value of the pump intensity, above which the scattered wave is amplified.

Unlike stimulated scattering of sound by sound, this process is capable in principle of competing with harmonic generation. In other words, the typical scattering length could be shorter than the wave-breaking distance.

In this paper we attempt to explain why scattering of this type has not been observed experimentally. We find the conditions under which it might become possible to detect this effect.

Previous studies have ignored an important distinction between the propagation of sound and the propagation of light: The propagation of sound is accompanied by high- and low-frequency transport in the medium. Large-scale acoustic motions are particularly important.[7-9] The Doppler corrections to the frequencies of the quasisteady waves which are excited by the transport via acoustic streaming can reach the values of these frequencies themselves and can even exceed them. The velocity of acoustic streaming is proportional to the pump intensity and has a highly nonuniform spatial distribution. As a result, the resonance condition is violated over the greater part of the interaction volume. Under these conditions the effective nonlinear growth rate is no longer a linear function of the pump intensity, and the scattering is suppressed. It follows from the calculations below that saturation of the growth rate is unavoidable under steady-state conditions. It would apparently be possible to detect the scattering under nonsteady conditions, but not just any liquid or gas would be suitable for the observation. It would instead be necessary to carefully select the medium. In addition to having a viscosity and a thermal conductivity, it would have to have an additional slow relaxation mechanism. The temporal and spatial characteristics of the conditions would have to meet some stringent requirements.

2. BASIC EQUATIONS

For clarity we examine a very simple medium which can be described by the standard hydrodynamic equations and by an equation of state of a general type. It is convenient to put this system of equations in a form such that the left sides of the equations describe independent, undamped modes of the medium, while the right sides describe damping of these modes and the interaction:

$$\frac{\partial}{\partial t}\tilde{\rho}+\rho_0 \operatorname{div} \mathbf{v}=-\operatorname{div}(\tilde{\rho}\mathbf{v}),$$

$$-\frac{1}{c^2}\frac{\partial^2 \tilde{P}}{\partial t^2}+\Delta\tilde{P}=\left(\frac{\partial}{\partial t}\right)^2\left(\tilde{\rho}-\frac{\tilde{P}}{c^2}\right)-\partial_i\partial_k(\rho v_i v_k-\sigma_{ik}'), \tag{2.1}$$

$$\sigma_{ik}'=\eta(\partial_i v_k+\partial_k v_i-{}^2/_3\delta_{ik}\operatorname{div}\mathbf{v})+\zeta\delta_{ik}\operatorname{div}\mathbf{v}, \tag{2.2}$$

$$\frac{\partial}{\partial t}W_i=\frac{1}{\rho^2}\left[\nabla\left(\tilde{\rho}-\frac{\tilde{P}}{c^2}\right)\nabla\tilde{P}\right] + (\operatorname{rot}[\mathbf{v}\mathbf{W}])_i+\varepsilon_{ijk}\partial_j\left(\frac{1}{\rho}\partial_l\sigma_{lk}'\right), \tag{2.3}$$

$$\frac{\partial\tilde{S}}{\partial t}=-(\mathbf{v}\nabla\tilde{S})+\frac{1}{\rho T}\operatorname{div}(\varkappa\nabla T)+\frac{1}{\rho T}\sigma_{ik}'\partial_i v_k. \tag{2.4}$$

Here ρ, P, T, S, c, $\mathbf{v}$, and $\mathbf{W}$ = curl $\mathbf{v}$, are the density, pressure, temperature, entropy per unit mass, adiabatic sound velocity in the medium, mass flow velocity, and vorticity.

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The tensor σ'_{ik} is the viscous stress tensor, which contains η and ζ (the shear and bulk viscosity coefficients), and $\varkappa$ is the thermal conductivity. A tilde ($\tilde{}$) means the deviation of the parameter of the medium from its thermodynamic equilibrium value, which is in turn marked by the subscript 0. Equations (2.1)–(2.4) close series expansions of the equations of state $\rho = \rho(P,S)$, $T = T(P,S)$ in the perturbations $\tilde{P}$ and $\tilde{S}$.

We consider the scattering of an intense sound wave 1, with frequency Ω_1 and wave vector $\mathbf{k}_1 = (\Omega_1/c)\mathbf{n}_1$, into a weak sound wave 2, with Ω_2 and $\mathbf{k}_2 = (\Omega_2/c)\mathbf{n}_2$. Here $\mathbf{n}_1$ and $\mathbf{n}_2$ are normalized vectors along the propagation directions of waves 1 and 2, respectively. Causing the scattering is the quasisteady interference wave 3, with $\omega = \Omega_1 - \Omega_2$ and $\mathbf{k}_3 = \mathbf{k}_1 - \mathbf{k}_2$ ($\omega \ll \Omega_1, \Omega_2$; $|\mathbf{k}_3| \sim |\mathbf{k}_1|, |\mathbf{k}_2|$).

In deriving a perturbation theory we make use of the circumstance that the damping is weak for all the waves and that the amplitude of the given pump is small[1)]: χk^2, $\nu k^2 \ll \Omega_{1,2}$, $|A|^2 = |\mathbf{v}_1/c|^2 \ll 1$, where χ and ν are the thermal diffusivity and kinematic viscosity, respectively, of the medium.

3. NONLINEAR GEOMETRIC-ACOUSTICS EQUATIONS

Let us derive equations for the slow evolution of waves 2 and 3 superposed on the pump wave. These equations will be linear in the amplitudes of these waves. The interaction of waves 2 and 3 with the pump wave 1 and the damping of these waves are described here in the first nonvanishing order in the amplitude of the pump wave and in the dimensionless damping rates for these waves (these damping rates are expressed in units of the frequency of the pump wave). Assuming that the frequency of the sound changes little due to scattering, we ignore the distinction between Ω_1 and Ω_2. We omit the corresponding subscript where we can do so without causing any confusion.

Wave 1. It is customary to express the amplitudes of the oscillations of the various physical quantities in the pump wave in terms of a dimensionless amplitude A:

$$\tilde{P}_1 = \rho_0 c^2 A, \quad \tilde{\rho}_1 = \rho_0 A, \quad \mathbf{v}_1 = cA\mathbf{n}_1. \tag{3.1}$$

The velocity distribution in the wave is irrotational. The entropy perturbation is a small quantity of first order,

$$\tilde{S}_1 = -i(\chi\Omega/c^2)C_P(\beta c^2/C_P)A, \tag{3.2}$$

where β is the bulk expansion coefficient, and c_p is the specific heat (per unit mass of the medium) at constant pressure.

