Hamiltonian dynamics of charged particles in a magnetic field and the field of an obliquely propagating finite wave packet

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An analytical and numerical investigation is performed of the Hamiltonian dynamics of a charged particle in a homogeneous magnetic field and the field of a packet of electrostatic waves that consists of one or a finite number of harmonics and propagates obliquely to the magnetic field. The system under consideration has $2\frac{1}{2}$ degrees of freedom and possesses properties characteristic of multidimensional systems. In the absence of perturbations the system is degenerate, as a result of which, diffusion along the channels of a stochastic web is possible in the system. The process of stochastic acceleration of the particles is bounded, and this distinguishes the case under consideration from the case of motion in the field of an infinitely broad wave packet. Estimates are obtained for the maximum magnitude of the stochastic heating. The transformation of the stochastic web with change of the number of harmonics of the wave packet is considered.

1. INTRODUCTION

The interaction of charged particles with a packet of electrostatic waves in a magnetic field has been investigated for a long time in connection with diverse applications in problems of plasma physics. In Ref. 1 the problem of the transverse propagation of one plane wave was discussed in application to the problem of the paradox of the vanishing of the Landau damping in a magnetic field. In Ref. 2 a description was given of the stochastic acceleration of particles that occurs in the presence of an obliquely propagating wave packet as a result of overlap of first-order longitudinal cyclotron resonances. A serious investigation of the motion in the field of a perpendicular wave was carried out in Refs. 3–5. In Ref. 3 the dynamics of particles in the case when the wave frequency is close to a resonance harmonic of the cyclotron frequency was considered, and it was shown that a stochastic layer always exists in the neighborhood of the separatrices and that global chaos arises as a result of the overlap of resonances. In Ref. 4 the case of a high wave frequency (much higher than the cyclotron frequency) was considered, and in Refs. 5 the stochastic dynamics of charged particles in a strong magnetic field was investigated. Various applications to problems in astrophysics were considered in Refs. 6 and 7. The investigation of the problem of the paradox of the vanishing of the Landau damping (motion in a weak magnetic field) was continued in Refs. 8 and 9, and a review of many of the results of the analysis of the stochasticity of the particles can be found in Refs. 10 and 11.

A substantially new understanding of the problem developed after the discovery of the stochastic web in the case of perpendicular resonance propagation of a plane wave[12,13] and a very broad wave packet[14] (see also the review in Ref. 15). The principal new result relates to the existence (when the resonance conditions are fulfilled) of unbounded diffusion of particles along channels of the stochastic web. This diffusion is similar in character to Arnold diffusion, although it is not exactly the latter, since unbounded escape of particles occurs for $N_0 = 1\frac{1}{2}$ degrees of freedom whereas Arnold diffusion is possible only for $N_0 > 2$. The possibility of unbounded diffusion for $N_0 = 1\frac{1}{2}$ is related to the degeneracy of the unperturbed problem.

As was shown in Ref. 12, if the packet contains only one harmonic the thickness of the stochastic web decreases exponentially with distance from the coordinate origin on the phase plane. Therefore, truly unbounded diffusion does not occur in this case. In the case of a uniform and infinitely broad wave packet the thickness of the stochastic web is approximately constant over the whole phase plane, and unlimited stochastic heating of the particles is possible.[14] Nevertheless, the diffusion does not occur very rapidly, since the thickness of the web is proportional to $\exp(-\text{const}/\varepsilon)$, where ε is the amplitude of the perturbation.[14] [We note that in the case of Arnold diffusion the web thickness is of order $\exp(-\text{const}/\varepsilon^{1/2})$.] If the wave packet propagates obliquely to the magnetic field, one more degree of freedom, corresponding to the longitudinal motion, appears in the system. In Ref. 16 it was shown that in the case of an infinitely broad uniform packet this leads to a sharp enhancement of the diffusion, even when the direction of propagation of the wave packet deviates only very slightly from the orthogonal direction ($\beta^2 \sim 10^{-4}$–10^{-6}, where $\beta \equiv k_z/k_x$).

In this paper we consider the problem of the motion of a charged particle in a magnetic field and the field of a packet of electrostatic waves that consists of a finite number of harmonics (including the case of one harmonic) and propagates obliquely to the magnetic field. It should be noted that this case is the most general in real conditions. The dynamics of the particle is investigated in conditions close to longitudinal and transverse resonance between the characteristic frequency of the wave packet and the cyclotron frequency. Here we consider the region of parameter values in which overlap of the first-order longitudinal cyclotron resonances does not occur.

In the case when the packet consists of a single harmonic, the motion of the particle is specified by the system of equations

$$\ddot{x}+x=\varepsilon \sin (x+z-\nu t),$$

$$\ddot{z}=\varepsilon\beta^2 \sin (x+z-\nu t).$$

Despite the comparative simplicity of these equations, the dynamics of the particle is extraordinarily complicated. For $\beta^2 = 0$, because of the degeneracy of the unperturbed problem, a stochastic ("bare") web exists on the phase plane (x,

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$\dot{x}$). For small, nonzero values of β a further degree of freedom is added, and a complicated structure, which may be called a spiral web, arises in the phase space of the system. If the packet consists of several harmonics, the stochastic web has an even more complicated geometry, since in different plane sections perpendicular to the external magnetic field its cells have different shapes. It is indisputable that the investigation of the diffusion of particles in this complicated dynamical system is of interest. In the paper it is shown that the stochastic heating of the particles is limited and that its character is determined to a considerable extent by the properties of the "bare" web.

It should also be noted that the investigation of the dynamics of a particle in a wave packet propagating obliquely to a magnetic field and having a finite number of harmonics is the first analysis of this kind that approximates very closely to the real situation. The considerable dynamical complexity described in the present article for certain cases permits one to assess the degree of difficulty of investigating the general problem for arbitrary conditions.

In Sec. 2 we derive the resonance Hamiltonian of the system. In Sec. 3 we consider the motion of a particle in the case of longitudinal cyclotron resonance. Section 3 also contains estimates of the thickness of the stochastic web and of the maximum magnitude of the stochastic heating of the particles. Motion in the field of a packet consisting of several harmonics is investigated in Sec. 4.

