Theory of nonlinear stationary states of a ring parametric oscillator

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Three-dimensional nonlinear steady states of wave fields in a ring parametric stimulated-Brillouin-scattering oscillator are found. A classification is given of nonlinear oscillator modes and a study is reported of the stability of the fundamental mode. It is shown that the fundamental mode is stable near the excitation threshold. When the threshold is exceeded significantly, the fundamental mode becomes unstable in the presence of distortions of the spatial distributions of the intensities of the interacting light beams and of phase distortions of their wavefronts, but the instability thresholds are different for these two types of perturbation.

1. INTRODUCTION

Development of efficient methods for transformation of light beams is one of the important tasks in modern nonlinear optics. The energy relationships governing this transformation have been investigated in great detail. The transverse structure of the interacting beams has for a long time eluded a proper description, particularly under conditions of strong depletion of such beams.

We propose an approach to an analysis of the transverse structure of beams when a four-wave interaction in the specific case of a ring parametric stimulated-Brillouin-scattering (STBS) oscillator, which is one of the promising devices for phase self-conjugation of the wavefronts of light beams. In this case (Fig. 1) a signal beam $\mathscr{E}_{01}$ which is to be phase-conjugated passes through a nonlinear medium and is returned, by external optical elements forming a feedback loop, to the nonlinear medium in the form of a beam $\mathscr{E}_{0-1}$ and it intersects the beam $\mathscr{E}_{01}$. At some specific value of the signal beam intensity the STBS process of a nonlinear medium converts the wave $\mathscr{E}_{01}$ into a wave $\mathscr{E}_{-11}$ propagating opposite to the wave $\mathscr{E}_{0-1}$. After passing along the feedback loop, the beam $\mathscr{E}_{-11}$ forms a scattered-light beam $\mathscr{E}_{-1-1}$, which acquires its energy due to STBS from the beam $\mathscr{E}_{0-1}$ and in turn amplifies the beam $\mathscr{E}_{-11}$. Therefore, scattering in such a system represents an absolute instability (oscillation).

A ring parametric oscillator had been realized experimentally using STBS-active liquids[1,2] and photorefractive crystals.[3] A theoretical analysis of this oscillator had been made on the basis of a one-dimensional model,[4,5] but this ignores completely the transverse structure of light beams and, consequently, cannot be used to find the spatial distribution of the scattered radiation or criteria for selection of the phase-conjugated component. A method for calculating the three-dimensional structure of fields under nonlinear conditions in a ring parametric oscillator was proposed in Ref. 6; the excitation thresholds were determined and the structure of normal longitudinal–transverse modes of a ring oscillator was found. In the case of the fundamental mode its nonlinear characteristics were determined also above the excitation threshold.

The question whether nonlinear stationary states predicted for various four-wave interaction systems can be realized may be answered by an analysis of their stability. Analytic investigations of this stability[7–9] were published recently and were limited to the one-dimensional approach. For example, the stability of nonlinear states of a ring parametric STBS oscillator was discussed in Ref. 9.

We shall use the approach of Refs. 6 and 7 to investigate the stability of the fundamental mode in a ring parametric STBS oscillator. We shall show that when the pumping rate exceeds greatly the excitation threshold, the fundamental mode becomes unstable due to amplitude distortions of spatial distributions of the intensities of the interacting beams and due to phase distortions of their wavefronts. We shall compare the results of our three-dimensional analysis with those obtained by the one-dimensional approach. We shall demonstrate that the range of stability of stationary states becomes narrower outside the one-dimensional approximation. In Sec. 2 we shall formulate mathematically the problem and derive certain general relationships. In Sec. 3 we shall find a nonlinear three-dimensional stationary state of the fields corresponding to the fundamental scattering mode and we shall discuss higher modes. In Sec. 4 we shall analyze the ground state in the presence of amplitude perturbations and in Sec. 5 we shall do the same in the case of phase perturbations. In the final section we shall consider the results obtained and compare them with those of the one-dimensional analysis given in Ref. 9.

2. PRINCIPAL EQUATIONS

We shall give a theoretical description of the system in Fig. 1 for fairly wide slightly aberrated electromagnetic radiation beams. The interaction of such beams in a nonlinear medium due to the excitation of a reflecting grating of the refractive index and their propagation along a feedback loop will be described in the geometric-optics approximation. We shall assume that the thickness of a nonlinear medium and the angle of convergence of the beams are relatively small. This makes it possible to ignore the relative displacement of the beams in the thickness of the nonlinear medium and to represent the electric field of the pump $\mathscr{E}_{0\pm1}$ and scattered $\mathscr{E}_{-1\pm1}$ radiation beams in the form

$$\mathscr{E}_{mn}=E_{mn}(z,\ \rho;\ t)\exp[ik_m\rho^2/2R_{mn}(z)+ik_mz-i\omega_mt]. \quad (1)$$

Here, z is the coordinate in the direction of propagation of the beam; ρ are the transverse coordinates; t is time; k_m and ω_m are, respectively, the wave number and frequency of the pump ($m=0$) or scattered ($m=-1$) beams; $R_{mn}(z)$ is

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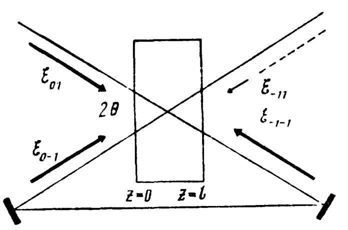

FIG. 1. Schematic representation of the interaction of four waves in a ring parametric oscillator.

