Energy of a system of gravitating bosons

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The ground state energy of a system of identical neutral bosons bound to each other by gravitational forces is investigated. An analysis of the Schrödinger equation shows that the lower bound on the energy of such a system for an arbitrary number of particles N can be determined from the solution of a purely geometrical problem on the maximum of the sum of cosines of all angles θ_{jkl}, formed by all triplets j, k, and l of particles that can be formed from the N particles of the system. In this way the purely geometrical lower bound is found on the energy of a gravitating boson system for $N = 3, 4, 6, 8, 12$, and 20 and also in the limit $N \to \infty$. A more refined calculation of the upper bound on the energy of a three-particle gravitational system is also carried out: $E(3) \leqslant -1.07143\,G^2m^5/\hbar^2$. It is much lower than the bound calculated previously by the Hartree method $[E(3) \leqslant -0.6511\,G^2m^5/\hbar^2.]$ This indicates that the lower energy bounds $[E(3) \geqslant -1.125\,G^2m^5/\hbar^2]$ determined by the geometrical method proposed here are much closer to the exact energy than the upper bounds obtained by integrating the Hartree self-consistent field equations. The geometrical method for the determination of the energy bounds proposed here can be modified to apply to a system of charged particles.

INTRODUCTION

A system of a large number of microscopic particles, bound to each other by gravitational forces, is of interest both in field theory in connection with the study of possible gravitational formation of particles,[1] and in astrophysics in connection with the problem of distribution of dark nonradiating matter in galactic neighborhoods.[2] Along with this application aspect the problem of a system of bosons bound by gravitational forces presents considerable methodological interest, being one of the fundamental problems of theoretical physics.

In the papers of Refs. 1 and 2 the ground state of the system of N identical bosons with gravitational interaction was investigated in the Hartree method. In that method any correlation in the motion of the particles is ignored and the trial wave function for the system is taken in the form of the product

$$F(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \prod_{j=1}^{N} f(\mathbf{r}_j), \tag{1}$$

where $f(\mathbf{r})$ is the spherically-symmetric real one-particle wave function. The condition that the mathematical expectation value of the energy of the system be a minimum leads to the equation for the self-consistent Hartree field

$$-\frac{\hbar^2}{2m}\Delta f(\mathbf{r}) + U(\mathbf{r}) f(\mathbf{r}) = \varepsilon f(\mathbf{r}). \tag{2}$$

We have introduced here the potential energy of the particle in the gravitational field due to all the other particles:

$$U(\mathbf{r}) = -Gm^2(N-1)\int \frac{f(\mathbf{r}')^2 d\tau'}{|\mathbf{r}-\mathbf{r}'|}, \tag{3}$$

where m is the mass of the particle and G is the gravitational constant. Further, the eigenvalue of the Hartree equation (2) and the mathematical expectation value of the total energy of the system calculated with the help of the trial function (1), are connected by the relation

$$\bar{E} = N\varepsilon/3. \tag{4}$$

It was found in Ref. 1 on the basis of a numerical solution of the integro-differential equation (2) that

$$N\varepsilon = -0{,}1626N(N-1)^2\gamma, \tag{5}$$

where γ equals $G^2m^5/\hbar^2$ and represents a natural unit of energy for the problem under consideration. Here the quantity (5) was taken (incorrectly) to be the total energy of the system, which is in contradiction with relation (4); this error was pointed out in Ref. 2. In order to check the results of Ref. 1 and avoid the difficulties associated with the procedure of solving the integro-differential equation (2) by iteration, this equation was replaced in Ref. 2 by a fourth-order differential equation, which can be obtained by equating to each other the following two expressions for ΔU:

$$\Delta U(\mathbf{r}) = 4\pi G m^2 (N-1) f(\mathbf{r})^2 \tag{6}$$

and

$$\Delta U(\mathbf{r}) = \frac{\hbar^2}{2m}\Delta\left[\frac{\Delta f(\mathbf{r})}{f(\mathbf{r})}\right]. \tag{7}$$