2. THE RESONANCE HAMILTONIAN

The initial equations of motion of a particle with mass m_0 and charge e in the field of a wave packet propagating at an angle to a magnetic field have the form

$$\ddot{\mathbf{r}}=\frac{e}{m_0}\mathbf{E}(\mathbf{r},t)+\frac{e}{m_0c}[\dot{\mathbf{r}}\mathbf{B}_0], \tag{1}$$

where the constant magnetic field $\mathbf{B}_0$ is in the direction of the z axis. The electric field $\mathbf{E}$ lies in the (x, z) plane and is an electrostatic wave

$$\mathbf{E}(\mathbf{r},\ t)=\mathbf{E}_0\sin\ (k_xx+k_zz-\nu_0t). \tag{2}$$

Because the electric field is assumed to be potential, the wave vector $\mathbf{k}$, like the amplitude vector $\mathbf{E}_0$, has only two components k_x and k_z, connected by the relation

$$\frac{k_z}{k_x}=\frac{E_{0z}}{E_{0x}}=\beta=\text{const}. \tag{3}$$

We write out Eq. (1) in its components, taking into account the representation (2) for the electric field:

$$\begin{aligned}\ddot{x}&=\frac{e}{m_0}E_{0x}\sin(k_xx+k_zz-\nu_0t)+\omega_H\dot{y},\\ \ddot{y}&=-\omega_H\dot{x},\\ \ddot{z}&=\frac{e}{m_0}E_{0z}\sin(k_xx+k_zz-\nu_0t),\end{aligned} \tag{4}$$

where $\omega_H=eB_0/m_0c$ is the cyclotron frequency. The second equation of the system (4) can be integrated:

$$\dot{y}+\omega_Hx=\text{const}. \tag{5}$$

Setting this constant equal to zero without loss of generality, we reduce the equations of motion of the particle to a system of two equations

$$\begin{aligned}\ddot{x}+\omega_H{}^2x&=\frac{e}{m_0}E_{0x}\sin(k_xx+k_zz-\nu_0t),\\ \ddot{z}&=\frac{e}{m_0}E_{0z}\sin(k_xx+k_zz-\nu_0t).\end{aligned} \tag{6}$$

It follows from the equations of motion (6) that in the case of an obliquely propagating wave ($k_x,\ k_z\neq0$) the longitudinal and transverse degrees of freedom are coupled. The Hamiltonian of the system (6) has the following form:

$$H=\frac{p_x{}^2+p_z{}^2}{2m_0}+\frac{m_0}{2}\omega_H{}^2x^2+e\phi_0\cos(k_xx+k_zz-\nu_0t), \tag{7}$$

where $\phi_0=E_{0x}/k_x=E_{0z}/\text{kHz}$ is the amplitude of the potential of the electric field and p_x and p_z are the corresponding components of the momentum of the particle:

$$p_x=m_0\dot{x},\quad p_z=m_0\dot{z}. \tag{8}$$

For the numerical analysis we used the dimensionless form of Eqs. (6). To obtain this form we set $m_0=1$ and go over to dimensionless variables by means of the replacements $x\to k_xx$, $z\to k_zz$, $t\to\omega_Ht$. Introducing the notation $\varepsilon=e\phi_0k_x^2/\omega_H^2$ and $\nu=\nu_0/\omega_H$, we obtain the dimensionless equations

$$\begin{aligned}\ddot{x}+x&=\varepsilon\sin\ (x+z-\nu t),\\ \ddot{z}&=\varepsilon\beta^2\sin(x+z-\nu t).\end{aligned} \tag{6'}$$

Let the resonance condition

$$\nu_0=q\omega_H, \tag{9}$$

be fulfilled, where q is an integer. The condition (9) implies that during one revolution of the particle in the magnetic field it passes through exactly q periods of the wave. When the condition (9) is fulfilled in the case of strictly transverse propagation of the wave the phase plane (x, p_x) of Eq. (6) is covered by a stochastic web for arbitrarily small wave amplitudes.[12] Below, we obtain the conditions for preservation of the stochastic web in the case of nonorthogonal propagation of the wave and investigate the changes of the phase portrait with change of the parameters of the system.

When the resonance condition (9) is fulfilled the Hamiltonian (7) can be represented in a more convenient form. With this aim we go over from the variables x, p_x to the action-angle variables $(J,\ \theta)$ for the transverse degree of freedom:

$$x=\rho\sin\theta,\ p_x=\rho\omega_H\cos\theta,\ J=\rho^2\omega_H/2, \tag{10}$$

where ρ is the Larmor radius (we have set $m_0=1$). In the new variables the Hamiltonian of the problem has the form

$$H=p_z{}^2/2+J\omega_H+V_0\cos\ (k_x\rho\sin\theta+k_zz-\nu_0t), \tag{11}$$

where we have introduced the notation $V_0=e\phi_0$. We rewrite (11), using expansion in Bessel functions:

$$H=\frac{p_z{}^2}{2}+J\omega_H+V_0\sum_{n=-\infty}^{+\infty}J_n(k_x\rho)\cos(n\theta+k_zz-\nu_0t). \tag{12}$$

By means of the generating function

$$F=(\theta-\omega_Ht)I \tag{13}$$

we make the change to new variables $I = J$, $\varphi = \theta - \omega_H t$ in the coordinate frame rotating with the cyclotron frequency ω_H. In these variables the Hamiltonian has the form

$$\tilde{H}=H+\frac{\partial F}{\partial t}=\frac{p_z^2}{2}+V_0\sum_{n=-\infty}^{+\infty} J_n(k_x\rho)\cos[n(\varphi+\omega_H t)+k_z z-\nu_0 t]. \quad (14)$$

Separating out the resonance term, which does not depend explicitly on the time, we write the following representation of this Hamiltonian:

$$\tilde{H}=H_q+V_q,$$
$$H_q=p_z^2/2+V_0J_q(k_x\rho)\cos(q\varphi+k_z z), \quad (15)$$
$$V_q=V_0\sum_{\substack{n=-\infty,\\ n\neq q}}^{+\infty} J_n(k_x\rho)\cos[n\varphi+k_z z+(n\omega_H-\nu_0)t].$$

We shall call the expression for H_q the averaged Hamiltonian. It is the expression for $\tilde{H}$, averaged over the period of the Larmor rotation. For a sufficiently large value of the cyclotron frequency ω_H the terms appearing in the expression for V_q are rapidly oscillating in comparison with H_q, and their contribution to $\tilde{H}$ is small. Thus, the averaged Hamiltonian describes particle motion that differs little from the true motion under the condition

$$\omega_H\gg k_z\dot{z}, \quad \omega_H\gg\Omega_\perp, \quad (16)$$

where we have introduced the transverse bounce frequency

$$\Omega_\perp^2=V_0k_x^2, \quad (17)$$

corresponding to the frequency of small oscillations of the particle in the field of a plane transverse wave with amplitude V_0.

