Branching bifurcations of vector envelope solitons and integrability of Hamiltonian systems

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Branching bifurcations of stationary vector envelope solitons by solitons with a given polarization are analyzed. Both creation and annihilation bifurcations are considered. The relationship between bifurcations of this sort and the integrability of the dynamic system with two degrees of freedom which arises upon the reduction of the system of nonlinear Schrödinger equations is discussed. Branching bifurcations might be utilized to control the structure of soliton signals.

Current research on optical solitons in optical fibers in situations in which the nonlinear medium has birefringent properties, or in which there is a finite number of waveguide modes, requires an analysis of soliton solutions of a system of nonlinear Schrödinger equations.[1] A question which unavoidably arises here is whether there can be substantial changes in the soliton states (signals) which are related to changes in structural parameters (the parameters of the nonlinear medium or the phase velocities or frequencies of envelope waveguide modes). In birefringent optical fibers, for example, both solitons with a given polarization and "vector" solitons, in which both polarizations are represented, can propagate.

In the present paper we take up the branching bifurcations, of the creation and annihilation types, of stationary vector envelope solitons by solitons with a given polarization which arise when the structural parameters change.

The dynamic system of equations [Eqs. (1.6) below] with two degrees of freedom which arises when the class of stationary envelope waves [Eqs. (1.5)] of the original system of nonlinear Schrödinger equations, (1.1), is identified is generally not integrable. It is integrable only if the structural parameters satisfy certain relations. Observations of the bifurcation curves of the branchings of complex vector solitons from solitons with a given polarization, along with the known cases of complete integrability of the dynamic system of equations, (1.6), indicate that those values of the structural parameters for which complete integrability prevails belong to bifurcation branching curves. This relationship suggests the unexpected possibility of utilizing data generated from analyzing the branching bifurcations of complex solitons (homoclinic saddle loops) from simple solitons in order to search for cases of complete integrability of Hamiltonian dynamic systems. The realization of this possibility might lead to the formulation of an integrability criterion for a certain class of Hamiltonian systems with two degrees of freedom which arise when the original field equations are reduced and which (a particularly important point) lead to the existence of soliton states and branching bifurcations thereof near an integrability point or integrability region.

At present we do not know just how bifurcations of the branching of stationary envelope solitons are manifested in evolution problems. The observation of such bifurcations in problems of this sort might suggest new possibilities for controlling soliton signals.

1. The system of nonlinear Schrödinger equations

$$\begin{aligned} &i\psi_{1,t}+iv_1\psi_{1,x}+\psi_{1,xx}+2(R_{11}|\psi_1|^2+R_{12}|\psi_2|^2)\psi_1=0,\\ &i\psi_{2,t}+iv_2\psi_{2,x}+\psi_{2,xx}+2(R_{21}|\psi_1|^2+R_{22}|\psi_2|^2)\psi_2=0 \end{aligned} \tag{1.1}$$

determines the dynamics of envelope waves with polarizations (ψ_1,ψ_2) and phase velocities (v_1,v_2) in a nonlinear medium with nonlinearity parameters R_{ij} (Ref. 1, for example).

Setting $R_{12}=R_{21}>0$, and transforming independent variables,

$$x\to\xi=R_{12}^{1/2}(x-v_1t),\qquad t\to\tau=R_{12}t, \tag{1.2}$$

we find that Eqs. (1.1) become

$$\begin{aligned} &i\psi_{1,\tau}+\psi_{1,\xi\xi}+2(a|\psi_1|^2+|\psi_2|^2)\psi_1=0,\\ &i\psi_{2,\tau}+i\delta\psi_{2,\xi}+\psi_{2,\xi\xi}+2(c|\psi_2|^2+|\psi_1|^2)\psi_2=0. \end{aligned} \tag{1.3}$$

Here

$$a=R_{11}/R_{12},\qquad c=R_{22}/R_{12},\qquad \delta=(v_2-v_1)/R_{12}^{1/2}. \tag{1.4}$$