Wave 2. The relations between the oscillation amplitudes of the pressure, the density, and the velocity are again given by (3.1), where the subscript 1 is to be replaced by 2 in all cases, and the amplitude A is to be replaced by a. The entropy and the vorticity, on the other hand, acquire some additional perturbations, because of the interaction of the pump wave 1 with the quasisteady wave 3:

$$\tilde{S}_2 = -i(\chi\Omega/c^2)C_P(\beta c^2/C_P)a - A\tilde{S}_3^*[1-(\mathbf{n}_1\mathbf{n}_2)], \tag{3.3}$$

$$\mathbf{W}_2 = -[\mathbf{n}_2[\mathbf{n}_1\mathbf{W}_3^*]]A - i\Omega\frac{\beta T_0}{C_P}[\mathbf{n}_1\mathbf{n}_2]\tilde{S}_3^*A. \tag{3.4}$$

Beats of waves 1 and 2. These beats (which we will call wave 4) occur at the doubled frequency and with the resultant wave vector. At the accuracy level of this analysis, this wave is irrotational and adiabatic. The perturbations of the pressure, the density, and the velocity are

$$\tilde{P}_4 = \rho_0 c^2 a A[1-(\mathbf{n}_1\mathbf{n}_2)]^{-1}\{2\mathscr{C} + [1+(\mathbf{n}_1\mathbf{n}_2)]^2\}, \tag{3.5}$$

$$\tilde{\rho}_4 = \rho_0 a A[1-(\mathbf{n}_1\mathbf{n}_2)]^{-1}[\mathscr{C}+1+(\mathbf{n}_1\mathbf{n}_2)][1+(\mathbf{n}_1\mathbf{n}_2)], \tag{3.6}$$

$$\mathbf{v}_4 = caA[1-(\mathbf{n}_1\mathbf{n}_2)]^{-1}[\mathscr{C}+2(\mathbf{n}_1\mathbf{n}_2)](\mathbf{n}_1+\mathbf{n}_2), \tag{3.7}$$

where

$$\mathscr{C} = \left[\frac{\partial(\ln c^2)}{\partial(\ln\rho)}\right]_S. \tag{3.8}$$

Beats of waves 1 and 3. These beats (wave 5) occur at the fundamental frequency Ω and with the wave vector $\mathbf{k}_5 = 2\mathbf{k}_1 - \mathbf{k}_2$. Only the perturbations of the entropy and the vorticity in this wave will be important to the discussion below:

$$\tilde{S}_5 = A\tilde{S}_3[1-(\mathbf{n}_1\mathbf{n}_2)], \tag{3.9}$$

$$\mathbf{W}_5 = -[(2\mathbf{n}_1-\mathbf{n}_2)[\mathbf{n}_1\mathbf{W}_3]]A - i\Omega\frac{\beta T_0}{C_P}[\mathbf{n}_1\mathbf{n}_2]\tilde{S}_3 A. \tag{3.10}$$

Low-frequency, large-scale perturbations of the medium. The propagation of intense sound is accompanied by heating of the medium and by excitation of vortex flows. These effects are quadratic in the sound amplitude. The deviations of the physical properties of the medium from their equilibrium values are determined by the geometry of the particular problem. The spatial and temporal variations of these deviations are also determined by this geometry. The effect of the nonuniformity of the medium which stems from heating the medium is unrelated to the mechanism for the suppression of scattering which we will be discussing below. That topic requires a separate study, and we intend to return to it in another place. The time-independent part of the velocity, $\mathbf{v}_0$, is related to the acoustic-streaming velocity $\mathbf{U}_a$ by[7–9]

$$\mathbf{v}_0 = \mathbf{U}_a - 2|A|^2 c\mathbf{n}_1. \tag{3.11}$$

The last term on the right side of Eq. (3.11) arises from the formal averaging over the oscillation period. The velocity of steady-state acoustic streaming usually reaches a value on the order of $c|A|^2(kR)^2$, where R is a characteristic transverse dimension of the acoustic sound beam. Acoustic streaming is suppressed if the sound is propagating in a closed channel and if the amplitude of the acoustic vibrations is uniform in any cross section. The velocity of the suppressed acoustic streaming is $c|A|^2$ in order of magnitude.

Dynamic equations for the interacting waves. Using the values calculated above for the perturbations, we find the following equations to describe the slow evolution of the dimensionless amplitudes of the interacting waves. For the envelope of sound wave 2, we find

$$\frac{1}{c}\frac{\partial a}{\partial t} + (\mathbf{n}_2\nabla)a - \frac{i}{2}\frac{c}{\Omega}\Delta a = -\Gamma a + i\frac{\Omega}{c}FA\frac{\tilde{S}_3^*}{C_P} + \frac{1}{2c}\frac{(\mathbf{n}_1\mathbf{n}_2\mathbf{W}_3^*)}{1-(\mathbf{n}_1\mathbf{n}_2)}A + i\frac{\Omega}{c}G|A|^2 a - i\frac{\Omega}{c^2}(\mathbf{n}_2\mathbf{U}_a)a, \tag{3.12}$$

where the form factors F and G are given by

$$F=\frac{1}{2}\left[\beta T_0(1-(\mathbf{n}_1\mathbf{n}_2))-\left(\frac{\partial(\ln c^2)}{\partial(\ln T)}\right)_P\right], \quad (3.13)$$

$$G=\frac{1}{2}\Big\{(\rho_0c^2)^2\frac{\partial^2(c^2)}{\partial P^2}-2(\mathbf{n}_1\mathbf{n}_2)$$
$$-\mathscr{C}(1-(\mathbf{n}_1\mathbf{n}_2))-[\mathscr{C}(1+(\mathbf{n}_1\mathbf{n}_2))^2$$
$$+2\mathscr{C}^2+2(\mathbf{n}_1\mathbf{n}_2)(1+(\mathbf{n}_1\mathbf{n}_2))(\mathscr{C}+2(\mathbf{n}_1\mathbf{n}_2))](1-(\mathbf{n}_1\mathbf{n}_2))^{-1}\Big\}. \quad (3.14)$$

The linear damping rate Γ is given by the well known expression

$$\Gamma=(\gamma-1)\frac{\chi\Omega^2}{c^3}+\frac{\zeta\Omega^2}{\rho_0c^3}+\frac{4}{3}\frac{\eta\Omega^2}{\rho_0c^3}, \quad (3.15)$$

where $\gamma=C_P/C_V$ is the ratio of specific heats.

The first term on the right side of Eq. (3.12) describes linear damping of the weak sound wave. The second and third terms describe the conversion of, respectively, the entropy wave and the vorticity wave into the weak wave, as a result of the interaction with the pump wave. Finally, the last two terms contribute nonlinear corrections to the dispersion relation.

On the left side of Eq. (3.12) there is a term with a Laplacian operating on the amplitude. This term may prove important in studies of diffraction and nonlinear-dispersion effects.

The conversion of the entropy wave into a scattered sound wave results from (first) the transport of the low-frequency perturbation of the entropy density by the high-frequency streaming in the pump wave and (second) the modulation of the sound velocity in the quasisteady wave. In expression (3.13) for the form factor F, these mechanisms are described by respectively the first and second terms in square brackets. This transport effect has been ignored in previous studies[2)] (Refs. 3, 4, and 10).