The averaged Hamiltonian H_q corresponds, generally speaking, to a nonintegrable Hamiltonian system. The dynamics of this system is determined in many respects by the relative magnitudes of the frequencies characterizing the longitudinal and transverse degrees of freedom.

Suppose that, besides the conditions (16), the conditions

$$|\dot{z}/V_0^{1/2}|\lesssim 1, \quad \Omega_\perp/\beta\omega_H\ll 1 \quad (18)$$

are fulfilled. The first of the inequalities (18) implies that through particles with high energies are not considered. The second inequality is fulfilled, e.g., if the wave propagates at a small angle to the direction of the magnetic field $\mathbf{B}_0$. In this case the averaged transverse motion is slow in comparison with the longitudinal motion, and in the system there is an additional approximate first integral. To find it, we write the expression (15) for H_q in the form of the Hamiltonian of a pendulum:

$$H_q=p_z^2/2-A(\rho)\cos[k_z z-\Phi(\varphi)] \quad (19)$$

with a slowly varying amplitude

$$A(\rho)=-V_0J_q(k_x\rho) \quad (20)$$

and with phase

$$\Phi(\varphi)=-q\varphi. \quad (21)$$

The slowness of the transverse motion leads to the appearance of an adiabatic invariant

$$\mathcal{I}(H_q,A(\rho))=\frac{1}{2\pi}\oint p_z\,dz =\frac{1}{2\pi}\oint\{2[H_q+A(\rho)\cos k_z z]\}^{1/2}dz\approx\text{const}. \quad (22)$$

Since on a trajectory we have $H_q = \text{const}$, the approximate integral $I = \text{const}$ follows from this.

An analytical investigation is also possible in another limiting case, when the second of the conditions (18) is replaced by the opposite condition:

$$\dot{z}/V_0^{1/2}\lesssim 1, \quad \Omega_\perp/\beta\omega_H\gg 1, \quad (23)$$

which corresponds, e.g., to the motion of a particle in the field of a wave propagating at almost a right angle to the direction of the magnetic field. In this case the averaged transverse motion is fast in comparison with the longitudinal motion. The system has an adiabatic invariant corresponding to the action of the averaged transverse motion. However, with drift of the slow variables z, p_z the phase point can intersect the separatrix of the fast motion on the (x, p_x) plane and the conditions for adiabaticity are violated. This leads to violation of the adiabatic invariance and cause the motion to become chaotic. The phenomena that occur in this case are considered below.

3. THE MOTION FOR LONGITUDINAL CYCLOTRON RESONANCE

In this section we consider the properties of the resonance Hamiltonian (14) in conditions of longitudinal cyclotron resonance:

$$k_z\dot{z}=s\omega_H, \quad (24)$$

where s is an integer. We shall assume that the motion of the particle along the magnetic field is close to the resonance motion, and that the cyclotron rotation is high-frequency:

$$\omega_H\gg|k_z\dot{z}-s\omega_H|, \quad \omega_H\gg\Omega_\perp. \quad (25)$$

In the Hamiltonian (14), when these conditions are fulfilled, it is possible to extract the resonance terms, i.e., to perform averaging in conditions of cyclotron resonance. For the averaged Hamiltonian we find

$$\langle H\rangle=p_z^2/2+V_0J_{q-s}(k_x\rho)\cos[(q-s)\varphi+k_z z-s\omega_H t]. \quad (26)$$

By means of the generating function

$$F_1(z,\bar{p}_z,t)=\bar{p}_z\left(z-\frac{s\omega_H}{k_z}t\right) \quad (27)$$

we go over to the coordinate frame moving along the direction of the magnetic field with velocity $v_z = s\omega_H/k_z$. In the new variables

$$Z=z-\frac{s\omega_H}{k_z}t, \quad \bar{p}_z=p_z$$

the Hamiltonian (26) can be written in the form

$$\langle\tilde{H}\rangle=\langle H\rangle+\frac{\partial F}{\partial t} =\frac{\bar{p}_z^2}{2}-\bar{p}_z\frac{s\omega_H}{k_z}+V_0J_{q-s}(k_x\rho)\cos[(q-s)\varphi+k_z Z].$$

Adding the constant $(s\omega_H/2k_z)^2$, we finally have

$$\langle \bar{H} \rangle = P_z^2/2 + V_0 J_{q-s}(k_x\rho) \cos [(q-s)\varphi + k_z Z], \tag{28}$$

where

$$P_z = p_z - s\omega_H/k_z. \tag{29}$$

The system (28), generally speaking, is not integrable. Simple to investigate is the case when the conditions

$$|\dot{Z}/V_0^{1/2}| \leqslant 1, \quad \Omega_\perp/\beta\omega_H \ll 1,$$

analogous to the conditions (18), are fulfilled. In this case the variables P_z, Z are fast and I,φ are slow. In the same way as in the preceding section, we find the approximate integral $I = \text{const}$ of the slow motion.