For Eqs. (1.3) we can identify stationary solutions

$$\psi_1(\xi,\tau)=e^{i\lambda_1\tau}X_1(\xi),\qquad \psi_2(\xi,\tau)=e^{i\lambda_2\tau+i\delta\xi/2}X_2(\xi) \tag{1.5}$$

with the real parameters λ_1, λ_2 and functions X_1, X_2. The functions X_1, X_2, satisfy the system of equations

$$\begin{aligned} &X_{1,\xi\xi}-\lambda_1X_1+2(aX_1^2+X_2^2)X_1=0,\\ &X_{2,\xi\xi}-\nu\lambda_1X_2+2(cX_2^2+X_1^2)X_2=0, \end{aligned} \tag{1.6}$$

where

$$\nu=(\lambda_2-\delta^2/4)\lambda_1. \tag{1.7}$$

In the phase space $\Gamma(P_1=X_{1,\xi},X_1,P_2=X_{2,\xi},X_2)$, Eqs. (1.6) correspond to the following system of canonical equations with Hamiltonian

$$H={}^1/_2P_1^2+{}^1/_2P_2^2-{}^1/_2\lambda_1X_1^2-{}^1/_2\nu\lambda_1X_2^2+{}^1/_2aX_1^4+X_1^2X_2^2+{}^1/_2cX_2^4. \tag{1.8}$$

Both the solutions of Eqs. (1.6) and Hamiltonian (1.8) depend on essentially the three structural parameters a, c, and ν. The parameters a and c are determined by the parameters of the nonlinear medium, while ν is determined by the difference between phase velocities, v_1-v_2, and by the parameters λ_1, λ_2 of the class of stationary solutions which have been singled out.

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According to (1.5), solutions of Eqs. (1.6) which satisfy the following conditions correspond to stationary nontopological envelope solitons:

$$\lim_{\xi\to\pm\infty} X_1=0, \qquad \lim_{\xi\to\pm\infty} X_2=0. \tag{1.9}$$

Let us assume that the parameters of Eqs. (1.6) satisfy the inequalities

$$a>0, \quad c>0, \quad \nu>0, \quad \lambda_1>0. \tag{1.10}$$

In this case, the problem (1.6), (1.9) allows the existence of solitons of the forms $(X_1,0)$ and $(0,X_2)$, where

$$\begin{gathered} X_1=\pm(\lambda_1/a)^{1/2}/\mathrm{ch}(\lambda_1^{1/2}\xi), \\ X_2=\pm(\nu\lambda_1/c)^{1/2}/\mathrm{ch}([\nu\lambda_1]^{1/2}\xi). \end{gathered} \tag{1.11}$$

We will refer to such states as soliton states with given polarizations $(\psi_1,0)$ or $(0,\psi_2)$, respectively. In the phase space Γ, the mappings of these states are separatrices: homoclinic loops of a singular saddle point $O(X_1=P_1=X_2=P_2=0)$, which lie in the invariant planes (P_1,X_1) and (P_2,X_2), respectively, on the Hamiltonian level $H=0$.

From this point of view, homoclinic saddle loops O, which are distinct from simple loops which lie in the (P_1,X_1) and (P_2,X_2) invariant planes, correspond to stationary nontopological envelope solitons of a general type which are determined by (1.5), (1.9) with $\lambda_1>0, \lambda_2>0$ in the phase space Γ. We will call representatives of soliton states of this type "vector solitons."

It was shown in Refs. 2 and 3 that in the case $\nu=1$ the following solution of Eqs. (1.6) corresponds to the simplest vector soliton:

$$\begin{gathered} X_2=\pm\mu(a,c)X_1, \quad \mu^2(a,c)=(a-1)/(c-1)>0, \\ X_1=1/(a+\mu^2)^{1/2}\,\mathrm{ch}\,\xi, \end{gathered} \tag{1.12}$$

with a region of existence $a>1, c>1$ or $a<1, c<1$.

In the case $\nu\neq1$, simple vector solitons correspond, in particular, to a single-parameter family of solutions of Eqs. (1.6) determined by

$$t-t_0=\int_{X_1}^{(X_1)m} dX/X\left[1-aX^2-\frac{2B^2}{1+\nu^{1/2}}X^{2\nu^{1/2}}\right]^{1/2}, \quad X_2=BX_1^{\nu^{1/2}}. \tag{1.13}$$

Here B is the parameter of the family, and $(X_1)_m$ is the solution of the equation

$$1-aX^2-\frac{2B^2}{1+\nu^{1/2}}X^{2\nu^{1/2}}=0. \tag{1.14}$$

The region in which the solutions (1.13) exist is defined by

$$a=2/[\nu^{1/2}(\nu^{1/2}+1)], \quad c=2/[\nu^{-1/2}(\nu^{-1/2}+1)]. \tag{1.15}$$