the radius of curvature of the beam wavefront; E_{mn} is the complex amplitude of the beam varying slowly compared with an exponential function. When beams propagate along their feedback loop, their radii of curvature vary in accordance with the formulas of the geometric-optics approximation and the slow amplitude E_{mn} undergoes a scaling transformation along a transverse coordinate ρ. It is shown in Ref. 10 that the pump radiation contracts in the external optical channel, i.e., when the diameter of the beam E_{0-1} in the nonlinear medium is less than the diameter of the beam E_{01}, the strong phase dependences of the scattered radiation in Eq. (1) become reversed relative to the corresponding pump beams so that $R_{-1-1} = -R_{01}$ and $R_{-11} = -R_{0-1}$. The weak phase and amplitude dependences are governed by the complex amplitudes E_{mn}. These amplitudes obey a system of dynamic equations describing the interaction of four electromagnetic waves E_{mn} in a nonlinear medium of thickness l ($0 \leqslant z \leqslant l$):

$$\frac{\partial}{\partial z}E_{0\pm1} = -\nu E_{-1\pm1}, \quad \frac{\partial}{\partial z}E^*_{-1\pm1} = -\nu E^*_{0\pm1}, \tag{2a}$$

$$\left(\tau\frac{\partial}{\partial t}+1\right)\nu = g(E_{01}E^*_{-11}+E_{0-1}E^*_{-1-1}). \tag{2b}$$

Here, ν is the amplitude of an acoustic wave excited in a medium and corresponding to STBS through an angle $\pi - 2\theta$, where $2\theta \ll 1$ is the angle between the pump beams E_{01} and E_{0-1} in the medium, g is the nonlinear coupling coefficient, and τ is the relaxation time.

The electromagnetic fields $E_{0\pm1}$ and $E_{-1\pm1}$ and the amplitude ν of an acoustic grating are functions of the longitudinal coordinate z in the direction of wave propagation (Fig. 1), of the transverse coordinates ρ, and of time t. The system (1) is identical with the equations used in the one-dimensional theory[5] and it depends explicitly only on the longitudinal coordinate z and on time t. The distinction from the one-dimensional theory is that the system (1) contains a transverse coordinate ρ as a parameter. The details of the derivation of the system of equations (2) and of the conditions of its validity can be found in Ref. 6.

The system (2) will be supplemented by two boundary conditions[6,10] corresponding to a given transverse distribution of the pump beam E_{01} at the entry to the nonlinear medium in the absence of the entry beam of the scattered radiation E_{-11}:

$$E_{01}(z=0,\ \rho;\ t) = E_0(\rho), \quad E_{-11}(z=l,\ \rho;\ t) = 0. \tag{3a}$$

Two other boundary conditions correspond to mapping of a transverse section of the pump beam E_{01}, at the exit from the nonlinear medium where $z = l$, onto the entry face of the medium $z = 0$ characterized by a scaling transformation coefficient γ and a corresponding mapping of a transverse section of the exit scattered-radiation beam E_{-11} at the boundary $z = 0$ onto the boundary $z = l$:

$$\begin{aligned} E_{0-1}(0,\ \rho;\ t) &= (r/\gamma)E_{01}(l,\ \rho/\gamma;\ t), \\ E_{-1-1}(l,\ \rho;\ t) &= r\gamma E_{-11}(0,\ \gamma\rho;\ t). \end{aligned} \tag{3b}$$

Here, r is the amplitude transmission coefficient of the feedback loop and γ is the scaling coefficient of the transformation. We shall consider the case when $\gamma < 1$ corresponding to compression of a transverse section of the pump beam as it passes along the feedback loop, because this condition ensures selection of modes in the system under consideration.[10]

The system of equations (2) has integrals of motion $C_j(\rho;t)$, where $j = 1$–4 (see, for example, Refs. 6 and 10):

$$\begin{aligned} C_1 &= |E_0(\rho=0)|^{-2}(|E_{01}|^2 - |E_{-11}|^2), \\ C_2 &= |E_0(0)|^{-2}(|E_{0-1}|^2 - |E_{-1-1}|^2), \\ C_3 &= |E_0(0)|^{-2}(E_{01}E_{-1-1} - E_{0-1}E_{-11}), \\ C_4 &= |E_0(0)|^{-2}(E_{01}E^*_{0-1} - E_{-11}E^*_{-1-1}). \end{aligned} \tag{4}$$

The use of the boundary conditions (3a) and (3b) in the expressions for the integrals of motion (4) allows us to derive the following systems of equations for these integrals:[6]

$$\begin{aligned} C_3(\rho;\ t) &= F(\rho;\ t)f(\rho) - \gamma^{-2}C_3(\rho/\gamma;\ t), \\ |C_3(\rho;\ t)|^2 &= (r\gamma)^2C_1(\rho;\ t)\,[f(\gamma\rho) - C_1(\gamma\rho;\ t)], \\ C_2(\rho;\ t) &= (r/\gamma)^2C_1(\rho/\gamma;\ t) - f(\rho)\,|F(\rho;\ t)|^2. \end{aligned} \tag{5}$$

Here, $f(\rho) = |E_0(\rho)/E_0(0)|^2$ is a function describing the transverse distribution of the intensity of the pump beam entering the system, whereas $F(\rho;t) = E^*_{-11}(z=0,\rho;t)/E_0(\rho)$ is a form factor containing detailed information on the structure of the scattered radiation field and enabling us to determine all the output characteristics of the oscillator such as, for example, the total nonlinear reflection coefficient and the quality of phase conjugation of the pump beam E_{01}. The form factor $F(\rho;\ t)$ cannot be derived however simply with the aid of the integrals of motion (4) and the boundary conditions (3); we have to solve the dynamic system (2).