The opposite limiting case, when the motion in the (P_z, Z) plane is slow and that in the (I, φ) plane is fast, corresponds to wave propagation almost perpendicular to the magnetic field. The parameter $\beta = k_z/k_x$ is small, and the second of the inequalities (18) is replaced by the opposite inequality

$$\beta\omega_H/\Omega_\perp \ll 1. \tag{30}$$

We rewrite the Hamiltonian (28) of the averaged motion in dimensionless form, for which we change to the dimensionless variables

$$P_z = \frac{\hat{P}_z}{k_z}, \quad I = \frac{\hat{I}}{k_x^2}, \quad t = \frac{\hat{t}}{\omega_H}, \quad Z = \frac{\hat{Z}}{k_z}. \tag{31}$$

Omitting the carets for simplicity, we find the dimensionless Hamiltonian

$$\mathscr{H} = \frac{k_x^2}{\omega_H^2} \langle \bar{H} \rangle = \frac{P_z^2}{2\beta^2} + V J_{q-s}(\rho) \{\cos[(q-s)\varphi] \cos Z - \sin[(q-s)\varphi] \sin Z\}, \tag{32}$$

where $V = k_x^2 V_0/\omega_H^2$. The corresponding Hamilton equations of motion have the form

$$\frac{dI}{dt} = -\frac{\partial \mathscr{H}}{\partial \varphi}, \quad \frac{d\varphi}{dt} = \frac{\partial \mathscr{H}}{\partial I},$$
$$\frac{dP_z}{dt} = -\beta^2 \frac{\partial \mathscr{H}}{\partial Z}, \quad \frac{dZ}{dt} = \beta^2 \frac{\partial \mathscr{H}}{\partial P_z}. \tag{33}$$

As was shown in Ref. 12, for strictly perpendicular propagation of the wave ($\beta = 0$) the equations of motion (33) determine a stochastic web on the (I, φ) plane. This web is a network of finite thickness, inside which the dynamics of the particles is stochastic, and outside which (i.e., in the cells of the network) the dynamics is regular. The geometry of the web as a whole is also preserved for $\varepsilon \sim 1$. An example of such a web is shown in Fig. 1, in which a Poincaré section of the phase trajectory of the system (6′) for $\beta = 0$, $\varepsilon = 2.2$, and $\nu = 4$. The points are calculated after equal time intervals $\Delta t = 2\pi/\nu$.

For a small nonzero value of $\beta \ll 1$ the variables Z and P_z vary slowly. Thus, in the system described by the Hamiltonian $\mathscr{H}$ the variables φ and I are fast and the variables Z and P_z are slow. A particle rotates rapidly in the cells of the web while executing slow motion (drift). The equations of motion have the following form:

$$\begin{aligned} \dot{Z} &= P_z, \\ \dot{P}_z &= \beta^2 V J_{q-s}(\rho) \{\cos[(q-s)\varphi] \sin Z + \sin[(q-s)\varphi] \cos Z\} \\ \dot{\varphi} &= V \left(\frac{1}{I\omega_H}\right)^{1/2} J'_{q-s}(\rho) \cos[(q-s)\varphi + Z], \\ \dot{I} &= V(q-s) J_{q-s}(\rho) \sin[(q-s)\varphi + Z], \end{aligned} \tag{34}$$

where $J'_{q-s}(\rho)$ denotes the derivative of a Bessel function with respect to its argument. For $\beta \ll 1$ we can consider the second pair of equations (34) with a slowly varying parameter Z. The separatrix network of the Hamiltonian is specified by the set of equations

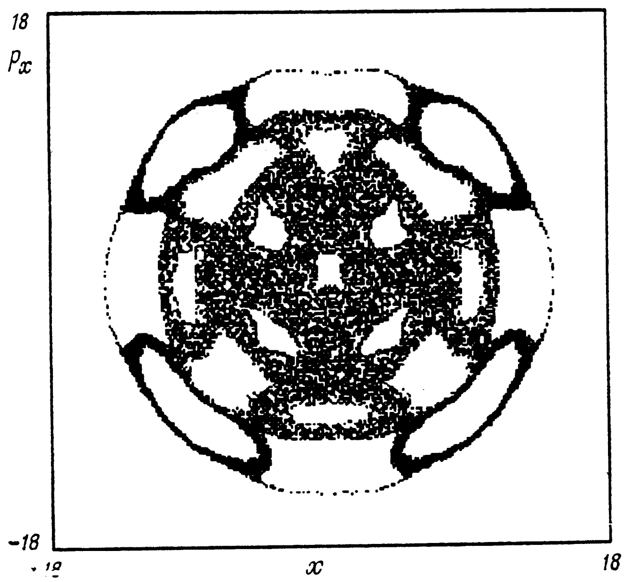

FIG. 1. Stochastic web for $\beta = 0$, $\varepsilon = 2.2$, $\nu = 4$.

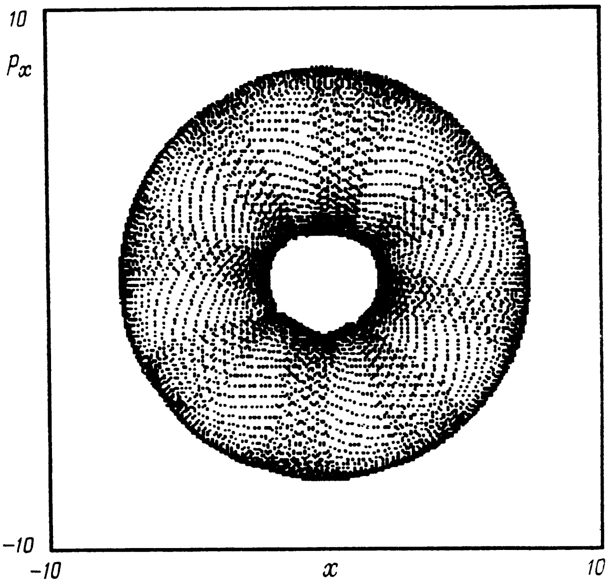

FIG. 2. Regular trajectory for $\beta^2 = 10^{-4}$, $\varepsilon = 0.5$, $\nu = 4$.

$$\rho=\rho_*^{(j)},\ J_{q-s}(\rho_*^{(j)})=0,\ j=1,2,\ldots,$$
$$\varphi=\varphi_*,\ (q-s)\varphi_*+Z=\pi/2+\pi n, \tag{35}$$

where n is an integer. The equations (35) specify a spiral web in the three-dimensional space (x, p_x, Z). In the section cut by a plane $Z=$ const the web is a system of concentric circles, and straight lines passing through the coordinate origin, on the (x, p_x) plane. The cells of the web form concentric "belts" around the coordinate origin. From Eqs. (35) it can be seen that, when Z is varying slowly as a result of detuning from longitudinal resonance, the separatrix network and, consequently, the stochastic web rotate about the coordinate origin with angular velocity $P_z\sim\beta^2$, while the shape and area of the shells remain constant. Because of this, a particle that is executing rapid rotation inside a cell sufficiently far from the separatrix of the averaged motion will never intersect the stochastic web and will move regularly. An example of such a regular trajectory is shown in Fig. 2.