The development of the entropy wave is determined by the thermal conductivity in the field of the nonlinear heat sources and by convection in the acoustic streaming:

$$\frac{1}{\Omega}\left\{\frac{\partial}{\partial t}-i[\omega-(\mathbf{k}_3\mathbf{U}_a)]+\chi k_3^2\right\}\frac{\tilde{S}_3}{C_P}=D_Sa^*A, \quad (3.16)$$

where

$$D_S=2\frac{\chi\Omega}{c^2}\frac{\beta c^2}{C_P}\Big\{[1-(\mathbf{n}_1\mathbf{n}_2)]$$
$$\times\left[1+\left(\frac{\partial(\ln\varkappa)}{\partial(\ln\rho)}\right)_S(1-(\mathbf{n}_1\mathbf{n}_2))\right]+\frac{\beta c^2}{C_P}\Big\}$$
$$+2\frac{c^2}{T_0C_P}\left\{\frac{\zeta\Omega}{\rho_0c^2}+2\frac{\eta\Omega}{\rho_0c^2}\left[(\mathbf{n}_1\mathbf{n}_2)^2-\frac{1}{3}\right]\right\}. \quad (3.17)$$

Vorticity is generated by nonlinear stresses, which are opposed by linear viscous forces. The vorticity is transported by the acoustic streaming:

$$\frac{1}{\Omega}\left\{\frac{\partial}{\partial t}-i(\omega-(\mathbf{k}_3\mathbf{U}_a))+\nu k_3^2\right\}\frac{\mathbf{W}_3}{\Omega}=2i[\mathbf{n}_1\mathbf{n}_2]D_Wa^*A, \quad (3.18)$$

where

$$D_W=\frac{c}{\Omega}\Gamma+2\frac{\nu\Omega}{c^2}\left[\frac{\partial(\ln\eta)}{\partial(\ln\rho)}\right]_S[1-(\mathbf{n}_1\mathbf{n}_2)]. \quad (3.19)$$

In Eqs. (3.17) and (3.19) we have discarded some terms which are associated with differentiation of the oscillation amplitudes of respectively the entropy and vorticity with respect to the spatial coordinates. The legitimacy of this simplification will be discussed below.

4. STEADY STATE

We seek steady-state solutions of the dynamic equations for the interacting waves. In this case, Eqs. (3.16) and (3.18) become algebraic equations for the quantities $\tilde{S}_3$ and $\mathbf{W}_3$. Substituting them into the equation for the envelope of the sound wave, (3.12), we find a nonlinear dissipative correction k_{tot} to the wave number k_2:

$$-i\frac{c}{\Omega}k_{tot}=-\frac{c}{\Omega}\Gamma+2i\left[G|A|^2-\frac{1}{c}(\mathbf{n}_2\mathbf{U}_a)\right]$$
$$-\frac{|A|^2FD_S\Omega}{i\chi k_3^2+\omega-(\mathbf{k}_3\mathbf{U}_a)}-\frac{|A|^2D_W\Omega}{i\nu k_3^2+\omega-(\mathbf{k}_3\mathbf{U}_a)}[1+(\mathbf{n}_1\mathbf{n}_2)]. \quad (4.1)$$

An amplification with a low threshold could be established locally in, for example, a very viscous liquid ($\nu\gg\chi$).[10] In this case—which we will, for definiteness, have in mind below—viscous terms dominate the linear damping rate and the form factor D_S. At the optimum value of the frequency detuning $\delta=\omega-(\mathbf{k}_3\mathbf{U}_a)$, i.e., under the condition $\delta\approx\chi k_3^2$, the contribution to the damping rate from the interaction with the entropy wave is on the order of $\Gamma|A|^2\Omega/\delta$. The threshold is reached as early as $|A|^2\approx\delta/\Omega$, and significant amplification can be achieved at $|A|^2\gg\delta/\Omega$.

If the acoustic streaming were uniform, incorporating it would have only one result: an additional Doppler frequency shift of the scattered wave. Since the acoustic streaming is nonuniform, however, the detuning δ depends on the coordinates, and the resonance condition can be satisfied only in a bounded region with a size which becomes smaller as the pump intensity becomes larger. Outside the resonance zone, the growth rate is small (on the order of the linear damping rate). Furthermore, the amplification gives way to an attenuation when the sign of δ is reversed.

In evaluating the total amplification, we restrict the analysis to the nonuniformity of the acoustic streaming along directions transverse with respect to the pump beam. In this approximation,[3)] the acoustic-streaming velocity is directed along the beam axis and is a quadratic function of the transverse coordinate r. The nonlinear growth rate thus has the profile

$$\Gamma_{NL}=\frac{\Omega}{c}\frac{\delta(r)\Omega}{(\chi k_3^2)^2+[\delta(r)]^2}|A|^2FD_S,$$
$$\delta=\delta_0-\delta_2(r/R)^2, \quad (4.2)$$

where δ_0 is the detuning at the beam axis, and δ_2 is the difference between the Doppler shifts at the axis and periphery of the beam.

The total amplification can reach a significant level only if Γ_{NL} is positive over the entire interaction volume. In turn, that situation is possible if the resonance zone is at the axis or periphery of the pump beam. Just where depends on the direction of the streaming velocity and on which wave—Stokes or anti-Stokes—is being amplified. If the resonance zone lies somewhere else, it consists of regions differing in the sign of the detuning and, correspondingly, in the sign of the growth rate. The amplification and attenuation in these regions cancel out. Using (4.2), we can easily estimate the dimensions of the resonance zone:

$$L_{res}^{a}=R\left(\frac{\chi k_3^2}{\delta_2}\right)^{1/2},\quad L_{res}^{b}=R\frac{\chi k_3^2}{\delta_2};\tag{4.3}$$

The superscripts a and b refer to the cases in which the resonance zone is at the axis and boundary, respectively, of the beam.

The direction of the streaming velocity and its order of magnitude differ for developed and suppressed acoustic streaming. We will accordingly discuss these cases separately.

Well-developed acoustic streaming is the typical form of large-scale acoustic streaming. The direction of the streaming velocity at axis of the pump beam coincides with the beam direction. In order of magnitude, the streaming velocity is[7–9]

$$U_a\approx c|A|^2\Gamma cR^2/\nu.\tag{4.4}$$

If it is the Stokes wave which is being amplified, the resonance zone lies at the beam axis ($\delta_0=0$); if it is the anti-Stokes wave, this zone is at the beam periphery ($\delta_0=\delta_2$). According to (4.3), the dimensions of the resonance zones in these cases are, in order of magnitude,

$$L_{res}^{a}\approx\frac{1}{k}\left(\frac{\chi k_3^2}{\Omega}\frac{\nu k_3^2}{c\Gamma}\frac{1}{|A|^2}\right)^{1/2},\quad L_{res}^{b}\approx\frac{1}{k}\frac{\chi k_3^2}{\Omega}\frac{\nu k_3^2}{c\Gamma}\frac{1}{|A|^2}.\tag{4.5}$$

In each case the width of the zone is much smaller than the wavelength, telling us that pronounced variations arise in the resonance zone, and Eqs. (3.16) and (3.18) must be modified. We will return to this topic in the following section of the paper; at this point we turn to an analysis of the suppressed acoustic streaming.