The first of these expressions follows from the Poisson equation for the gravitational potential, and the second can be obtained by dividing Eq. (2) term by term by the one-particle function $f(\mathbf{r})$ and applying the Laplacian to the result. From (6) and (7) we obtain the fourth-order differential equation

$$\frac{\hbar^2}{2m}\Delta\left[\frac{\Delta f(\mathbf{r})}{f(\mathbf{r})}\right] - 4\pi G m^2 (N-1) f(\mathbf{r})^2 = 0. \tag{8}$$

Numerical solution of this equation, followed by a calculation of the mathematical expectation value of the energy of the system, gave for the self-consistent energy of the ground state of the system of N bosons in gravitational interaction, the value[2]

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$$\bar{E}(N)=-0{,}05426N(N-1)^2\gamma. \tag{9}$$

Taken together with Eqs. (4) and (5) this result is in complete agreement with the results of Ref. 1. Thus two independent calculations of the ground-state energy of a system of mutually gravitating bosons by the Hartree method are in beautiful agreement with each other. It should be noted that only the case of a large number of particles was considered in Refs. 1 and 2, and in this connection the coefficient on the right-hand side for the formula for the gravitational potential was taken equal to N [in place of $(N-1)$]. Similarly, the expression for the energy given in Ref. 2 contains in place of the coefficient $N(N-1)^2$ the coefficient N^3. This shortcoming is easily removed by scaling the function $f(\mathbf{r})$; the result is formula (9), correct for all N.

In view of the variational character of the Hartree method the quantity (9) represents an upper bound on the exact energy $E(N)$ of the system of particles. The exact energy certainly lies lower than the Hartree energy (9), and its lowering is entirely due to the correlations in the particles motion that are ignored in the self-consistent field Hartree method. To establish the maximum possible error in the Hartree value of the energy and to estimate the correlation energy it is necessary to know a lower bound on the exact energy of the system of mutually gravitating particles. We will show here that such a lower bound can be found very simply, without solving differential equations of any kind, from considerations of a purely geometrical problem. This represents a unique case, when the solution of a quantitative quantum-mechanical problem can be obtained by a geometrical method. The lower bound on the energy of a many-boson gravitational system obtained in this way turns out to be quite close to the (unknown) exact energy.

LOWER BOUND ON THE ENERGY OF A SYSTEM OF GRAVITATING BOSONS

A system of N identical bosons interacting gravitationally is described by the Hamiltonian

$$H_N=-\frac{\hbar^2}{2m}\sum_{j=1}^{N}\Delta_j-Gm^2\sum_{j>k}^{N}\frac{1}{r_{jk}},\qquad r_{jk}=|\mathbf{r}_j-\mathbf{r}_k|. \tag{10}$$

To eliminate the unquantized uniform motion of the center of mass of the system it is sufficient to require that all wave functions depend only on relative coordinates of the particles. For these coordinates we can take the $N-1$ vectors

$$\boldsymbol{\rho}_j=\mathbf{r}_j-\mathbf{r}_N;\qquad j=1,\ 2,\ \ldots,\ N-1. \tag{11}$$

Admissible wave functions should then depend only on the vectors $\boldsymbol{\rho}_j$ from (11), and the integrals encountered in the calculations of matrix elements should be evaluated by integration over the space of these vectors. The interparticle distances are expressed in terms of the relative coordinates (11) as follows:

$$r_{jN}=|\boldsymbol{\rho}_j|,\ r_{jk}=|\boldsymbol{\rho}_j-\boldsymbol{\rho}_k|\quad\text{for}\quad j\neq N,\ k\neq N. \tag{12}$$