A trajectory with initial conditions sufficiently close to the separatrix of the averaged motion is captured by a stochastic layer and wanders chaotically along the channels of the stochastic web. In this case, of course, the particle energy can vary significantly, and, thus, the particle will execute diffusional motion. This is, in essence, the usual mechanism of diffusion over a stochastic web—a mechanism that is described, e.g., in Ref. 12 and also occurs when the wave propagation is strictly perpendicular to the magnetic field. Thus, there is no substantial enhancement of the diffusion in comparison with that in the case of a perpendicularly propagating wave.

These conclusions were confirmed by numerical integration of the system (6′). Figure 3a depicts the projection of the Poincaré section on to the (x, p_x) plane for the same values of the parameters as in Fig. 1 but with $\beta^2=10^{-5}$. The initial conditions are the same in the two cases. It can be seen that there is some increase in the energy to which the particle is "heated" during the observation time, but this increase is slight. The slow drift of orbits as a result of the detuning of the longitudinal resonance can also be seen. We note that stochastic dynamics occurs in the system despite the fact that the criterion that was used in Refs. 2 and 17 for overlap of cyclotron resonances is not fulfilled. This is demonstrated by Fig. 3b, which shows the projection of the Poincaré cross section on to the (Z, p_z) plane for the same parameter values as in Fig. 3a. It can be seen that the amplitude of the variations of the momentum p_z is too small for the criterion for overlap of resonances to be fulfilled, although the dynamics of the system is stochastic. This is due to a certain roughness (noted, e.g., in Ref. 18) in the criterion for overlap of the primary resonances.

We shall estimate the energy to which the particles can be heated stochastically as a result of random walks of the phase trajectories inside the stochastic web. When the trajectory falls inside the web the probability of heating of the particle, i.e., the probability that it moves chaotically over the web in the radial direction away from the coordinate origin, is smaller than the probability of motion in the direction of lower values of the action. This is due to the exponential decrease of the thickness of the web with increase of ρ. In the case of wave propagation perpendicular to the magnetic field the thickness of the stochastic layer is estimated as follows:[12]

$$H_c=(2\pi)^{3/2}\frac{4\omega_H^2}{k_x^2\varepsilon q^2}(k_x\rho_0)^{5/2}\exp\left\{-\frac{\omega_H^2}{k_x\varepsilon q}\left(\frac{\pi}{2}k_x\rho_0\right)^{3/2}\right\}, \tag{36}$$

where ρ is the coordinate of the nearest stationary elliptic point. This estimate can also be used in the case of a longitudinal resonance of order s for oblique propagation of the wave for $\beta\ll 1$ (with the replacement $q\to|q-s|$). If we deem the diffusion to cease for those values of ρ_0 for which the argument of the exponential is equal to -2, we obtain an approximate estimate for the value ρ_c up to which diffusion of particles occurs:

$$\rho_c\sim\frac{4}{\pi k_x^{1/3}}\left[\frac{\varepsilon|q-s|}{\omega_H^2}\right]^{2/3}. \tag{37}$$

As is shown by numerical analysis, the diffusion of particles is also limited in the case when the longitudinal-resonance condition (24) is not fulfilled.

4. DYNAMICS OF A PARTICLE IN THE FIELD OF A PACKET CONSISTING OF SEVERAL HARMONICS

We shall consider the dynamics of a charged particle in a homogeneous magnetic field and the field of a wave packet consisting of a finite number M of harmonics. The electric field $\mathbf{E}$ of the packet lies in the (x, z) plane and has the form

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\sum_{m=1}^{M}\sin(k_x x+k_z z-m\nu_0 t), \tag{38}$$

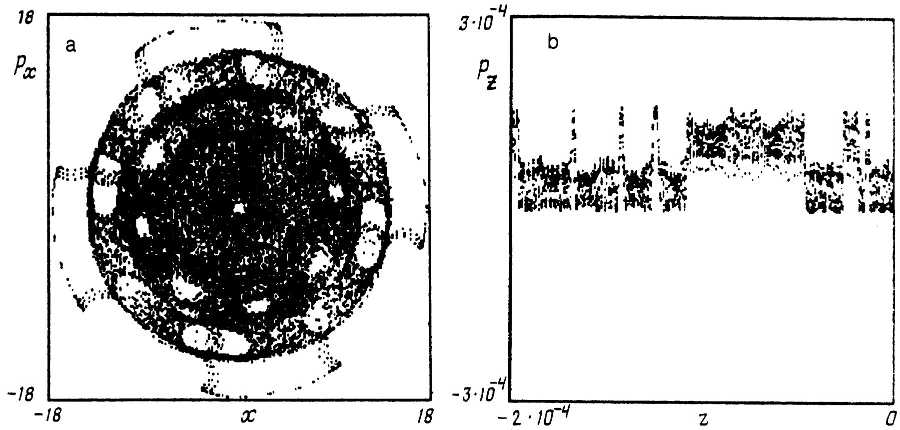

FIG. 3. a) Phase trajectory for $\beta^2=10^{-5}$, $\varepsilon=2.2$, $\nu=4$: projection on to the (x, p_x) plane; b) the same, but in projection on to the (Z, p_z) plane.