If the characteristic streaming velocity is to be reduced, the pump wave must propagate in a waveguide and must be completely uniform over the cross section of the waveguide. In addition, any possibility of an average mass transport over the cross section must be eliminated.

For simplicity we consider the case of a planar one-dimensional waveguide. In this case the velocity of the acoustic streaming is directed opposite the pump wave and has a parabolic profile

$$U_a=-{}^3/_8c[1-3(x/R)^2]|A|^2.\tag{4.6}$$

If it is the Stokes wave which is being amplified, the resonance zone is at the waveguide wall (superscript b), while if it is the anti-Stokes wave the zone is at the waveguide axis (a). The dimensions of the resonance zone are, in order of magnitude,

$$L_{res}^{a}=R\left[\frac{\chi k_3^2}{\Omega}\frac{\nu k_3^2}{c\Gamma}\frac{1}{|A|^2}\right]^{1/2},\quad L_{res}^{b}=R\frac{\chi k_3^2}{\Omega}\frac{\nu k_3^2}{c\Gamma}\frac{1}{|A|^2}.\tag{4.7}$$

Since the width of the resonance zone in this case is considerably larger than the wavelength, Eq. (3.12) gives a valid description of the evolution of the scattered wave. This evolution is determined by the interplay of three competing processes: the amplification of the wave, the escape of this wave from the resonance zone, and the phase distortions of the wave. We can look at two typical situations: the amplification of a probe beam and propagation of a waveguide mode.

We first consider the amplification of a probe beam directed at some angle from the axis. If this angle is not too small, phase distortions do not have time to build up over the time taken by the probe beam to traverse the resonance zone. The overall amplification is found by integrating the local growth rate along the path of the probe beam. The effective amplification along the waveguide axis is characterized by a growth rate

$$\Gamma_{eff}=\frac{1}{\cos\theta}\langle\Gamma_{NL}\rangle,\quad \langle\Gamma_{NL}\rangle=\frac{1}{R}\int_0^R\Gamma_{NL}(x)dx,\tag{4.8}$$

where θ is the angle between the direction of the probe beam and the axis of the waveguide. The nonlinear growth rate for the Stokes wave (superscript b), averaged over the cross section, is, according to (4.6), (4.2), and (4.8),

$$\langle\Gamma_{NL}^{b}\rangle=\frac{4}{9}\frac{\Omega}{c}FD_s\ln\left(\frac{\Omega}{\chi k_3^2}|A|^2\right),\tag{4.9a}$$

That for the anti-Stokes wave (superscript a) is

$$\langle\Gamma_{NL}^{a}\rangle=\frac{4}{9}\frac{\Omega}{c}FD_s\left(\frac{\Omega}{\chi k_3^2}|A|^2\right)^{1/2}.\tag{4.9b}$$

It can be seen from these expressions that the dependence of the effective growth rate on the pump amplitude is weakened by the nonuniformity of the amplification. Because of this factor and because the amplification length cannot be greater than the breaking distance for the pump wave, which is known to be inversely proportional to the amplitude of this wave, the total growth rate reaches saturation at a point far above the threshold. Although amplification of the scattered wave is possible, it remains at the threshold level in the case $\cos\theta\approx 1$.

On the other hand, we can conclude from (4.8) that the growth rate increases without bound as $\theta\to\pi/2$. The reason is that in the case of nearly transverse propagation the probe beam can intersect the resonance zone repeatedly while traversing a unit length along the axis of the waveguide. Actually, expression (4.8) breaks down at angles close to $\pi/2$. The physical reason is that the minimum difference $\delta\theta=\pi/2-\theta$ cannot be smaller than the diffractive divergence angle of the beam which is built up over the amplification distance $l_{\rm amp}\approx\langle\Gamma_{NL}\rangle^{-1}$. This minimum difference is

$$\delta\theta\approx(kl_{amp})^{-1/2}\approx(\langle\Gamma_{NL}\rangle/k)^{1/2}.$$

Substituting this expression for $\cos\theta$ in (4.8), and multiplying by the breaking distance $l_{br} \approx (k|A|)^{-1}$, we find the maximum integral growth rate to be

$$\Gamma_{int}^{max} \approx |A|^{-1}(\langle\Gamma_{NL}\rangle/k)^{1/2}.$$

Using the values for the average growth rate in (4.9), we conclude that the total amplification can reach a level on the order of $(\nu/\chi)^{1/2}$ if the pump amplitude is near the threshold level ($|A| \approx \chi\Omega/c^2$).

These qualitative considerations can be supported by the following calculation. We assume that the probe beam is propagating in a direction nearly transverse to the waveguide axis (along the x axis). In the term with the Laplacian in (3.12) we retain the second derivative of the amplitude with respect to z. It is a straightforward matter to find a solution of the resulting equation in the form

$$a(x,z) = \exp\left\{iq_z z - i\frac{cQ^2}{2\Omega}x - i\int_0^x k_{tot}(\xi)\,d\xi\right\}, \tag{4.10}$$

where q_z is the z component of the wave vector q. When the fast phase dependence $\exp(i\Omega x/c)$ is taken into account, we see that the solution describes a wave which is propagating from one wall of the waveguide toward the other. The reflected wave corresponds to a change in the sign of x. The superposition of these waves is a waveguide mode if the following quantization condition is satisfied[4]:

$$\frac{\Omega}{c} - \frac{cq_z^2}{2\Omega} - \langle k_{tot}\rangle = q_n, \qquad q_n = \frac{\pi n}{R}, \tag{4.11}$$

where the average has the same meaning as in Eq. (4.8). We thus find

$$\Gamma_{eff} = -\mathrm{Im}\, q_z, \qquad q_z^2 = 2\frac{\Omega}{c}(k_z - \langle k_{tot}\rangle), \qquad k_z = \frac{\Omega}{c} - q_n. \tag{4.12}$$

Under the condition $k_z \gg \langle k_{tot}\rangle$, the result from (4.8) matches with (4.12). In the opposite case $k_z \ll \langle k_{tot}\rangle$, we obtain the diffraction limit, which was discussed qualitatively above.