Consider the following wave function of the system of N particles:

$$\Phi_N=\exp\left(-\frac{Gm^3}{2\hbar^2}\sum_{j>k}^{N}r_{jk}\right). \tag{13}$$

It represents an exponentially decreasing function of the sum of the distances between all the particles, depends only on the relative coordinates (11) and is quadratically integrable over the space of these coordinates. The function (13) is endowed with the useful property of being the exact solution of the auxiliary Schrödinger equation

$$\bar{H}_N\Phi_N=\bar{E}(N)\Phi_N, \tag{14}$$

where the auxiliary energy operator and its eigenvalues are given by

$$\bar{H}_N=-\frac{\hbar^2}{2m}\sum_{j=1}^{N}\Delta_j-Gm^2\sum_{j>k}^{N}\frac{1}{r_{jk}}+\frac{G^2m^5}{4\hbar^2}\sum_{j,k,l}^{N}{}'\cos\theta_{jkl}, \tag{15}$$

$$\bar{E}(N)=-N(N-1)\gamma/8. \tag{16}$$

We denote by θ_{jkl} the angle between the vectors $\mathbf{r}_k-\mathbf{r}_j$ and $\mathbf{r}_l-\mathbf{r}_j$ joining the jth particle with the kth and lth particles. The prime on the summation symbol in (15) indicates that it is to be calculated under the conditions: $k>l, j\neq k, j\neq l$. The cosines of the angles entering this sum can be expressed in terms of the interparticle distances:

$$\cos\theta_{jkl}=(r_{jk}^2+r_{jl}^2-r_{kl}^2)/2r_{jk}r_{jl}. \tag{17}$$

It can be verified directly that Eq. (14) is satisfied—it is sufficient to substitute into it the wave function (13) and perform the differentiations.

We shall refer to the energy operator (15) as the model Hamiltonian, to distinguish it from the Hamiltonian of the real system (10). Since its eigenfunction (13) has no nodes it describes the ground state of the model system, and the quantity (16) represents the energy of the ground state of this system.

It can be seen from Eqs. (10) and (15) that the model Hamiltonian differs from the real Hamiltonian by the term

$$\bar{H}_N-H_N=\frac{\gamma}{4}L_N(\theta), \tag{18}$$

where we have introduced the notation

$$L_N(\theta)=\sum_{j,k,l}^{N}{}'\cos\theta_{jkl} \tag{19}$$

for the sum of the cosines of all the angles θ_{jkl} formed by the N particles. The total number of these angles is equal to

$$n(N)=N(N-1)(N-2)/2. \tag{20}$$

The argument θ is symbolic of the totality of all angles θ_{jkl}.

Let us denote by Ψ_N the exact eigenfunction normalized to unity and by $E(N)$ the energy of the ground state of the real system described by the Hamiltonian (10). Let us form with this function the mathematical expectation value of the model Hamiltonian (15). In view of the variational principle it lies no lower than the energy of the ground state of the model system. We therefore have

$$\begin{aligned}\bar{E}(N)&\leqslant\langle\Psi_N|\bar{H}|\Psi_N\rangle=\langle\Psi_N|H_N+(\bar{H}_N-H)|\Psi_N\rangle\\&=E(N)+\frac{\gamma}{4}\langle\Psi_N|L_N(\theta)|\Psi_N\rangle.\end{aligned} \tag{21}$$

Including (16) we obtain from this the following lower bound on the energy of the ground state of the system of N mutually gravitating bosons:

$$E(N) \geqslant -\frac{\gamma}{4}\left|\frac{N(N-1)}{2}+\langle\Psi_N|L_N(\theta)|\Psi_N\rangle\right|. \quad (22)$$

The exact eigenfunction Ψ_N of the real system is unknown, hence so is the mathematical expectation value of the sum of the cosines of the angles that enters (22). However this sum is bounded for an arbitrary geometrical arrangement of the N particles since each of the quantities $\cos\theta_{jkl}$ has magnitude not exceeding unity. Therefore we can always find numbers $L_N^{(-)}$ and $L_N^{(+)}$ that bound the function $L_N(\theta)$ on both sides for an arbitrary arrangement of the N particles:

$$L_N^{(-)} \leqslant L_N(\theta) \leqslant L_N^{(+)}. \quad (23)$$

It is obvious that

$$\langle\Psi_N|L_N(\theta)|\Psi_N\rangle \leqslant \langle\Psi_N|L_N^{(+)}|\Psi_N\rangle = L_N^{(+)}. \quad (24)$$