3. STATIONARY STATES

Stationary states correspond to a monochromatic scattered wave and to the substitution $\partial/\partial t \to i\delta\omega$ in Eq. (2b), where $\delta\omega$ is a possible frequency offset of the scattered radiation from the center of the gain profile. Substituting the expression for the amplitude of an acoustic grating from Eq. (2b) into Eq. (2a), we obtain a system of four nonlinear ordinary differential equations, the solution of which is known.[4] This general solution, together with the integrals of motion (4), allows us to find all the electromagnetic fields inside the nonlinear medium and, in particular, to write down the following expression for the form factor $F(\rho;\ t)$ of the scattered radiation:[6]

$$F(\rho;\,t)=G[-C_1+C_2+d+(C_1-C_2+d)P]/2C_4^{*}(|G|^2-P),$$
$$G(\rho;\,t)=-(2|C_3|^2+C_1C_2+C_1^2-C_1d)/2C_3C_4,$$
$$d(\rho)=[(C_1+C_2)^2+4|C_3|^2]^{1/2}, \qquad (6)$$
$$P(\rho)=\exp[-\varkappa d/(1-i\delta\omega\tau)].$$

Here, $\varkappa=g|E_0(\rho=0)|^2l$ is the coefficient representing convective amplification of the scattered radiation during passage through the center of the pump beam. In stationary solutions the integrals C_1, C_2, and C_4 are independent of time, whereas C_3 and F depend harmonically on time: $C_3\propto F\propto\exp(-i\delta\omega\tau)$.

The system of equations (5) for the integrals of motion C_1 to C_3 and the expression (6) for the form factor determine all the characteristics of the investigated oscillator under steady-state conditions. The solution of the system (5)–(6) is a denumerable set of longitudinal–transverse nonlinear modes, but the most interesting (from the practical point of view) is the family of modes corresponding to an oscillation at the center of the gain band where $\delta\omega=0$, because such modes have the lowest excitation thresholds. They differ in respect of the transverse structure of the scattered radiation beam. They can be called transverse and can be labeled by the index $n=0, 1, 2,\ldots$. In the case of radially symmetric beams the structure of the scattered field of a mode with an index n at its excitation threshold corresponds to the asymptotic dependence $F(\rho)\propto|\rho|^n$ in the limit when $|\rho|\to 0$. The excitation thresholds of such modes are given by[6]

$$\varkappa_{th,n}=(1+r^2/\gamma^2)^{-1}\ln[1+\gamma^{-n}(r^{-2}+\gamma^{-2})]. \qquad (7)$$

The lowest excitation threshold belongs to the fundamental scattering mode $n=0$ described in Ref. 6. Typical radial distributions of the field of the scattered radiation of this mode are shown in Fig. 2, whereas the dependences of the nonlinear reflection coefficient R and of the coefficient H representing the phase-conjugation quality on the convective gain are illustrated in Fig. 3.

The fundamental scattering mode corresponds to complete phase conjugation of the pump beam when the amplitude structure is reconstructed subject to some distortions. Complete reconstruction of the phase structure of the pump radiation means that, in the case of the fundamental mode under steady-state conditions, we can regard all the fields $E_{0\pm1}$ and $E_{-1\pm1}$ as purely real without any limitations on the validity of this conclusion.

In an analysis of the stability of the dynamic system (2) we shall linearize the system near its ground state. We then obtain a system of differential equations for complex perturbations of electromagnetic fields $\delta E_{0\pm1}$ and $\delta E_{-1\pm1}$, and of perturbation of the acoustic field δv. The coefficients of this system are purely real. Therefore, a linearized system of equations can be split into two independent subsystems. One of them describes the stability of a nonlinear stationary solution against excitation of amplitude distortions of the spatial distributions of the intensity of the interacting beams, and the other describes stability against excitation of purely phase distortions of their wavefronts. We shall give the results of an investigation of the stability of stationary states in the presence of these two classes of perturbations.

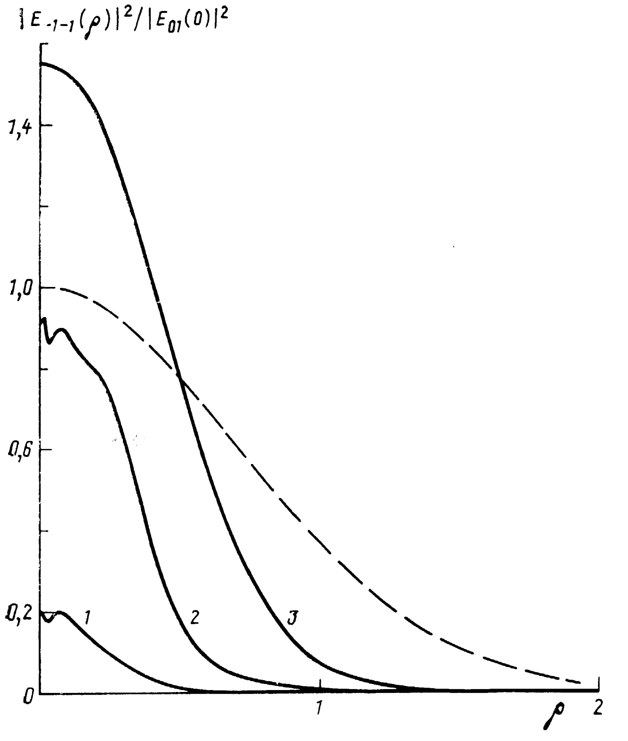

FIG. 2. Radial distributions of the intensity of a phase-conjugated wave (fundamental stationary scattering mode) calculated for $\gamma=0.5$, $\varkappa=0.38$ (curve 1), 0.5 (curve 2), and 1.0 (curve 3); $|r|^2=1$. The dashed curve represents the signal beam.