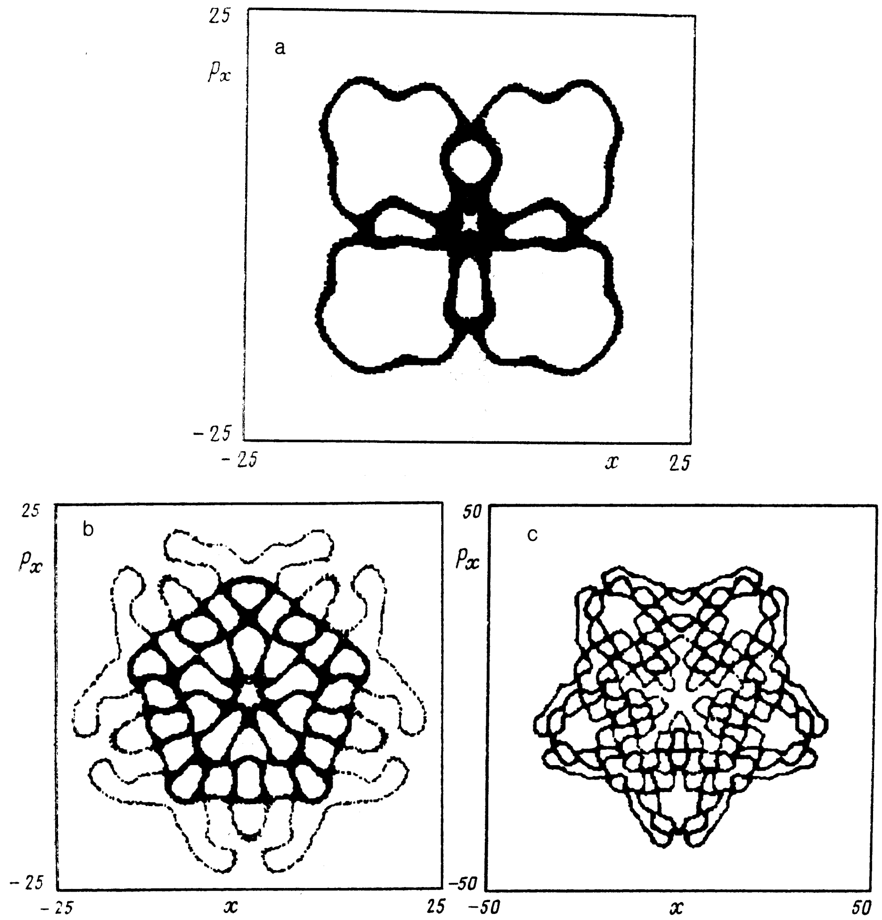

FIG. 4. Stochastic web formed for motion in the field of a wave packet for $\beta=0$, $\varepsilon=1.0$: a) $M=2, q=4$; b) $M=3, q=5$; c) $M=5, q=5$.

i.e., for simplicity the amplitudes of all the harmonics are taken to be the same. As before, let the resonance condition $\nu_0=q\omega_H$ be fulfilled. The Hamiltonian of the system is written in the form

$$H=\frac{p_x^2+p_z^2}{2}+\frac{1}{2}\omega_H^2x^2+V_0\sum_{m=1}^{M}\cos(k_xx+k_zz-m\nu_0t). \tag{39}$$

Just as was done in Sec. 2, we find the resonance Hamiltonian:

$$\bar{H}=H_q+V_q,$$

$$H_q=\frac{p_z^2}{2}+V_0\sum_{m=1}^{M}J_{mq}(k_x\rho)\cos(mq\varphi+k_zz), \tag{40}$$

$$V_q=V_0\sum_{m=1}^{M}\sum_{\substack{n=-\infty,\\ n\neq mq}}^{+\infty}J_n(k_x\rho)\cos[n\varphi+k_zz+(n\omega_H-m\nu_0)t].$$

When the conditions (16) are fulfilled, the dynamics of the system is described in the main by the averaged Hamiltonian H_q. In particular, for $\beta=0$ the latter determines the geometry of the stochastic web and the arrangement of the elliptic and hyperbolic stationary points on the phase plane. Figure 4 shows Poincaré sections of phase trajectories of the particle in the case of strictly perpendicular ($\beta=0$) propagation of the wave packet for $\varepsilon=1.0$ and $M=2$, $q=4$ (Fig. 4a), $M=3$, $q=5$ (Fig. 4b), and $M=5$, $q=5$ (Fig. 4c). The order of the symmetry of the stochastic web is determined by the quantity q, i.e., by the ratio of the characteristic frequency of the wave packet to the cyclotron frequency. The skeletons of the webs have a more complicated geometry than in the case (analyzed in Ref. 12) of a packet consisting of one harmonic, in which they are a set of concentric circles and rays issuing from the coordinate origin.

For $\beta\neq0$ the number of degrees of freedom in the system becomes equal to $2\frac{1}{2}$, and this leads to substantial changes in the dynamics of the particles. We shall consider the particular case $q=4$, $M=2$. The Hamiltonian H_q takes the form

$$H_4=p_z^2/2+V_0\{J_4(k_x\rho)\cos(4\varphi+k_zz)+J_8(k_x\rho)\cos(8\varphi+k_zz)\}. \tag{41}$$

For oblique propagation of the wave packet ($\beta^2\ll1$) the variable z in (41) is a slowly varying parameter. With increase of z the geometry of the separatrix network of the averaged Hamiltonian changes. This is demonstrated in Figs. 5a and 5b, which present the numerically obtained separatrices of the Hamiltonian H_4 for $k_zz=0$ and $k_zz=\pi/2$, respectively. It can be seen that with change of z the shape and area of the cells of the separatrix network change. Therefore, a particle executing rapid rotation inside a cell not too close to an elliptic point will, in time, intersect the separatrix of the averaged motion and fall into a stochastic layer. As a result, the effective width of the web turns out to be considerably greater than for $\beta=0$. This increase of the region of chaotic dynamics is analogous to that described in Ref. 16 for an infinitely wide wave packet, when it leads to a sharp

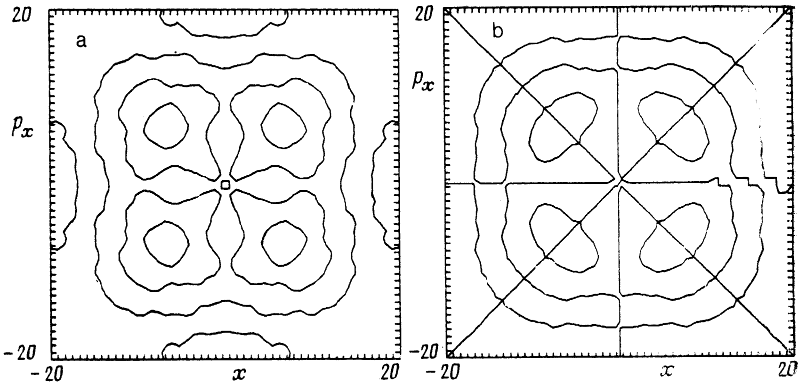

FIG. 5. Separatrices of the Hamiltonian H_4 for $k_z z = 0$ (a) and $k_z z = \pi/2$ (b).