Hence the inequality (22) remains in force if we replace in it the unknown mathematical expectation value of the operator $L_N(\theta)$ by the upper bound on this operator $L_N^{(+)}$. As a result we obtain

$$E(N) \geqslant -\frac{\gamma}{4}\left[\frac{N(N-1)}{2}+L_N^{(+)}\right]. \quad (25)$$

Therefore to find a lower bound on the energy of the ground state of the system of N mutually gravitating bosons it is necessary to solve the purely geometrical problem: to find the quantity $L_N^{(+)}$ which bounds from above the sum (19) of cosines of the angles θ_{jkl}, at whose vertices lie the N particles ($j = 1, 2, ..., N$) and whose sides connect the given particle j with all the remaining particles k and l. Substitution of the resulting quantity $L_N^{(+)}$ into the inequality (23) then gives the desired lower bound on the energy.

CALCULATION OF THE GEOMETRICAL LOWER BOUND ON THE ENERGY OF A SYSTEM OF MUTUALLY GRAVITATING BOSONS

Replacing in (19) all cosines of the angles by their upper bound, equal to unity, and taking into consideration the total number of these angles (20) we obtain the simplest estimate:

$$L_N(\theta) \leqslant N(N-1)(N-2)/2. \quad (26)$$

Together with (22) this gives the following lower bound on the energy of a system of mutually gravitating bosons:

$$E(N) \geqslant -0{,}125N(N-1)^2\gamma. \quad (27)$$

The estimate (26) is quite rough since it ignores any geometrical relations between the angles θ_{jkl}. To take such relations into account (and obtain a better estimate of the quantity $L_N(\theta)$ and also of the lower bound on the energy) we regroup the terms in (19). From p particles it is possible to form

$$\binom{N}{p} = \frac{N!}{(N-p)!p!} \quad (28)$$

groups g, each of which contains p particles, where $3 \leqslant p \leqslant N$. Let us associate with the instantaneous configuration of the group of p particles a polyhedron at whose vertices lie the particles. In each such polyhedron there are

$$n(p) = p(p-1)(p-2)/2 \quad (29)$$

angles θ_{jkl} formed by all pairs of straight lines connecting each particle (each vertex of the polyhedron) with all remaining particles in the group (with all remaining vertices of the polyhedron). The sum of cosines of these angles for the given group of particles will be denoted by $S_p(g)$. Both the geometry of the polyhedron and the sum of the cosines of the angles $S_p(g)$ depend on the instantaneous arrangement of the p particles forming the given group. Going over in (19) to summation over the described groups of particles we obtain

$$L_N(\theta) = \frac{(p-3)!(N-p)!}{(N-3)!}\sum_g S_p(g). \quad (30)$$

The factor in front of the summation compensates for the possible effect of multiple counting of the contribution from one and the same angle θ_{jkl} because the same particle can enter into different groups g.

Different numbers p of particles in groups lead to different upper bounds on the sum of cosines (19) and different lower bounds on the energy (25) of the ground state of the system of N gravitating bosons.

We start with the case $p = 3$. From (30) we obtain

$$L_N(\theta) = \sum_g S_3(g). \quad (31)$$

The quantity $S_3(g)$ represents the sum of cosines of the internal angles of the triangle formed by the three particles entering the group g. The internal angles of an arbitrary triangle are equal to α, β and $\gamma = \pi - \alpha - \beta$. The sum of their cosines equals

$$S_3 = \cos\alpha + \cos\beta - \cos(\alpha+\beta). \quad (32)$$

By differentiation of this expression with respect to α and β one readily finds that the maximum value of the sum of the cosines of the internal angles of a triangle equals $\frac{3}{2}$ and that this value is achieved for an equilateral triangle so that $\alpha = \beta = \gamma = \pi/3$. Replacing now in (31) the sums S_3 for each group of three particles by their upper bound, which is $\frac{3}{2}$, and taking into account the number (28) of such groups, we obtain the following upper bound for the sum of cosines of all angles θ_{jkl}:

$$L_N(\theta) \leqslant N(N-1)(N-2)/4. \quad (33)$$

Substitution of (33) into (25) gives for the ground-state energy of the system of mutually gravitating bosons the improved lower bound

$$E(N) \geqslant -0{,}0625N^2(N-1)\gamma. \quad (34)$$

For large N it gives an improvement by a factor 2 over the original rough bound (27). The reason is clear: the maximum possible value of the sum of cosines of the internal angles of a triangle (equal to $\frac{3}{2}$) is smaller by a factor 2 than the sum of the maximum values of the cosines of these angles (equal to 3). Consequently an improved lower bound (34) is obtained by taking into account the relations among the angles θ_{jkl}.