4. AMPLITUDE PERTURBATIONS

In discussing amplitude perturbations we can assume that all the functions $E_{0\pm1}$ and $E_{-1\pm1}$ in the system of equations (2) can be regarded as purely real even when the conditions are not steady-state. We can then introduce a function

$$\Psi(z,\rho;t)=\ln\left[\left(\frac{(C_4+C_3)^2+C_1^2}{(C_4-C_3)^2+C_1^2}\right)^{1/2}\frac{E_{01}+E_{-11}}{E_{01}-E_{-11}}\right], \qquad (8)$$

and reduce the system (2) to one nonlinear partial differential equation:[7,8]

$$\left(\tau\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial z}\Psi+\frac{\varkappa}{l}d\,\mathrm{sh}\,\Psi=0. \qquad (9)$$

The stationary solution of the system (9) is identical with Eq. (6).

In an analysis of the stability we shall represent the function Ψ as follows:

$$\Psi(z,\ \rho;\ t)=\Psi(z,\ \rho)+\mathrm{Re}[\delta\Psi(z,\ \rho)\exp(-ift/\tau)], \qquad (10)$$

where f is the complex dimensionless frequency $f=f'+if''$ and $\delta\Psi$ is a small correction. We can similarly represent deviations from stationary values of all the electromagnetic fields and integrals of motion. An instability of a stationary state corresponds to $f''>0$, whereas at the instability threshold we have $f''=0$.

Linearization of Eq. (9) relative to the steady state and solution of the resultant equation for perturbations $\delta\Psi$ yields, as in the one-dimensional theory,[7] the following equation:

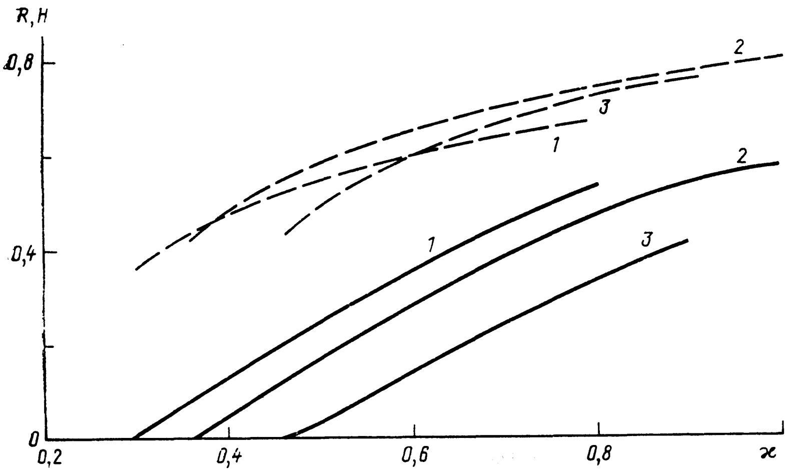

FIG. 3. Dependences of the total nonlinear backreflection coefficient R (continuous curves) and of the phase-conjugation quality coefficient H (dashed curves) on the convective gain $\varkappa$, calculated for $\gamma=0.3$ (curve 1), 0.5 (curve 2), and 0.7 (curve 3); $|r|^2=1$.

$$\delta\Psi(l,\rho)=\delta\Psi(0,\rho)\left[\frac{\operatorname{sh}\Psi(l,\rho)}{\operatorname{sh}\Psi(0,\rho)}\right]^{1/(1-if)}+\frac{1}{1-if}\frac{\delta d}{d}\int\limits_{\Psi(0,\rho)}^{\Psi(l,\rho)}dy\left[\frac{\operatorname{sh}\Psi(l,\rho)}{\operatorname{sh}y}\right]^{1/(1-if)}. \tag{11}$$

We can obtain a closed dispersion equation for the determination of the complex frequency f by finding the relationship between the perturbations $\delta\Psi$ (0) and $\delta\Psi$ (l) with the aid of variation of the integrals of motion C_1–C_3 and using the expression (8) for Ψ; these expressions then have to be substituted in Eq. (11). It follows from Eq. (8) subject to the boundary conditions $E_{-11}(l,\rho;t)=0$ that

$$\delta\Psi(\rho,l)=2(C_1d^2)^{-1}[\delta C_3C_4(C_1+C_2)+\delta C_4C_3(C_1-C_2)-2C_3C_4\delta C_1]. \tag{12}$$

The relationship (12) is local, i.e., it relates perturbations of the function Ψ and of the integrals C_1–C_4 at the same point. Similarly, variation of the expression for the function Ψ at the point $z=0$, subject to the definition of the integral C_1 given by Eq. (4), yields

$$\delta\Psi(0,\rho)=\delta\Psi(l,\rho)-\delta C_1[C_1(1-C_1)^{1/2}]^{-1}. \tag{13}$$

Variation of the function d which occurs on the right-hand side of Eq. (11) gives

$$\delta d=d^{-1}[(C_1-C_2)(\delta C_1-\delta C_2)+4C_4\delta C_4]. \tag{14}$$

Finally, variation of the system (5) gives

$$\begin{aligned}&\delta C_3(\rho)=\delta F(\rho)f(\rho)-\gamma^{-2}\delta C_3(\rho/\gamma),\\&2C_3\delta C_3(\rho)=(r\gamma)^2\delta C_1(\rho)[f(\gamma\rho)-C_1(\gamma\rho)]\\&\qquad-(r\gamma)^2\delta C_1(\gamma\rho)C_1(\rho),\\&\delta C_2(\rho)=(r/\gamma)^2\delta C_1(\rho/\gamma)-2f(\rho)F(\rho)\delta F(\rho).\end{aligned} \tag{15}$$

Variation of the integral C_4 may be excluded from the system of equations (12), (15), since it follows from the general relationship $C_4^2-C_3^2=C_1C_2$ that

$$2C_4\delta C_4=2C_3\delta C_3+C_1\delta C_2+C_2\delta C_1. \tag{16}$$

Therefore, we obtain for the quantities δC_1, δC_2, δC_3, δF. a system of four linear homogeneous equations (11) and (15), which are nonlocal in respect of the coordinate ρ. The condition of solvability of this system of equations is the dispersion equation for the determination of the complex frequency f.