enhancement of the diffusion of particles. The difference between the system under consideration and that investigated in Ref. 16 is manifested at sufficiently large values of the energy. For large values of ρ we make use of the asymptotic expansions of the Bessel functions and their derivatives:

$$J_{mq}(k_x\rho) = \left(\frac{2}{\pi k_x\rho}\right)^{1/2} \cos\left(k_x\rho - \frac{mq\pi}{2} - \frac{\pi}{4}\right) + \ldots,$$

$$J_{mq}'(k_x\rho) = -\left(\frac{2}{\pi k_x\rho}\right)^{1/2} \sin\left(k_x\rho - \frac{mq\pi}{2} - \frac{\pi}{4}\right) + \ldots$$

We confine ourselves to the leading terms of the asymptotic forms, which is admissible when

$$k_x\rho \gg (4m^2q^2 - 1)/8 \equiv k_x\rho_m. \tag{42}$$

The Hamiltonian (41) takes the form

$$H_4 = \frac{p_z^2}{2} + V_0 \left(\frac{2}{\pi k_x\rho}\right)^{1/2} \cos\left(k_x\rho - \frac{\pi}{4}\right) \times \{\cos(4\varphi + k_z z) + \cos(8\varphi + k_z z)\}. \tag{43}$$

From this we find the set of equations that specify the geometry of the separatrix network for large ρ:

$$k_x\rho_* = \frac{3\pi}{4} + \pi n, \quad \varphi_* = \frac{\pi}{4} + \frac{\pi k}{2}, \quad \varphi_* = \frac{\pi}{6} + \frac{\pi l}{2} - \frac{k_z z}{6}, \tag{44}$$

where n, k, and l are integers. Thus, for large ρ the separatrix network is a set of concentric circles and rays issuing from the coordinate origin, as in the case of motion in the field of one wave. Therefore, for such ρ the arguments of the preceding section concerning the mechanism of the limitation of diffusion over the stochastic web can be repeated. In the process of slow drift the phase trajectory cannot leave the "belt" in which it finds itself, and the diffusion is weakened on account of the exponential decrease of the thickness of the web with increase of ρ. Thus, in practice, diffusion ceases at values of ρ at which the cells of the web form concentric circular belts. The limitation of the stochastic heating can be seen from Figs. 6a–d, which show consecutive stages of the development of the diffusion process in the field of the packet (38) consisting of two harmonics ($\beta^2 = 10^{-4}$, $\varepsilon = 1.0$,

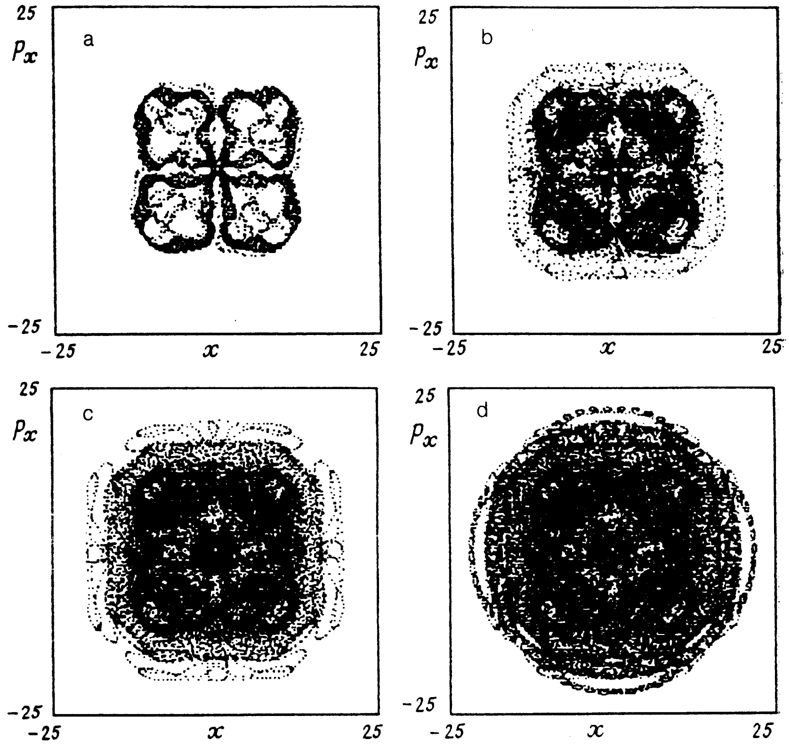

FIG. 6. Stages in the development of the diffusion process in the field of the packet (38) for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 2$, $q = 4$ (one phase trajectory is represented) for various values of the calculation time T: a) $T = 8000$; b) $T = 16\,000$; c) $T = 24\,000$; d) $T = 40\,000$.

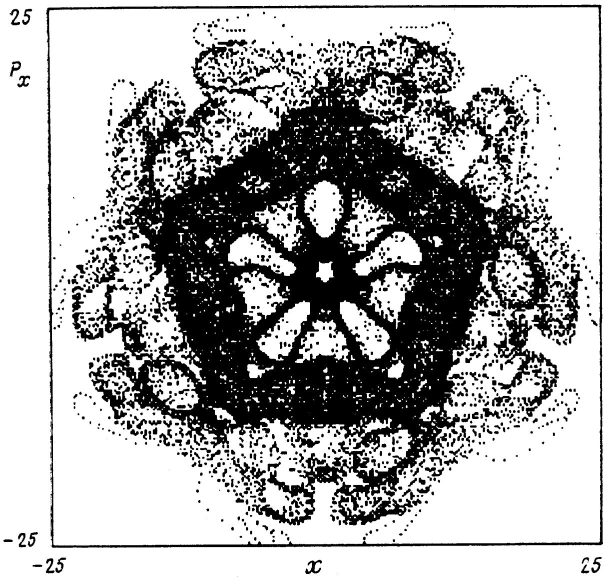

FIG. 7. Poincaré section of a phase trajectory of a particle moving in the field of the packet (38) for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 3$, $q = 5$.