In the space of the structural parameters (a,c,ν) the region in which vector solitons of this sort exist is a curve with the following projection onto the (X_1,X_2) plane:

$$a+c+6=8/ac. \tag{1.16}$$

The existence of a single-parameter family of vector solitons (1.13)–(1.15) is a consequence of the coalescence in the phase space Γ of the stable and unstable manifolds of the singular point O at the Hamiltonian level $H=0$. This coalescence of the stable and unstable saddle manifolds of a singular point is one necessary condition for the complete integrability of a Hamiltonian dynamic system (i.e., for integrability at an arbitrary level of Hamiltonian H).

A first observation: On the curve (1.15) on which the saddle manifolds coalesce, there are three points at which Eqs. (1.6) are completely integrable. They correspond to the following values of the structural parameters of these equations:

$$a=c=\nu=1; \tag{1.17}$$

$$a=8/3, \quad c=1/3, \quad \nu=1/4; \quad a=1/3, \quad c=8/3, \quad \nu=4. \tag{1.18}$$

In case (1.17), the complete integrability is associated with the invariance of the system under the rotation group and the presence of the obvious supplementary integral $M=X_1P_2-X_2P_1$. For the values of the structural parameters in (1.18), the existence of this supplementary integral (in the form of a fourth-degree polynomial in the momenta P_1,P_2) and the complete integrability of Eqs. (1.6) follow from the analysis of Ref. 4.

At the points of complete integrability, (1.17), (1.18), it is a simple matter to evaluate the integral in (1.13) and to write explicit expressions for the single-parameter family of vector solitons. For the point of complete integrability $a=c=1$, $\nu=1$, and for the value $\lambda_1=1$ (which can be reached through a simple scaling), for example, the family of vector solitons is given by

$$X_2=BX_1, \quad X_1=1/(1+B^2)^{1/2}\,\mathrm{ch}(\xi-\xi_0), \tag{1.19}$$

where B is the actual parameter of the family, while the parameter ξ_0 is associated with the invariance of the system under a shift along the trajectory in Γ.

2. It was shown in Refs. 2 and 3 for the case $\nu=1$ that $a=1$ and $c=1$ are bifurcation values which determine the creation or annihilation of simple vector solitons as in (1.12). To demonstrate the point, we assume $a<1, c>1$. As the parameter a increases at a constant value of the parameter c, the crossing of the value $a=1$ results in the creation (in the case $a=1$) of a pair of vector solitons as in (1.12); they branch off from a pair of solitons with the given polarization, $(X_1,0)$. If, at a constant value of the parameter a, we then reduce the value of the parameter $c>1$, we find that when the bifurcation value $c=1$ is crossed these vector solitons coalesce with a pair of solitons of a different given polarization, $(0,X_2)$.

In addition, a countable set of bifurcation values of the parameter a (or c) was determined in Ref. 2. When those bifurcation values are crossed, a countable set of complex vector solitons, which branch off from a pair of solitons with a given polarization, $(X_1,0)$ [or $(0,X_1)$], is created. Figure 1 shows projections of the trajectory of a vector soliton onto the (X_1,X_2) plane to illustrate various stages of the transformation of this trajectory with distance from the bifurcation point.

Corresponding to the vector solitons in the phase space Γ are homoclinic loops of the saddle O, i.e., trajectories along which stable and unstable manifolds of the saddle intersect.