The structure of the system (11), (15) shows that its solutions represent a denumerable set of normal nonstationary excitation modes, which differ in respect of the transverse structures of the fields. We shall number these modes by two indices (n, m), the first of which governs the nature of behavior of the fields in the vicinity of a fixed point $\rho=0$: $\delta C_j^{(n,m)}(\rho)$, $\delta E_{ij}^{(n,m)}(\rho)\propto|\rho|^n$, whereas the second is the number of a stationary mode from which the nonstationary mode in question arises. The structure of the system (11), (15) leads to the conclusion that the excitation threshold of a nonstationary mode n can be found from an analysis of this system in the vicinity of a fixed point $\rho=0$, where the system of equations (15) yields

$$\begin{aligned}&\delta C_3^{(n,m)}(\rho)[1+\gamma^{-2-n}]=\delta F^{(n,m)}(\rho),\\&2C_3^{(m)}(0)\delta C_3^{(n,m)}(\rho)=(r\gamma)^2\delta C_1^{(n,m)}(\rho)[1-C_1^{(m)}(0)(1+\gamma^n)],\\&\delta C_2^{(n,m)}(\rho)=r^2\gamma^{-2-n}\delta C_1^{(n,m)}(\rho)-2F^{(m)}(0)\delta F^{(n,m)}(\rho).\end{aligned} \tag{17}$$

We shall confine ourselves to an investigation of the stability of the fundamental stationary mode $m=0$. It will be clear from our later analysis that an instability of this mode appears when its excitation threshold is exceeded greatly, i.e., when $\exp(\varkappa d)\gg 1$. We shall use this condition to determine approximately the excitation threshold of nonstationary modes.

The solution of equations for the stationary state [Eqs. (5) and (6)] at a fixed point $\rho=0$ gives the following asymptotic expressions for the integrals of motion of the fundamental mode $m=0$:

$$\begin{aligned}&C_1^{(0)}(0)=\frac{\gamma^2}{1+\gamma^2}(1+\varepsilon),\\&C_2^{(0)}(0)=-\frac{r^2\gamma^2}{1+\gamma^2}[1-\gamma^2\varepsilon-(1+\gamma^2)\varepsilon^2],\\&C_3^{(0)2}(0)=\frac{r^2\gamma^2}{(1+\gamma^2)^2}[1+\varepsilon(1-\gamma^2)-\gamma^2\varepsilon^2],\\&C_4^{(0)2}(0)=\frac{r^2\gamma^4}{1+\gamma^2}\varepsilon^2(1+\varepsilon),\end{aligned} \tag{18}$$

where

$$\varepsilon=\frac{1+r^2}{r^2(1+\gamma^2)}\exp(-\varkappa d)\ll 1,\quad d=\frac{\gamma^2(1+r^2)}{(1+\gamma^2)}.$$

Substitution of the expressions (18) into Eq. (8) for Ψ makes it possible to determine the value of this function and the boundaries of the nonlinear medium in the ground state:

$$\Psi^{(0)}(0,l)=2(1+\gamma^2)^{1/2}r^2\varepsilon/(1+r^2),$$
$$\Psi^{(0)}(0,0)=\ln\{[(1+\gamma^2)^{1/2}+1][(1+\gamma^2)^{1/2}-1]^{-1}\}. \qquad (19)$$

The use of the relationship (18) together with (12)–(14) gives the following equations describing the behavior of $\delta\Psi$ and of perturbations of the integrals of motion in the vicinity of a fixed point in the case of a nonstationary nth mode:

$$\delta\Psi^{(n,0)}(\rho,l)\approx\frac{r^2}{1+r^2}\frac{(1+\gamma^2)^{1/2}}{\gamma^2}(1+\gamma^2+\gamma^{n+2}+\gamma^{-n})\delta C_1^{(n,0)}(\rho),$$

$$\delta\Psi^{(n,0)}(\rho,0)\approx-\frac{(1+\gamma^2)^{1/2}}{(1+r^2)\gamma^2}[1+\gamma^2-r^2(\gamma^{n+2}+\gamma^{-n})]\delta C_1^{(n,0)}(\rho),$$

$$\delta d^{(n,0)}(\rho)\approx(1-r^2\gamma^{n+2})\delta C_1^{(n,0)}(\rho). \qquad (20)$$

Substitution of Eqs. (19) and (20) into Eq. (11) in the limit $|\rho|\to 0$ shows that the integral term in this equation is small. Neglecting this term, we obtain the following dispersion equation describing excitation of a nonstationary mode (n, 0):

$$\left[\frac{1+\gamma^2}{\gamma^2}e^{\varkappa d}\right]^{1/(1-if)}=-\frac{1+\gamma^2-r^2(\gamma^{n+2}+\gamma^{-n})}{r^2(1+\gamma^2+\gamma^{n+2}+\gamma^{-n})}\equiv-e^{U_n}. \qquad (21)$$

At the excitation threshold of a nonstationary mode (corresponding to bifurcation of the ground stationary state) we have $f''_{\rm bif}=0$ and it follows from Eq. (21) that

$$\varkappa_{bif}^{(n,0)}=\frac{1+\gamma^2}{\gamma^2(1+r^2)}\left[\frac{U_n^2+\pi^2(2N+1)^2}{U_n}-\ln\frac{1+\gamma^2}{\gamma^2}\right],$$
$$f'_{bif,n}=\pi(2N+1)/U_n, \qquad (22)$$

where $N=0,\ \pm 1,\ \pm 2,\ldots$ is an arbitrary integer. The minimum excitation threshold of the amplitude mode (n, 0) corresponds to $N=0,\ -1$.