Further increase of the number of particles p in the groups g is accompanied by further improvement of the lower bound on the energy of the system of N mutually gravitating bosons. Indeed, when we pass to summing over groups containing more than three particles ($p > 3$) we encounter additional relations among the angles θ_{jkl} (besides the conditions on the internal angles of triangles). The mutual arrangement of p particles is determined by $3p - 6$ parameters: this is the number of degrees of freedom after subtraction of the translational and rotational degrees of freedom of the group as a whole. In addition the angles θ_{jkl} are unchanged by a simultaneous change in the scale of the coordinates of all p particles. Therefore the totality of the angles θ_{jkl} for the group of p particles is completely determined by $3p - 7$ parameters. At the same time the number of such angles, equal to $p(p-1)(p-2)/2$, grows rapidly as the number of particles increases; simultaneously the number of relations among the angles also grows. Thus, for the values $p = 4$, 6, 8, 12, and 20, corresponding to the number of vertices of the five regular polyhedra, the number of independent angles θ_{jkl} equals respectively 5, 11, 17, 29, and 53, the total number of these angles equals respectively 12, 60, 168, 660, and 3420, and the number of relations among the angles equals respectively 7, 49, 151, 631, and 3367. This means that as the number of particles p in the group increases the construction from these particles becomes ever more "rigid" with respect to a change of the angles θ_{jkl} and, as a consequence, in the configuration of particles corresponding to the maximum value of the sum of cosines of the angles θ_{jkl} the triangles formed from the particles are forced to deviate more and more from being equilateral. Therefore with increasing p the upper bound $L_N^{(+)}$ on the quantity $L_N(\theta)$ decreases and the lower bound on the energy of the system of gravitating bosons improves.

For $p = 4$ to each group of particles corresponds a polyhedron with four vertices, all four sides of which are triangles. The sum $S_4(g)$ reaches a maximum when this polyhedron is the regular tetrahedron, all sides of which are equilateral triangles. Therefore the maximum value of each sum $S_4(g)$ equals 6. We then obtain from (30) and (29) again the bound (33) for the sum $L_N(\theta)$. Therefore the cases $p = 3$ and 4 give the same lower bound (34) for the energy of a system of N gravitating bosons.

The cases when the groups contain $p = 6$, 8, 12, and 20 particles are treated similarly. The sums of cosines of the angles θ_{jkl} here reach maxima for the particles distributed at the vertices of the regular polyhedra (the platonic solids)—octahedron, cube, icosahedron and dodecahedron. The sums of cosines of the angles we calculated for the indicated regular polyhedra are given in Table I.

Along with the sum of cosines of the angles S_p we give in Table I the average value of the cosine of the angle for the regular polyhedra calculated from the formula

$$\langle \cos\theta\rangle_p = S_p/n(p). \tag{35}$$

This average value for the regular polyhedron represents the upper bound on the unknown true average value of $\cos\theta_{jkl}$, which could be calculated using the exact eigenfunction Ψ_N of the real multiboson system. Also included in Table I is the "planar regular polyhedron"—the equilateral triangle. In the limit that the number of particles grows without bound ($N \to \infty$) one can consider the group containing $p = N$ particles. To this group corresponds a regular polyhedron with an infinite number of vertices, i.e., a sphere with uniform distribution of particles on its surface. The average value of the cosine of the angle for this case is easily calculated. Indeed, in this case θ_{jkl} is the angle between two "stars" k and l, which is measured by an observer at a point on the surface of the sphere when the density of "stars" is distributed uniformly on the surface of the sphere. The point j can be taken without loss of generality as a pole of the sphere. Let the observed pair of stars have the spherical coordinates θ_k, φ_k and θ_l, φ_l. The cosine of the angle θ_{jkl} is then given by Eq. (17). Going over in that formula to spherical coordinates we obtain

$$\cos\theta_{jkl} = \sin\frac{\theta_k}{2}\sin\frac{\theta_l}{2} + \cos\frac{\theta_k}{2}\cos\frac{\theta_l}{2}\cos(\varphi_k - \varphi_l). \tag{36}$$