Expression (4.8) is also incorrect in the case of backscattering. In this case the scattered probe beam may lie entirely in the resonance zone. Because of the pronounced nonuniformity of the induced wave, however, the propagation of the probe beam is accompanied not only by amplification of this beam but also by substantial phase distortions. Actually, the probe wave decays into independent modes (when the nonuniform, complex "refractive index" is taken into account), and its amplification is limited by the maximum growth rate of these modes. For a mathematical description of the structure of these modes, we work from Eq. (3.12), in which the vector $\mathbf{n}_2$ is directed along the z axis, and from the Laplacian we retain $\partial^2/\partial x^2$. We seek a solution of the resulting equation in the semiclassical approximation:

$$a(z,x) = \frac{a_0}{[p(x)]^{1/2}}\exp\left\{i\delta kz + i\int_0^x p(\xi)\,d\xi\right\}, \tag{4.13}$$

where

$$p(x) = \pm[2(\Omega/c)(-\delta k + k_{tot})]^{1/2}. \tag{4.14}$$

In order to satisfy the boundary conditions, we need to use a superposition of two linearly independent solutions (4.13), with different signs of $p(x)$ in (4.14). It is a straightforward matter to derive quantization conditions which determine the spectrum δk:

$$\int_{-R}^{R} p(x)\,dx = \pi n, \tag{4.15}$$

where n is an integer.

We will first show how to find the effective growth rate in (4.8) from the quantization condition (4.15). If the strong inequality $|\delta k| \gg |k_{tot}|$ holds throughout the interaction volume, including the resonance zone, we can write the expansion

$$p(x) = \pm\left(-2\frac{\Omega}{c}\delta k\right)^{1/2}\left(1 - \frac{1}{2}\frac{k_{tot}}{\delta k}\right) \tag{4.16}$$

and seek a solution of the dispersion relation (4.15) by successive approximations. In the first two orders we find[5]

$$\delta k^{(1)} = -\frac{c}{2\Omega}\left(\frac{\pi n}{R}\right)^2, \qquad \delta k^{(2)} = \langle k_{tot}\rangle. \tag{4.17}$$

The imaginary part $\delta k^{(2)}$ is the same as Γ_{eff} from (4.8). The resulting expressions are valid for scattering angles

$$\theta = \frac{\pi n}{R}\frac{c}{\Omega}$$

which are so large that

$$\theta^2 \gg \frac{c}{\Omega}\Gamma\frac{\Omega}{\chi k_3^2}|A|^2.$$

If the last inequality does not hold, expansion (4.16) is not valid in the resonance zone.

We turn now to the directly opposite case:

$$\theta^2 \ll \frac{c}{\Omega}\Gamma\frac{\Omega}{\chi k_3^2}|A|^2,$$

which corresponds to $n = O(1)$. If the resonance zone is not at the center of the waveguide (it might be at the boundary of the waveguide), the solution of the dispersion relation would have to be found numerically. In order of magnitude, we have $|\delta k| \approx \Gamma$. If the resonance zone is instead at the center of the waveguide, we can find δk asymptotically, by assuming

$$\frac{\Omega}{\chi k_3^2}|A|^2 \gg \frac{\delta k}{\Gamma} \gg 1.$$

The integral in (4.15) can be evaluated approximately by breaking up the integration region into three parts: the resonance zone proper, of size L_{res}^a; the wings of the resonance zone ($R \gg |x| \gg L_{res}^a$); and the nonresonant region, which is the rest of the volume. The contribution from the resonance zone can be calculated by multiplying the size of the zone by the characteristic value of the momentum:

$$p \approx \frac{\Omega}{c}\left(\frac{\Omega}{\chi k_3^2}|A|^2 FD_s\right)^{1/2},$$

which is, in order of magnitude, $R(\Omega/c)(FD_s)^{1/2}$. In the nonresonant region, the momentum is $p(x) \approx (-2\Omega\delta k/c)^{1/2}$, and the size of the resonance zone is $2R$. Finally, in the wings of the resonance, we have[6)]

$$p(x)\approx\frac{2}{3}\frac{\Omega}{c}\frac{R}{|x|}(iFD_s)^{1/2}.$$

The corresponding contribution to the integral (4.15) is therefore logarithmic:

$$\frac{1}{3}\frac{\Omega}{c}R(iFD_s)^{1/2}\ln\left(\frac{R}{\delta k}\frac{\Gamma}{L^a_{res}}\right).$$

From the condition that the phase contributions in the wings and outside the resonance zone cancel out we find

$$\delta k=-\frac{i}{24}\frac{\Omega}{c}FD_s\left[\ln\left(\frac{\Omega}{\chi k_3^2}|A|^2\frac{\zeta^2}{\eta^2}\right)\right]^2. \qquad (4.18)$$

The amplitude dependence in (4.18) is seen to be even weaker than in (4.9b). For this reason, the growth of the scattered wave could hardly compete with harmonic generation, even though amplification is in principle possible above the threshold.

5. WELL-DEVELOPED ACOUSTIC STREAMING

We consider the propagation of a test sound wave superposed on the pump wave, with well-developed acoustic streaming. We assume that the plane pump wave, with a wave number $k = \Omega/c$, is propagating along the z axis and that its propagation direction is the same as that of the velocity of the acoustic streaming, $\mathbf{U}_a(x)$. Here, as in the preceding section of this paper, we consider the plane-geometry problem. We assume a coordinate dependence $\exp(iqz)a(x)$ for the amplitude of the oscillations of the test sound wave. Correspondingly, for the low-frequency entropy oscillations we have

where $Q = k - q$. The conversion of the test sound wave into the entropy wave and vice versa are described by modified dynamic equations for the interacting waves[7)]:

$$\left(\frac{d^2}{dx^2}+K_\perp^2\right)a(x)=-k^2FA\sigma^*(x), \qquad (5.1)$$

$$\frac{1}{\Omega}\left\{-\chi\frac{d^2}{dx^2}-i[\omega-QU_a(x)]+\chi Q^2\right\}\sigma(x)=D_sAa^*(x), \qquad (5.2)$$

where $K_\perp^2 = k^2 - q^2 + 2ik\Gamma$. This is a simplified system of equations; it totally ignores the low-frequency oscillations of the vorticity. The justification for this simplification is that the vorticity contribution enters the nonlinear growth rate and the nonlinear diffractive distortions additively and differs from the entropy contribution in that χ is replaced by ν. For this reason, the former is always smaller than the latter in accordance with the assumptions made in the preceding section of this paper (more on this below).

Equations (5.1) and (5.2) must be supplemented with boundary conditions, the vanishing of $da(x)/dx$ and $\sigma(x)$ at $x = \pm R$. We seek a solution of these equations by an iterative method in the small amplitude A of the pump wave. In zeroth order we find

$$a^{(0)}(x)=\alpha\psi_\nu(x), \quad \sigma^{(0)}(x)=0, \quad K_\perp^{(0)}=K_\nu, \qquad (5.3)$$

$$\psi_\nu(x)=(2R)^{-1/2}\begin{cases}\cos(K_\nu x), & \nu\in Z\\ \sin(K_\nu x), & \nu\in Z+1/2\end{cases}, \quad K_\nu=\nu\pi/R, \qquad (5.4)$$

where α is a dimensionless amplitude. In first order in the pump amplitude we have $a^{(1)}(x) = 0$, and we find $\sigma^{(1)}(x)$ from Eq. (5.2) after we substitute the solution (5.3) into it.