It follows from Eq. (21) that an instability of the ground state against excitation of the nth mode may appear when

$$r^2<r^2_{max,n}=(1+\gamma^2)[1+\gamma^2+2(\gamma^{n+2}+\gamma^{-n})]^{-1}.$$

The range of values of r where the instability is possible becomes narrower on increase in the mode number n and is widest for the mode $n=0$: $0<r^2<1/3$. If $r^2>1/3$ the ground state is stable against excitation of amplitude perturbations with any number n. The minimum excitation threshold for the first few modes is approximately the same and corresponds to the relationship $U_n\propto\pi$. A rough estimate carried out using Eq. (22) subject to $\gamma\sim 1$ gives $\varkappa^{(n,0)}_{\rm bif,min}\sim 2\pi$. This threshold is reached at low values of r^2:$r^2\ll 1$, which decrease on increase in the mode number n. We must also bear in mind that an increase in the mode number n results in breakdown of the condition $\varepsilon\ll 1$ of Eq. (18), so that Eq. (22) describes well the bifurcation thresholds for several low numbers n. Typical dependences of the bifurcation threshold on the beam compression coefficient γ and on the transmission coefficient r^2 of the channel are plotted in Figs. 4 and 5.

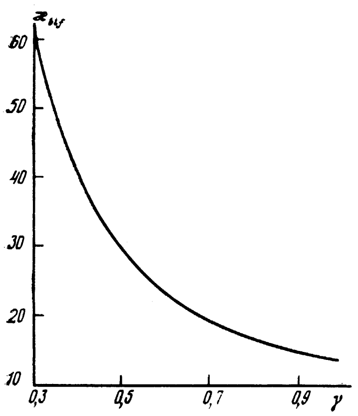

FIG. 4. Dependence of the stability loss threshold (bifurcation threshold) of the fundamental scattering mode on the beam compression coefficient in a feedback loop, calculated for $|r|^2=0.1$; amplitude perturbations, (0, 0) mode.

5. PHASE PERTURBATIONS

In discussing the instability of a stationary state against phase perturbations of the wavefronts of the interacting waves, we shall represent the deviations of the complex amplitudes of electromagnetic fields from their stationary values as follows:

$$E_{m,n}(z,\rho;t)=E_{m,n}(z,\rho)+i\,{\rm Im}\,[\delta E_{mn}(z,\rho)e^{-ift/\tau}]. \qquad (23)$$

We shall similarly describe deviations of the amplitude of the acoustic wave and of the integrals of motion C_3 and C_4 from their stationary values. The integrals C_1 and C_2 remain unperturbed, because they are always purely real.

The system (2) leads to equations similar to those used in Ref. 9:

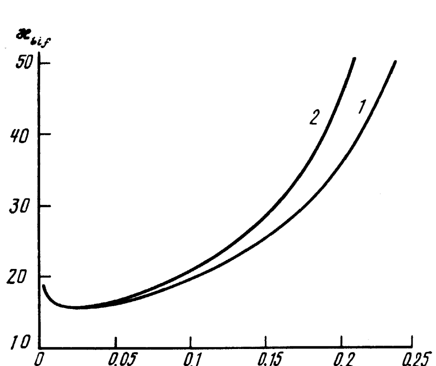

FIG. 5. Dependences of the bifurcation thresholds of the fundamental scattering mode on the transmission coefficient of a feedback loop in the case when $\gamma=0.7$; amplitude perturbations: 1) (0, 0) mode; 2) (1, 0) mode.

$$\frac{\partial}{\partial z}(E_{01}E^{*}_{-11}) = -\nu(I_{01}+I_{-11}),$$
$$\frac{\partial}{\partial z}(E_{0-1}E^{*}_{-1-1}) = -\nu(I_{0-1}+I_{-1-1}), \tag{24}$$

where

$$I_{ij}(z,\rho;t) = |E_{ij}(z,\rho;t)|^2,$$

and to the following equation for the amplitude of the acoustic wave

$$\left(\tau\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial z}\nu = -g\nu I_\Sigma, \tag{25}$$

where

$$I_\Sigma(z,\rho;t) = I_{01}+I_{0-1}+I_{-11}+I_{-1-1}$$

is the sum of the intensities of all four waves interacting in the nonlinear medium.

Variation of Eq. (25) subject to the condition that $\delta I_{ij} = 0$, gives

$$\delta\nu(z,\rho) = \delta\nu(l,\rho)\exp\left[\frac{g}{1-if}\int_z^l dz' I_\Sigma(z',\rho)\right]. \tag{26}$$

The function I_Σ on the right-hand side of Eq. (26) is governed by the distributions of the intensities of the interacting waves in the stationary state. Variations of the equations in the system (24), subject to Eq. (26), obey the following relationships:

$$\delta(E_{01}E^{*}_{-11})_{z=0} = J_1\delta(E_{0-1}E^{*}_{-1-1})_{z=l},$$
$$\delta(E_{0-1}E^{*}_{-1-1})_{z=l} - \delta(E_{0-1}E^{*}_{-1-1})_{z=0} = -J_2\delta(E_{0-1}E^{*}_{-1-1})_{z=l}, \tag{27}$$

where

$$J_1(\rho) = \frac{g}{1-if}\int_0^l dz(I_{01}+I_{-11})\exp\left[\frac{g}{1-if}\int_z^l dz' I_\Sigma\right],$$
$$J_2(\rho) = \frac{g}{1-if}\int_0^l dz(I_{0-1}+I_{-1-1})\exp\left[\frac{g}{1-if}\int_z^l dz' I_\Sigma\right]. \tag{28}$$

The boundary condition $E_{-11}(l,\rho;t) = 0$ is allowed for in the derivation of Eq. (27).