The average value of this expression

$$\langle\cos\theta\rangle = \frac{1}{16\pi^2}\int \cos\theta_{jkl}\sin\theta_1\sin\theta_2\, d\theta_1\, d\theta_2\, d\varphi_1\, d\varphi_2 \tag{37}$$

is easily evaluated and equals $\frac{4}{9} = 0.44444$.

It is seen from Table I that the average value of the cosine of the angle θ_{jkl} decreases with increasing number of vertices of the polyhedron (or the number of particles p in the group). Further, to each value of the number of particles in the group corresponds a lower bound for the energy of the system of gravitating bosons, that follows directly from (25) including (35) and (20):

TABLE I. Sum of cosines of angles θ_{jkl} and the average value of the cosine of the angle $\langle\cos\theta\rangle$ for regular polyhedra.

Polyhedron	Number of vertices	Number of angles	Sum of cosines	Average value of cosines
Triangle	3	3	1,5	0,50000
Tetrahedron	4	12	6	0,50000
Octahedron	6	60	28,9706	0,48284
Cube	8	168	79,3935	0,47258
Icosahedron	12	660	306,7470	0,46477
Dodecahedron	20	3420	1561,9340	0,45671
Sphere	∞	∞	∞	0,44444

$$E(N) \geqslant -[(N-2)\langle\cos\theta\rangle_p+1]N(N-1)\gamma. \tag{38}$$

These bounds are valid under the condition $N \geqslant p$. For each fixed value of the number of particles N in the system the optimum lower bound on the energy of the ground state is obtained by using the largest of the numbers p for which the average value (35) of the cosine of the angle is known and which does not exceed N. With this rule we obtain from Eq. (38), including the average values $\langle\cos\theta\rangle_p$ from Table I, the following formulas for the lower bound of the ground-state energy of the system of N mutually gravitating bosons:

$$E(N) \geqslant -0{,}0625\ N^2(N-1)\gamma \quad \text{for } N=3 \text{ and } 4, \tag{39a}$$

$$E(N) \geqslant -(0{,}063553N-0{,}004289)N(N-1)\gamma \quad \text{for } N \geqslant 6, \tag{39b}$$

$$E(N) \geqslant -(0{,}059073N-0{,}006854)N(N-1)\gamma \quad \text{for } N \geqslant 8, \tag{39c}$$

$$E(N) \geqslant -(0{,}058096N-0{,}008808)N(N-1)\gamma \quad \text{for } N \geqslant 12, \tag{39d}$$

$$E(N) \geqslant -(0{,}057088N-0{,}010824)N(N-1)\gamma \quad \text{for } N \geqslant 20, \tag{39e}$$

$$E(N) \geqslant -(0{,}055556N-0{,}013889)N(N-1)\gamma \quad \text{for } N \to \infty. \tag{39f}$$

Let us compare the lower bounds on the energy (39), obtained from purely geometrical estimates of the cosines of the angles, with the upper bound on the energy (9), calculated by the Hartree method. It is seen from Table II that the disagreement between these bounds, amounting to 73% for $N=3$, decreases with increasing number of particles and tends to 2.4% as $N \to \infty$. The unknown exact energy of the ground state of a system of mutually gravitating bosons lies for all N in the gap between these bounds. The width of this energetic gap is determined by the sum of the absolute values of the deviation of the Hartree energy (9) from the unknown exact energy and the deviation of the lower bound (39) from that same exact energy.