We seek a solution for the entropy wave in the form

$$\sigma(x)=\alpha^*D_sA\mathcal{P}(x)\psi_\nu(x). \qquad (5.5)$$

Over the greater part of the volume we can ignore the second derivative with respect to x, so we can write

$$\mathcal{P}(x)=i\Omega/\omega(x), \quad \omega(x)=\omega-QU_a(x). \qquad (5.6)$$

This approximation breaks down near roots of the function $\omega(x)$. The behavior of $\mathcal{P}(x)$ near these roots is determined by reference equations, to whose derivation we now turn. We denote by x_0 a root of $\omega(x)$. We define the length scale $\mathcal{L}$ in such a way that the two terms on the left side of Eq. (5.2) are of the same order of magnitude at distances $|x - x_0| \leqslant \mathcal{L}$. We set $x = x_0 + \mathcal{L}\xi$, where ξ is a new local coordinate, if $x_0 \gg \mathcal{L}$. Near x_0 we have $\omega(x) = \omega'\mathcal{L}\xi$. If $x_0 = O(\mathcal{L})$, then $\omega(x) = \omega''\mathcal{L}^2(\xi^2 - \xi_0^2)$. After the substitution

$$\mathcal{P}(x)=\frac{\Omega}{\chi}\mathcal{L}^2\mathfrak{S}(\xi) \qquad (5.7)$$

we find the following reference equations for the function $\mathfrak{S}(\xi)$:

$$\left\{\frac{d^2}{d\xi^2}-i\xi+\delta\right\}\mathfrak{S}(\xi)=1, \quad \mathcal{L}=\left(\frac{\omega'}{\chi}\right)^{-1/3}, \qquad (5.8a)$$

$$\left\{\frac{d^2}{d\xi^2}+i(\xi_0^2-\xi^2)+\delta\right\}\mathfrak{S}(\xi)=1, \quad \mathcal{L}=\left(\frac{\omega''}{\chi}\right)^{-1/4}. \qquad (5.8b)$$

In each case we have $\delta = (Q\mathcal{L})^2$.

In order of magnitude [see (4.4)], we have $\omega' \approx \Omega|A|^2Rk^2$ and $\omega'' \approx \Omega|A|^2k^2$. We thus have $\delta \ll 1$ and $K_\nu\mathcal{L} \ll 1$. By virtue of the last inequality, the right sides of the reference equations reduce to unity. In other words, the function $a^{(0)}(x)$ varies negligibly over the interval $\mathcal{L}$.

Solutions of the reference equations can be found through quadratures with the help of the solutions of the corresponding homogeneous equations: Airy functions [for (5.8a)] and parabolic cylinder functions [for (5.8b)]. For our purposes, however, there is no need for explicit expressions for the corresponding integral representations. The only important point is that the functions $\mathfrak{S}(\xi)$ are regular near $\xi = 0$ and that the following asymptotic forms occur at $\xi \gg 1$:

$$\mathfrak{S}(\xi)\approx i\xi^{-1}, \quad \mathfrak{S}(\xi)\approx i(\xi_0^2-\xi^2)^{-1}. \qquad (5.9)$$

Since the parameter δ is small, and the function $\mathfrak{S}(\xi)$ regular, the behavior of this function depends only negligibly on δ. It follows that the order of magnitude of $\mathscr{S}(x)$ near the roots of this function is found from (5.8), where $\mathfrak{S}(\xi) = O(1)$. Comparing $\mathscr{L}$ with the dimensions of the resonance zone, (4.5), we easily see that if the resonance zone is at the center it undergoes broadening:

$$\frac{\mathscr{L}}{L_{res}} \approx \left(|A|^2 \frac{\Omega}{\chi k^2} \right)^{1/4}.$$

For an arbitrary position of the zone, in contrast, the zone is more likely to shrink:

$$\frac{\mathscr{L}}{L_{res}} \approx \left(\frac{|A|^2}{(kR)^2} \frac{\Omega}{\chi k^2} \right)^{1/4}.$$

In each case, the height of the $\sigma(x)$ peaks decreases by a factor of $(kR)^2$.

Using standard perturbation theory, we find corrections to the wave amplitude and to the transverse wave number:

$$a_\nu^{(2)}(x) = \alpha |A|^2 F D_S k^2 \sum_{\lambda \neq \nu} \frac{\langle \lambda | \mathscr{S} | \nu \rangle}{K_\nu^2 - K_\lambda^2} \psi_\lambda(x), \qquad (5.10)$$

$$2\delta K_\nu K_\nu^{(0)} + (\delta K_0)^2 \delta_{\nu,0} = k^2 |A|^2 F D_S \langle \nu | \mathscr{S} | \nu \rangle, \qquad (5.11)$$

where the matrix elements are

$$\langle \lambda | \mathscr{S} | \nu \rangle = \int_{-R}^{R} \psi_\nu(x) \psi_\lambda(x) \mathscr{S}(x) dx. \qquad (5.12)$$

Expression (5.10) can be used to evaluate the diffraction effects. From (5.11) we find the nonlinear growth rate, with the help of

$$2q\Gamma_{NL} = -\operatorname{Im}\{2\delta K_\nu K_\nu^{(0)} + (\delta K_0)^2 \delta_{\nu,0}\}. \qquad (5.13)$$

To evaluate the integrals in (5.12), we break the integration region up into parts in such a way that we can single out a pseudoresonance zone. The reference equations are still valid inside this pseudoresonance zone, but at its boundaries the function $\mathfrak{S}(\xi)$ goes into its asymptotic behavior. Outside the pseudoresonance zone, the asymptotic expression in (5.7) is valid.

We first note that for an arbitrary position of the pseudoresonance zone the basic contribution from this zone to the growth rate (5.13) vanishes. The reason is that the reference equation (5.8a) is invariant under a change in the sign of ξ and a simultaneous complex conjugation of the function $\mathfrak{S}(\xi)$. Consequently, Im $\mathfrak{S}(\xi)$ is an odd function of its argument, and the contributions to the imaginary part of integral (5.12) from left-hand and right-hand ε-neighborhoods of x_0 cancel out. The logarithmic contributions from the nonresonant regions at the boundaries of the pseudoresonance zone also cancel out.

In summary, as in the case of the suppressed acoustic streaming, we would not expect anything in the way of a significant amplification unless the pseudoresonance zone lies at the periphery or center. Let us examine both possibilities.