The additional equations for the determination of perturbations of electromagnetic fields at the boundaries of a nonlinear medium can be obtained by varying the expression in the system (4) for the integrals of motion C_3 and C_4;

$$\delta E_{-1-1}(0,\rho)E_{01}(0,\rho) - \delta E_{-11}(0,\rho)E_{0-1}(0,\rho)$$
$$-\delta E_{0-1}(0,\rho)E_{-11}(0,\rho)$$
$$= \delta E_{01}(l,\rho)E_{-1-1}(l,\rho) + \delta E_{-1-1}(l,\rho)E_{01}(l,\rho),$$

$$-\delta E_{0-1}(0,\rho)E_{01}(0,\rho) - \delta E_{-11}(0,\rho)E_{-1-1}(0,\rho)$$
$$+\delta E_{-1-1}(0,\rho)E_{-11}(0,\rho)$$
$$= \delta E_{01}(l,\rho)E_{0-1}(l,\rho) - \delta E_{0-1}(l,\rho)E_{01}(l,\rho), \tag{29}$$

and by varying the boundary conditions of Eq. (3b):

$$\delta E_{0-1}(0,\rho) = (r/\gamma)\delta E_{01}(l,\rho/\gamma),$$
$$\delta E_{-1-1}(l,\rho) = (r\gamma)\delta E_{-11}(0,\gamma\rho). \tag{30}$$

The system of equations (27), (29), and (30) has—as in the case of amplitude perturbations discussed in the preceding section—a denumerable set of eigensolutions (nonstationary excitation modes) which are again numbered by the indices (n, m). The index n describes the behavior of the fields of a nonstationary mode in the vicinity of a fixed point $\rho = 0$: $\delta E_{ij}(\rho) \propto |\rho|^n$; the index m identifies the number of the stationary mode from which a given nonstationary mode arises. As before, we shall consider only the stability of the fundamental stationary mode $m = 0$. The excitation threshold of a mode $(n, 0)$ can be found simply by analyzing the system of equations (27), (29), and (30) in the vicinity of the fixed point $\rho = 0$. The dispersion equation for the $(n, 0)$ mode obtained as a result of such analysis is

$$\left(J_1^{(0)}C_1^{(0)}r^2\frac{1+\gamma^2}{\gamma^2} - J_2^{(0)}\right)$$
$$\times\left[2\left(C_1^{(0)}\frac{1+\gamma^2}{\gamma^2}-1\right) + C_1^{(0)}\gamma^{-2}(\gamma^{-n}-1)\right]$$
$$-J_1^{(0)}C_1^{(0)}r^2\left(C_1^{(0)}\frac{1+\gamma^2}{\gamma^2}-1\right)(\gamma^{-n}-1)(1+2\gamma^n)$$
$$-2\left(C_1^{(0)}\frac{1+\gamma^2}{\gamma^2}-1\right) = 0. \tag{31}$$

In Eq. (31) the functions $J_{1,2}^{(0)}$ and the integral $C_1^{(0)}$ apply at the beam center ($\rho = 0$). The expressions for the intensities of the interacting waves in a stationary state with $\rho = 0$, which are represented by the integrands on $J_{1,2}^{(0)}$, follow from the solution of the system (1) in the stationary state and are described by the expressions

$$I_{01}(z,0) = I_0(1-Y)^{-1}[Y-M(z)]^2[M^2(z)-Y]^{-1},$$
$$I_{-11}(z,0) = I_0Y(1-Y)^{-1}[1-M(z)]^2[M^2(z)-Y]^{-1},$$

$$I_{0-1}(z,0) = I_0Y(1-Y)^{-1}(4C_3^2)^{-1}[(C_1+C_2+d) - (C_1+C_2+d)M(z)]\times[M^2(z)-Y]^{-1},$$

$$I_{-1-1}(z,0) = I_0Y(1-Y)^{-1}(4C_4^2)^{-1}[(-C_1+C_2+d) + (C_1-C_2+d)M(z)]\times[M^2(z)-Y]^{-1}, \tag{32}$$

where

$$Y = [2C_3^2+C_1(C_1+C_2-d)][2C_3^2+C_1(C_1+C_2+d)]^{-1},$$
$$M(z) = \exp[\varkappa d(z/l-1)],$$

all the integrals C_1–C_4 are taken at $\rho = 0$, and, finally, we have $I_0 = |E_0(\rho = 0)|^2$.

We shall first analyze the dispersion relationship (31) at the excitation threshold of the ground state $m = 0$. We then have $C_3^{(0)} \to 0$, $C_1^{(0)} \to 1$, $C_2^{(0)} \to (r/\gamma)^2$, and it follows from Eq. (31) that

$$\exp\left(\frac{\varkappa_{th,n}d}{1-if}\right)=(r^{-2}+\gamma^{-2}+\gamma^{n})\gamma^{-n}.$$

Since $\varkappa_{\text{th},n}d=\ln(1+r^{-2}+\gamma^{n})$ [see Eqs. (7) and (18)], the above expression can be written in the form

$$\exp\left(\frac{\varkappa_{th,n}d}{1-if}\right)=\frac{\exp(\varkappa_{th,n}d)-(1-\gamma^{n})}{1-(1-\gamma^{n})}. \tag{33}$$

The solution of Eq. (33) gives $f=0$ for the $n=0$ mode and $f=if''$, where $f''<0$, for the modes with nonzero values of n. In other words, at its excitation threshold the ground stationary state ($m=0$) is the state of neutral equilibrium relative to the mode $n=0$ and stable equilibrium relative to the modes $n>0$. The physical reason for the existence of a neutral equilibrium state is the circumstance that, within the framework of the system of equations (1) and (2), the phase of the scattered radiation is indeterminate and can have any value. The conclusion that the ground state ($n=0$) is the state of neutral equilibrium applies at any value of the pump intensity exceeding the threshold for the appearance of a stationary state.