To clarify the role played by the imprecision of the one-particle approximation in the origin of this gap it would be desirable to have more precise variational calculations of the upper bound with more precise trial wave functions, taking into account mutual correlations in the motion of the particles. As a simplest wave function of that type we can take the eigenfunction Φ_N (13) of the model system. Taking into account permutation symmetry we then obtain the following upper bound on the energy of the ground state of the N-particle system:

$$E(N) \leqslant \left\langle \Phi_N \left| -\frac{N\hbar^2}{2m}\Delta_1 - \frac{N(N-1)Gm^2}{2r_{12}} \right| \Phi_N \right\rangle \Big/ \langle \Phi_N | \Phi_N \rangle. \tag{40}$$

For $N=3$ the integrals in (40) are easily calculated in perimeter coordinates

$$u=r_{12}+r_{23}-r_{31}, \quad v=r_{12}-r_{23}+r_{31}, \quad w=-r_{12}+r_{23}+r_{31}, \tag{41}$$

where the volume element for our case can be taken equal to

$$d\tau=(u+v)(v+w)(w+u)\,du\,dv\,dw. \tag{42}$$

As a result we obtain the following improved upper bound on the ground-state energy of a system of three mutually gravitating bosons:

$$E(3) \leqslant -(15/14)\gamma = -1{,}07143\gamma. \tag{43}$$

It lies significantly deeper (by a factor of 1.65) than the Hartree energy (9). This means that the neglect of correlations in the particles motion is not sufficiently justified: the mutual correlation of the particles, which is taken into account by the wave function (13) dependent on interparticle distances, makes it possible for the particles to pack much more tightly than in the one-particle Hartree approximation, where they only feel each other's average fields. Hence it also follows that our geometrical bound on the energy (39a) is much more precise than the Hartree upper bound (9): the gap between the improved upper bound (43) and the geometric lower bound (39a) is nine times smaller than that between the Hartree upper bound and our lower bound.

An improved upper bound on the energy can be obtained from Eq. (40) for an arbitrary number of particles N. This, however, goes beyond the framework of the purely analytic approach we have developed, since for $N \geqslant 4$ numerical methods must be used to evaluate the integrals in (40).

With increasing number of particles the Hartree upper bound and our geometrical lower bound come closer to each other, but there remains a gap between them even for $N \to \infty$ (see Table II). This coming together of the bounds indicates

TABLE II. Bounds on the total energy $E(N)$ and specific energy $E(N)/N$ in the one-particle calculation for a system of mutually gravitating bosons (energy expressed in units of $G^2m^5/\hbar^2$).

N	Total energy: Lower bound	Total energy: Upper bound	Specific energy: Lower bound	Specific energy: Upper bound	Ratio of bounds
3	−1,125	−0,6511	−0,375	−0,2170	1,7278
4	−3,000	−1,9534	−0,75	−0,4883	1,5358
6	−10,9926	−8,1390	−1,8321	−1,3565	1.3506
8	−26,8484	−21,2690	−3.3560	−2,6587	1,2623
12	−93,1871	−78,7855	−7,7656	−6,5655	1,1828
20	−437,9871	−391,7572	−21,8994	−19,5879	1,1180
$N \to \infty$	$-0{,}055555\ N^3$	$-0{,}05426\ N^3$	$-0{,}05555\ N^2$	$-0{,}05426\ N^2$	1,0239

a decrease of the relative (but not absolute) size of the correlation energy: for large N each particle feels first the average gravitational field of all the other particles and reacts only comparatively weakly to the fluctuations of this field, due to the deviation of the instantaneous configuration of the particles from their average distribution. Here we have every reason to believe that for a large number of particles our geometrical lower bound (39) will be, as before, much more accurate than the Hartree upper bound on the energy.

In conclusion we note that the foundation of the present purely geometric approach to the calculation of the lower bound on the energy lies in the simple properties of the wave function (13), exponentially dependent on the inter-particle distances. Because of these properties, the model energy operator (15) differs from the true Hamiltonian (10) by terms that depend only on angular variables and can be estimated purely geometrically. Such an approach can be also applied to systems of charged particles with appropriate modifications to take into account the presence of not only attractive but also repulsive forces depending on the signs of the charges on the particle pair.

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