In examining the peripheral pseudoresonance zone, we need to take account of the boundary condition on the entropy. The effect of the boundary layer becomes important. We should not, on the other hand, expect that the contribution from this boundary layer would completely cancel out the contribution from the pseudoresonance zone. The latter can be estimated in order of magnitude by multiplying the value $\mathscr{S}(x_0 \approx R)$ by the size of the characteristic interval $\mathscr{L}$. As a result we find

$$\Gamma_{res} \approx \frac{\mathscr{L}}{R} \frac{\Omega}{\chi} \mathscr{L}^2 |A|^2 F D_S. \qquad (5.14)$$

Substituting in $\mathscr{L}$ from (5.8a), we find a small amplitude-independent growth rate $\Gamma_{res} \approx \Gamma/(kR)^2$. The contribution of the nonresonant region to Γ_{NL} comes primarily from near the boundary of this region, where $\omega(x)$ is small [see (5.8)]:

$$\Gamma_{NL} \approx k^2 \frac{\Omega |A|^2}{\omega'(0)} F D_S \ln \frac{R}{\mathscr{L}} \approx \frac{\Gamma}{3kR} \ln \left[\frac{\Omega |A|^2}{\chi k^2} (kR)^2 \right]. \qquad (5.15a)$$

Comparing this expression with (4.9a), we see that the amplification in this case is far less effective, and it is not possible to exceed the threshold.

If the pseudoresonance zone is at the center, the estimate (5.14) remains in force. Using (5.11), we find

$$\Gamma_{res} \approx \frac{k}{R} \frac{\Omega}{\chi} \left(\frac{\chi}{\omega''(0)} \right)^{3/4} |A|^2 F D_S \approx \Gamma \left(\frac{\Omega |A|^2}{\chi k^2} \right)^{1/4} (kR)^{-3/2}. \qquad (5.15b)$$

In order to evaluate the contribution from the nonresonant region, we need to substitute (5.7) into (5.12) and cut off this integral at its lower limit, at a scale $\mathscr{L}$. We find

$$\Gamma_{nonres} \approx \frac{k}{R\mathscr{L}} \frac{\Omega |A|^2}{\omega''(0)} F D_S, \qquad (5.16)$$

which is the same in order of magnitude as (5.15b). Comparing this result with (4.9b), we see that again in this case the amplification against the background of the well-developed acoustic streaming is far weaker than against the background of the suppressed streaming.

We conclude with a discussion of the validity of perturbation theory, i.e., with a discussion of the distortions of the seed sound wave. The correction $a_\nu^{(2)}(x)$ is given by (5.10), in which the matrix elements can be evaluated by the method used for the nonlinear growth rate. The denominator in (5.10) is k/R in order of magnitude for modes $\psi_\lambda(x)$ which are approximately the same as the fundamental mode $\psi_\nu(x)$, provided that the scattering occurs at an angle of order unity ($\nu \gg 1$). For backscattering [$\nu = O(1)$] we have $K_\nu^2 - K_\lambda^2 = O(1/R^2)$. The diffractive distortions are thus at a maximum for backscattering: Their level increases by a factor of kR. An estimate of this level for the case in which the pseudoresonance zone is at the periphery yields

$$\alpha F D_S \ln \frac{R}{\mathscr{L}} \approx \alpha \frac{\Gamma}{3k} \ln \left[\frac{\Omega |A|^2}{\chi k^2} (kR)^2 \right].$$

This result can be regarded as small, since the damping is weak. The corresponding estimate in the case of a central position for the pseudoresonance zone yields

$$\alpha F D_S \frac{R}{\mathscr{L}} \approx \alpha \Gamma R \left(\frac{\Omega |A|^2}{\chi k^2} \right)^{1/4}.$$

The parameter ΓR here can be assumed to be small, since it is not possible to create a uniform pump beam with a width on the order of the sound attenuation length. The parameter expressing the extent to which the threshold is exceeded, raised to the power 1/4, could hardly be a very large number. Consequently, even in the least favorable case the growth rate could be found correctly in order of magnitude by perturbation theory.

6. UNSTEADY CONDITIONS

As was shown in Secs. 4 and 5, stimulated scattering of sound is suppressed in the steady state, since it is superposed on a nonuniform acoustic streaming. Could simulated sound scattering be detected under time-varying conditions? We consider an experiment in which the pump applied to the medium is a short pulse. In this case it might be possible to avoid the suppressing effects of the acoustic streaming if the latter did not have time to arise during the pulse length. On the other hand, a necessary condition for the detection of stimulated sound scattering is that the rise time of this scattering be at least comparable to the length of the pump pulse.

It is a simple matter to estimate the duration of the transients involved in stimulated sound scattering under the assumption that the acoustic streaming is not developed and under the assumption that the pump pulse is long (Ref. 12, for example). For this purpose we need to find the response of Eqs. (3.12), (3.16), (3.18) to an instantaneous sound probe signal, applied to the exit of the scattering medium, and we need to determine at which instant this response reaches a maximum at the entrance. Considering (as before) the contribution only from the scattering by the entropy wave, we easily find

$$\tau_{IS}={}^1/_2\Gamma_{int}(\chi k_3^2)^{-1}, \tag{6.1}$$

where $\Gamma_{\rm int}$ is the total growth rate over the entire path of the probe signal in the steady state, with completely suppressed streaming.

The rise time of the acoustic streaming is R^2/ν in order of magnitude.[7] The suppressing effect of streaming on stimulated sound scattering, however, may become substantial even over a much shorter time. An increase in the characteristic value of the Doppler deviation $(\mathbf{k}_3\cdot\mathbf{U}_a)$ to a level equal to the width of the gain line would be sufficient to render any further amplification of the probe signal ineffective. In the two cases of well-developed and suppressed acoustic streaming, two different mechanisms operate on the stimulated sound scattering. Since the velocity of well-developed streaming is in the same direction everywhere, as the streaming develops over the entire cross section of the pump beam, the spectral gain line shifts away from its original position. As a result, the frequency of the signal which was initially being amplified moves outside this line. In the case of suppressed streaming, on the other hand, resonance zones form. These zones contract, while their profiles simultaneously become sharper. In both cases the time scale for the manifestation of the streaming is

$$\tau_{AF}=\frac{R^2}{\nu}\frac{\chi k_3^2}{(\mathbf{k}_3\mathbf{U}_a)}. \tag{6.2}$$

It follows from the limitation $\tau_{IS}\ll\tau_{AF}$ that the total growth rate is less than

$$\Gamma_{int}^{max}=\frac{R^2}{\nu}\frac{(\chi k_3^2)^2}{(\mathbf{k}_3\mathbf{U}_a)}. \tag{6.3}$$

Using the velocity estimate in (4.5), we find the following result for the case of well-developed acoustic streaming:

$$\Gamma_{int}^{max}=\frac{(\chi k_3/c)^2}{|A|^2\Gamma/k_3}. \tag{6.4}$$

Above the threshold we have

$$|A|^2(\Gamma/k_3)(\chi k_3/c)^{-1}\geqslant 1,$$

so again in the time-dependent case amplification would hardly be possible, unless measures were taken to suppress the acoustic streaming.