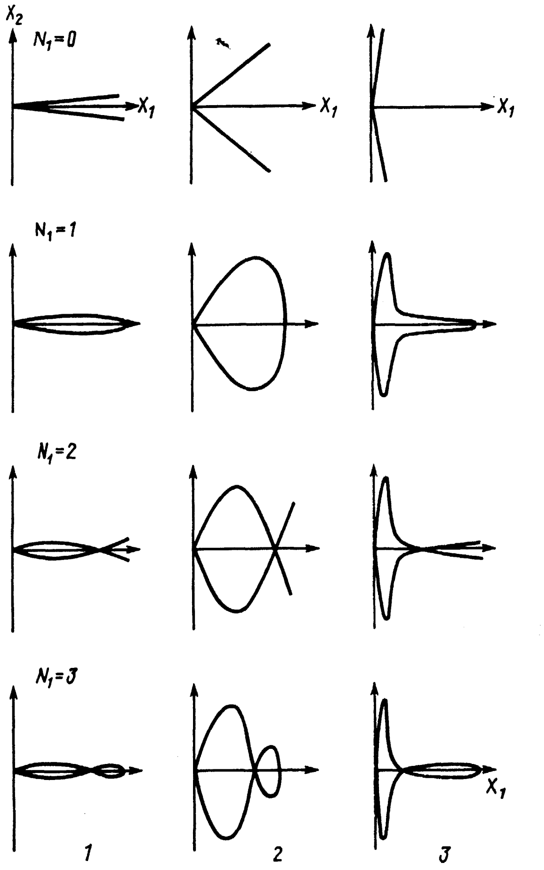

FIG. 1. Projections onto the (X_1,X_2) plane of homoclinic loops which branch off at various values of N_1 ($N_1=0,1,2,3$) from the simple loop corresponding to a soliton with a given polarization, $(X_1,0)$, in various stages of the transformation when the parameter a varies. 1—Immediately after the creation bifurcation; 2,3—when the parameter a increases.

A qualitative analysis, which is supported by numerical calculations, leads to the following assertion:[5] Bifurcation values of the parameters a and c correspond to tangency of the stable and unstable manifolds of saddle O along simple homoclinic loops which lie in the invariant planes (P_1,X_1) and (P_2,X_2) and which are inverse transforms of solitons with a given polarization $(X_1,0)$ and $(0,X_2)$.

It is shown below that a generalization of this assertion to the case $\nu\neq 1$ is the major step in a determination of those manifolds of lower dimensionality in the space of structural parameters (a,c,ν) to which points of complete integrability, or of an integrability at a certain level of Hamiltonian H, belong.

Setting $\lambda_1=1$ in Eqs. (1.6) (we do not detract from the generality of this analysis by doing so), and also setting

$$X_1=1/a^{1/2}\,\mathrm{ch}\,\xi+u_1,\quad X_2=0+u_2, \tag{2.1}$$

we find that to first order in the functions u_1 and u_2 the possibility that the vector solitons can branch off from a pair of solitons with a given polarization $(\pm 1/a^{1/2}\cosh\xi,0)$ is determined by the conditions under which the following problem can be solved:

$$u_{1,\xi\xi}-u_1+6u_1/\mathrm{ch}^2\,\xi=0, \tag{2.2}$$

$$u_{2,\xi\xi}-\nu u_2+2u_2/a\,\mathrm{ch}^2\,\xi=0. \tag{2.3}$$

For an arbitrary point (a,c,ν) in the space of structural parameters, the problem (2.2), (2.3) is overdetermined and generally unsolvable. The reason is that Eq. (2.2) has a unique solution $\sinh\xi/\cosh^2\xi$, which satisfies the necessary conditions and which corresponds to a displacement mode. However, Eq. (2.3) can be solved for an arbitrary value of ν only for values of a and c which satisfy the bifurcation relations

$$\nu^{1/2}=N_1(a)-N_1,\quad N_1=0,\,1,\,\ldots,\,[N_1(a)], \tag{2.4}$$

where $N_1(a)=\{-1+[1+8/a]^{1/2}\}/2$, and $[N_1(a)]$ is the greatest integer in $N_1(a)$. Correspondingly, we find that when vector solitons branch off from a pair of solitons with a different polarization $[0,\pm(\nu/c)^{1/2}/\cosh\xi]$ we obtain the bifurcation relations

$$\nu^{-1/2}=N_2(c)-N_2,\quad N_2=0,\,1,\,\ldots,\,[N_2(c)]. \tag{2.5}$$

Solving (2.4), (2.5) for a and c, we reach the conclusion that vector solitons branch off from two pairs of solitons of given polarizations, $(X_1,0)$ and $(0,X_2)$, on a countable set of bifurcation curves which arise upon the intersection of bifurcation surfaces in the parameter space (a,c,ν) (Fig. 2):

$$\begin{aligned} a&=2/[(\nu^{1/2}+N_1)(\nu^{1/2}+N_1+1)],\quad N_1=0,\,1,\ldots,\\ c&=2/[(\nu^{-1/2}+N_2)(\nu^{-1/2}+N_2+1)],\quad N_2=0,\,1,\ldots\,. \end{aligned} \tag{2.6}$$

According to the assertion above, the stable and unstable manifolds of saddle O are tangent along simple homoclinic loops on these bifurcation curves. These loops lie in the invariant planes (P_1,X_1) and (P_2,X_2) and are the inverse transforms of solitons with a given polarization. In the case $N_1=N_2=0$, the curve determined by (2.6) coincides with (1.15), on which the stable and unstable manifolds of saddle O coalesce, and there are three points of complete integrability, (1.17) and (1.18).