$M = 2$, $q = 4$). Figure 7 shows a Poincaré section of the phase trajectory of a particle moving in the field of the packet (38) for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 3$, $q = 5$. Figure 8 shows stages of the diffusion process for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 5$, $q = 5$; the corresponding calculation times are $T = 19\,000$ (Fig. 8a) and $T = 49\,000$ (Fig. 8b). Owing to the presence of higher harmonics in the packet, the "heating" of a particle is to higher values of the energy than in the case represented in Fig. 7. Figure 9 depicts Poincaré sections of a particle trajectory for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 3$, $q = 7$; the corresponding calculation times are $T = 14\,200$ (Fig. 9a) and $T = 46\,200$ (Fig. 9b). It is clear that the symmetry of the phase portraits on Figs. 6–9 is determined by the quantity $q = v_0/\omega_H$. The slow drift of the orbits as a result of the longitudinal motion can be seen.

To conclude the section we make a remark concerning the maximum magnitude of the stochastic heating. The energy to which a particle can be heated has been estimated from simple physical considerations. In Ref. 19 it is assumed that a particle can be accelerated to a velocity equal to the highest of the phase velocities of the harmonics of the wave packet. However, the above results from the numerical analysis show that this estimate is much too low. This can be seen, e.g., from Fig. 4c, obtained for $M = 5$, $q = 5$. The corresponding highest phase velocity in the wave packet is equal to (in dimensionless units) $v_{p,\max} = Mq\omega_H/k_x = 25$, whereas the maximum velocity to which the particle is accelerated in the observation time is $\dot{x}_{\max} \approx 40$, which is almost twice as large. This discrepancy is due, obviously, to the stochastic character of the acceleration.

5. CONCLUSION

The system that we have considered has $2\frac{1}{2}$ degrees of freedom and possesses the basic properties intrinsic to multidimensional Hamiltonian dynamical systems. Thus, investigation of this system permits us to make progress in our understanding of the complicated phenomena associated with the onset of stochastic dynamics in multidimensional systems. Owing to the degeneracy of the problem with respect to some of the degrees of freedom, a "bare" web exists in the phase space. In the case when the wave packet contains one harmonic, the skeleton of this web, as determined by the averaged Hamiltonian, possesses radial symmetry and is a set of concentric circles and rays issuing from the coordinate origin. The presence of the extra longitudinal degree of freedom gives rise to slow drift of the phase trajectories, and this leads to the appearance of a spiral web in the phase space. If the packet consists of several harmonics the structure of the "bare" web is more complicated: There exists a region around the coordinate origin in which the skeleton of the web is not a set of rays issuing from the coordinate origin and circles, as in the case of one wave, but is more like the skeleton of the stochastic web that arises in the field of an infinitely wide packet.[14] If for a stochastic web arising in a four-dimensional phase space we consider sections of it cut by planes perpendicular to the magnetic-field direction, it turns out that in this region the shape, area, and relative positions of the cells of the web vary from one section to another. Here, phenomena occur that are similar to those considered in Ref. 16—in particular, enhancement of diffusion. Nevertheless, the diffusion of particles is limited, since the geometry of the web at large values of the energy of the transverse motion is the same as in the field of one wave, but the thickness of the web decreases exponentially with increase of the energy.

We note that the slowing of the diffusion at large ener-

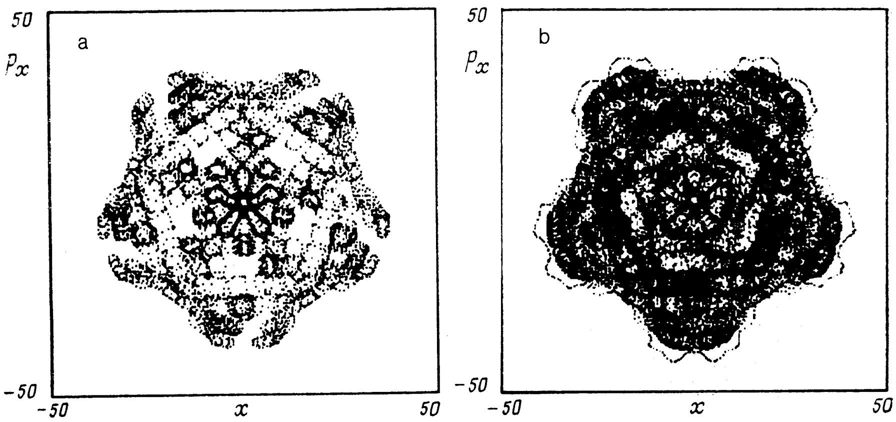

FIG. 8. Stages in the development of the diffusion process for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 5$, $q = 5$ (one phase trajectory is represented) for $T = 19\,000$ (a) and $T = 49\,000$ (b).

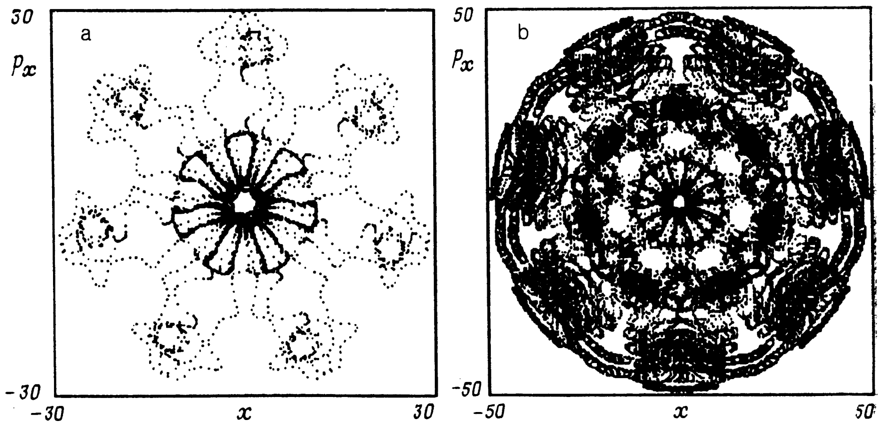

FIG. 9. Stages in the development of the diffusion process for $\beta^2 = 10^{-4}$, $\varepsilon = 1.0$, $M = 3$, $q = 7$ (one phase trajectory is represented) for $T = 14\,200$ (a) and $T = 46\,200$ (b).

gies of the transverse motion creates considerable difficulties in a numerical investigation of the dynamics of the particles on the web. In particular, the question of the rearrangements of stochastic webs with increase of the number of harmonics in the packet requires further investigation in the case of oblique propagation of the packet.

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