The corresponding estimate for the case of suppressed streaming, found with the help of (4.9), is

$$\Gamma_{int}^{max}=\frac{\chi k_3}{\Omega|A|^2}\frac{\chi}{\nu}(k_3R)^2. \tag{6.5}$$

Noting that the reciprocal of the first factor on the right side of Eq. (6.5) is equal to the ratio of the nonlinear growth rate to the linear growth rate, and expressing it in terms of the total gain over the sound breaking distance $l_{br}\approx(k|A|)^{-1}$, we finally find

$$\Gamma_{int}^{max}\leqslant k_3R\left(\frac{\Gamma\chi}{k|A|\nu}\right)^{1/2}. \tag{6.6}$$

In order to achieve a significant amplification under time-dependent conditions, it would thus be necessary to create, in a wide closed channel, an intense pump beam which was highly uniform over its cross section. Otherwise, acoustic streaming would develop as a result of the nonuniformity of the pump beam. In addition, the linear damping rate of the sound in the medium would have to be high, and the thermal diffusivity χ and the kinematic shear viscosity ν would have to be comparable in magnitude. These conditions can be met if the attenuation in the medium results from a bulk viscosity ζ, whose large value stems from the existence of additional relaxation mechanisms.

7. DISCUSSION

In summary, the detection of stimulated sound scattering in the usual formulation of the experiment—in a broad pump beam—is hardly feasible. If it is possible to observe stimulated sound scattering in gases and liquids at all, it would be necessary to satisfy some stringent conditions, to ensure suppression of the acoustic streaming.

We can point out two possibilities for observing stimulated scattering. The first is time-dependent scattering in the specially selected medium which we described in the preceding section. The second possibility (which we did not discuss above) is steady-state scattering, in a cylindrical waveguide, of a beam which is reflected from the walls at a small angle (a whispering-gallery effect of a sort). Such a beam could never escape from the resonance zone if this zone were near the wall, so the beam should be amplified effectively.

Steady-state stimulated sound scattering can be described in terms of waveguide modes with a large azimuthal wave number m [the angular dependence of the properties in the wave is expressed in terms of $\cos(m\varphi)$]. The energy of such a mode is concentrated in a narrow zone near the wall, with a width on the order of $m^{-1/3}R$. Although the number m cannot exceed kR, the ratio of the size of the resonance zone, L^b_{res}, to the length scale for the localization of the mode is given by

$$\frac{\chi k_3^2}{\Omega}\frac{(kR)^{1/3}}{|A|^2}.$$

Although this quantity could of course not be very large, it could definitely be of order unity. The question of a nonlinear stage for this exotic stimulated sound scattering remains an open question, however. The reason is that when the amplitude of the scattered mode becomes comparable to the pump wave, the sound field becomes nonuniform, and this effect will cause a growth of acoustic streaming.

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[1)] We are ignoring effects which stem from pump-wave depletion as a result of the transfer of energy to the scattered wave.

[2)] The study of stimulated scattering by acoustic streaming carried out in Refs. 10 and 11 needs correction. The equation for the acoustic streaming describes an inconsequential relaxation mode in both those studies. In Ref. 10, this follows from the dispersion relation $\omega = i(\zeta + 4\eta/3)\rho_0^{-1}k^2$ which was found, while in Ref. 11 it follows from the circumstance that the acoustic streaming was derived in one-dimensional hydrodynamics.

[3)] We have in mind the very simple case of a plane beam or a cylindrically symmetric beam, uniform over its cross section.[8,9] For a more complex geometry, we do not know the explicit profile of the acoustic-streaming velocity along the coordinate, and the nonuniformity of the beam would intensify the acoustic streaming.

[4)] The quantization condition arises when the boundary conditions—the vanishing of the normal component of the velocity at the wall—are taken into account. The conditions that the tangential components of the velocity and the temperature deviations vanish are satisfied, as usual, by virtue of boundary layers. The width of these layers for quasi-steady waves can reach several times the length of the sound waves; such widths are negligible in comparison with the width of the resonance zone for suppressed acoustic streaming.

[5)] In our approximation we have $\cos\theta \approx 1$. To take the deviation from unity into account, we would need to retain the second derivatives with respect to z in Eq. (4.1).

[6)] The semiclassical approximation is valid in this region only under the inequality $(\Omega/c)\Gamma R^2 \gg 1$.

[7)] The derivation of these equations is analogous to the derivation of the geometric-acoustic equations in Sec. 3. The calculations are simplified by the circumstance that there is no dependence on the transverse coordinate in the pump wave.

1. I. S. Starunov and I. L. Fabelinskiĭ, Usp. Fiz. Nauk **98**, 441 (1969) [Sov. Phys. Usp. **12**, 463 (1970)].
2. N. I. Pushkina and R. V. Khokhlov, Akust. Zh. **17**, 167 (1971) [Sov. Phys. Acoust. **17**, 144 (1971)].
3. F. V. Bunkin, D. V. Vlasov, E. A. Zabolotskaya, and Yu. A. Kravtsov, Pis'ma Zh. Tekh. Fiz. **7**, 560 (1981) [Sov. Tech. Phys. Lett. **7**, 239 (1981)]; Akust. Zh. **28**, 743 (1982) [Sov. Phys. Acoust. **28**, 440 (1982)].
4. F. V. Bunkin, K. I. Volyak, and G. A. Lyakhov, Akust. Zh. **28**, 607 (1982) [Sov. Phys. Acoust. **28**, 360 (1982)].
5. F. V. Bunkin, G. A. Lyakhov, and O. B. Shuman, Kratk. Soobshch. Fiz. No. 2, 389 (1983).
6. F. V. Bunkin, G. A. Lyakhov, and M. Yu. Romanovskiĭ, Akust. Zh. **31**, 159, 733 (1985) [Sov. Phys. Acoust. **31**, 93, 446 (1985)].
7. O. V. Rudenko and S. I. Soluyan (editors), *Theoretical Foundations of Nonlinear Acoustics*, Nauka, Moscow, 1976 (Plenum, New York, 1977).
8. L. K. Zarembo, in *High-Intensity Ultrasonic Fields* (ed. L. D. Rozenberg), Nauka, Moscow, 1968, p. 87 (Plenum, New York, 1971).
9. W. Nyborg, in *Physical Acoustics*, Vol. II, Part B (ed. W. P. Mason), Academic, New York, 1965 (Russ. Transl. Mir, Moscow, 1969, Vol. 2, Part B, p. 302).
10. F. V. Bunkin, K. I. Volyak, G. A. Lyakhov, and M. Yu. Romanovskiĭ, Zh. Eksp. Teor. Fiz. **86**, 140 (1984) [Sov. Phys. JETP **59**, 80 (1984)].
11. F. V. Bunkin, K. I. Volyak, G. A. Lyakhov, and M. Yu. Romanovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 389 (1982) [JETP Lett. **36**, 469 (1982)].
12. B. Y. Zel'dovich *et al.*, *Principles of Phase Conjugation*, Nauka, Moscow, 1985 (Springer-Verlag, New York, 1985).

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