In the cases $N_1=1$, $N_2=0$ and $N_1=0$, $N_2=1$, there are points of complete integrability on the bifurcation curves determined by (2.6): $(a=\frac{8}{3},\ c=\frac{1}{6},\ \nu=\frac{1}{4})$ and $(a=\frac{1}{6},\ c=\frac{8}{3},\ \nu=4)$, respectively.[6] In particular, at the point of

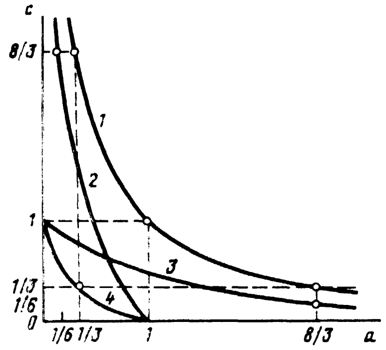

FIG. 2. Projections onto the (a,c) parameter plane of the bifurcation curves for simultaneous branching of complex loops from two pairs of simple loops in the invariant planes (P_1,X_1) and (P_2,X_2). Shown here are projections of the points of complete integrability of the dynamic system. 1—$N_1=0$, $N_2=0$; 2—$N_1=0$, $N_2=1$; 3—$N_1=1$, $N_2=0$; 4—$N_1=1$, $N_2=1$.

complete integrability $a=\frac{8}{3}, c=\frac{1}{6}, \nu=\frac{1}{4}$ the family of vector solitons is determined by

$$X_1=\frac{\operatorname{ch}(\xi-\xi_0)}{[\operatorname{ch}^4(\xi-\xi_0)+B^2]^{1/2}}, \quad X_2=\frac{B\operatorname{sh}2(\xi-\xi_0)}{2[\operatorname{ch}^4(\xi-\xi_0)+B^2]}, \tag{2.7}$$

where B parametrizes the family. Finally, in the case $N_1=N_2=1$, there is a point of complete integrability, $a=c=\frac{1}{3}, \nu=1$, on the corresponding bifurcation curve, (2.6). At this point the family of vector solitons is determined by

$$\begin{aligned} X_1&=(3^{1/2}/2)[1/\operatorname{ch}(\xi-\xi_1)+1/\operatorname{ch}(\xi-\xi_2)], \\ X_2&=(3^{1/2}/2)[1/\operatorname{ch}(\xi-\xi_1)-1/\operatorname{ch}(\xi-\xi_2)], \end{aligned} \tag{2.8}$$

at which the actual parameter of the family is $\xi_2-\xi_1$.

These comments lead to a second observation: In the known cases of complete integrability of a dynamic system with Hamiltonian (1.8), which satisfies conditions (1.10), the corresponding values of the structural parameters a, c, ν belong to the bifurcation curves (2.6), for which vector solitons branch off from two pairs of solitons with a given polarization.

These observations, along with the generalization of the assertion above to the case $\nu\neq 1$, indicate that it would be possible to select, in the space of structural parameters of Hamiltonian systems of a certain type, bifurcation sets of lower dimensionality to which points of complete integrability might belong.

For a dynamic system with Hamiltonian (1.8), these bifurcation sets are represented by curves in the parameter space (a,c,ν). Corresponding to each point of these curves in phase space Γ the stable and unstable manifolds of the saddle O are tangent along simple homoclinic loops which lie in invariant two-dimensional planes of the phase space.[7] In the course of motion along these bifurcation curves, the order of the tangency of the manifolds of saddle O may change. Numerical calculations indicate that the order of the tangency of the manifolds increases as the known points of complete integrability are approached; these manifolds coalesce (i.e., there is a tangency of infinite order) exactly at the point of complete integrability. Exceptional cases are the points of complete integrability which lie on the bifurcation curve (2.6) in the case $N_1=N_2=0$. In this case, the entire curve corresponds to a complete coalescence of the stable and unstable manifolds of the saddle (and, probably, to the case of integrability at an isolated level of the Hamiltonian H).

The search for cases of a complete integrability, or of integrability at an isolated level of H, of Hamiltonian systems can thus be linked with the identification of a certain set of (generally isolated) points on the bifurcation sets for which the stable and unstable manifolds of the saddle are tangent. As these points are approached in Γ, the order of the tangency of these manifolds increases without bound.

Clearly, this scenario for achieving complete integrability in the space of Hamiltonian systems is only one possible scenario, which is strongly linked with the existence of a nontrivial bifurcation set of the tangency of the stable and unstable manifolds of a saddle point. However, the related reduction in terms of the dimensionality of the structural parameters opens up the possibility of constructing algorithms for a numerical search for cases of complete integrability. Furthermore, the primary object of the search consists of homoclinic trajectories of a saddle, which can be directly related to stationary states of a system of nonlinear Schrödinger equations of the form (1.1). We also note that studies in the two-dimensional Hénon-Heiles Hamiltonian model show that a situation analogous to that described above may occur.

Let us consider a small neighborhood of a point of complete integrability as in (1.17). In this neighborhood, a single-parameter family of vector solitons is determined by

$$X_1=\frac{\cos\varphi}{\operatorname{ch}\xi}, \quad X_2=\frac{\sin\varphi}{\operatorname{ch}\xi}. \tag{2.9}$$

Setting

$$a=1+\delta a, \quad c=1+\delta c, \quad \nu=1+\delta\nu,$$
$$X_1=\frac{\cos\varphi}{\operatorname{ch}\xi}+u_1, \quad X_2=\frac{\sin\varphi}{\operatorname{ch}\xi}+u_2 \tag{2.10}$$

in Eqs. (1.6), we find that in the approximation linear in the perturbation the problem of the survival of vector solitons is related to the solvability of the inhomogeneous problem

$$w_{1,\xi\xi}-w_1+\frac{6}{\operatorname{ch}^2\xi}w_1=-2\delta a\frac{\cos^4\varphi}{\operatorname{ch}^3\xi}-2\delta c\frac{\sin^4\varphi}{\operatorname{ch}^3\xi}+\delta\nu\frac{\sin^2\varphi}{\operatorname{ch}\xi},$$

$$w_{2,\xi\xi}-w_2+\frac{2}{\operatorname{ch}^2\xi}w_2=2\delta a\frac{\cos^3\varphi\sin\varphi}{\operatorname{ch}^3\xi}+\cos\varphi\left[-2\delta c\frac{\sin^3\varphi}{\operatorname{ch}^3\xi}+\delta\nu\frac{\sin\varphi}{\operatorname{ch}\xi}\right], \tag{2.11}$$

under the conditions

$$\lim_{\xi\to\pm\infty} w_1=\lim_{\xi\to\pm\infty} w_2=0.$$

The functions w_1, w_2 here are related to the functions u_1, u_2 by a transformation consisting of a rotation through an angle of $\pi/4$. The first of Eqs. (2.11) can be solved for arbitrary perturbations w_1, w_2, since the solution of the homogeneous problem ($\pm 1/\cosh\xi$) is odd. The condition under which the second of Eqs. (2.11) can be solved—the condition that the solution of the corresponding homogeneous equation be orthogonal to the right side of the inhomogeneous equation—leads to the equation

$$\sin\varphi[{}^4/_3(\delta a\cos^2\varphi-\delta c\sin^2\varphi)+\delta\nu]=0.$$

In the parameter space (a,c,ν), this equation determines a plane which passes through the point of complete integrability, $a=c=\nu=1$, and which has the orientation determined by the parameter of the family of vector solitons, (2.9). Consequently, a vector soliton with parameter φ can "survive" only if we move away from the point of complete integrability in the space (a,c,ν) along directions which belong to a plane with a certain orientation in this space. Along the bifurcation curve determined by (2.6) with $N_1=N_2=0$, however, the entire family of vector solitons (2.9) survives. For this bifurcation curve, which passes through a point of complete integrability, we have

$$\delta a|_{\nu=1}=-{}^3/_4\delta\nu, \quad \delta c|_{\nu=1}={}^3/_4\delta\nu,$$

and the condition under which the problem (2.11) can be solved is satisfied for an arbitrary value of the parameter φ. This result indicates that on this curve on which the stable and unstable manifolds of saddle O are tangent these manifolds coalesce completely.

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