The situation for the $(n,0)$ modes when $n>0$ is different. When the pump intensity is sufficiently high, the ground stationary state is unstable and its modes are excited. We can demonstrate this by considering the dispersion equation (31) with $n\neq0$ on the assumption that $\varkappa\gg1$. Using the asymptotic values (18) of the integrals $C_1^{(0)}$–$C_4^{(0)}$ in this limit and calculating the functions J_1 and J_2 of Eq. (28), we can reduce the dispersion equation (31) to

$$\gamma^{-2/(1-if)}-2(1+\gamma^{2})(1-\gamma^{n+2})^{-1}J$$

$$-(1+\gamma^{2})^{-1/(1-if)}\exp\left[V_n-\frac{\varkappa d}{1-if}\right]=0, \tag{34}$$

where

$$J=\frac{1}{1-if}\int_0^\infty dz\left(\frac{e^{-z}}{1+\gamma^{2}-e^{-2z}}\right)^{(2-if)/(1-if)},$$

$$V_n=\ln\frac{1+\gamma^{2}+r^{2}(\gamma^{-n}+\gamma^{n+2})}{r^{2}(1-\gamma^{n+2})(\gamma^{-n}-1)}.$$

At the instability threshold of the $(n,0)$ mode (bifurcation of the ground state) the solutions of Eq. (34) correspond to high values of the frequency $f'_{\text{bif}}\gg1$. Bearing this point in mind, we find that

$$\varkappa_{bif}^{(n,0)}=\frac{1+\gamma^{2}}{\gamma^{2}(1+r^{2})}[V_n+(2\pi N)^{2}/V_n],\quad f_{bif}'=\frac{2\pi N}{V_n}$$

where $N=\pm1,\pm2,...$ is a nonzero integer; the minimum bifurcation threshold corresponds to $N=\pm1$. Since $V_n>1$, instability of the ground state due to excitation of phase perturbations is possible for any value of r and γ, but the bifurcation threshold is fairly high: $\varkappa_{\text{bif}}\gtrsim4\pi$. Characteristic dependences of the bifurcation threshold of the ground state in the case of excitation of the mode (1, 0) are plotted in Figs. 6 and 7.

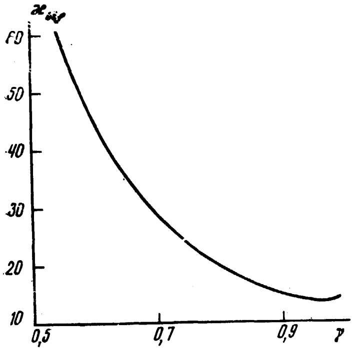

FIG. 6. Dependence of the bifurcation threshold of the fundamental scattering mode on the coefficient of beam compression in a feedback loop, calculated for $|r|^2=1$; phase perturbations, (1, 0) mode.

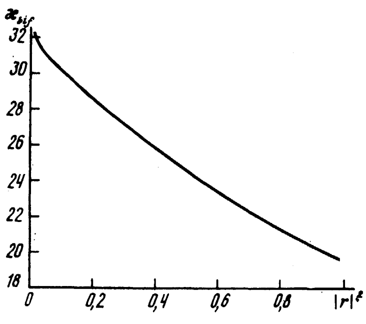

FIG. 7. Dependence of the bifurcation threshold of the fundamental scattering mode on the transmission coefficient of a feedback loop, calculated for $\gamma=0.8$; phase perturbations, (1, 0) mode.

6. CONCLUSIONS

The problem of stability of a stationary state of a ring parametric STBS oscillator in the presence of amplitude and phase perturbations was considered in Ref. 9 using the one-dimensional model. It was shown there that the fundamental mode of a stationary state may be unstable in the presence of amplitude perturbations if the transmission coefficient of a feedback loop is $r^2<1/3$, but it corresponds to a state of neutral equilibrium relative to phase perturbations. A comparison of these results with our results obtained by a three-dimensional analysis shows that the one-dimensional approach yields qualitative results compared with those of a study of the stability of stationary states, but only in the case of excitation of the fundamental nonstationary mode (0, 0) in the three-dimensional case. The exact values of the bifurcation thresholds in the presence of amplitude perturbations of the mode (0, 0) are different in the one- and three-dimensional cases, although they are of the same order of magnitude.

Some characteristics, such as the maximum value of the transmission coefficient of the optical channel at which a stationary state may be unstable due to excitation of amplitude perturbations $r^2_{\max}=1/3$, are the same for the one- and three-dimensional models. The higher modes of amplitude perturbations $(n,0)$ characterized by $n>0$ are not described by the one-dimensional theory, but their excitation thresholds are higher than the threshold of the (0, 0) mode and the range of existence in terms of the relevant parameter is narrower. Therefore, on the whole, we can say that the one-dimensional model gives a qualitatively correct answer on the subject of the stability of the ground stationary state and amplitude perturbations are excited. The reason for this is clearly the fact that the transverse structure of the field of the (0, 0) mode does not differ too greatly from the transverse structure of the fundamental stationary mode.

The situation is opposite in comparisons of the predictions of one- and three-dimensional models in the case of excitation of phase perturbations. In this case the presence of modes with $n \neq 0$ in the three-dimensional analysis, which have no analogs in the one-dimensional model, changes the situation qualitatively and gives rise to a new instability in the three-dimensional model, which is not predicted by the one-dimensional approach.

We shall now compare the relative role of amplitude and phase perturbations considered using the three-dimensional model (Secs. 4 and 5 in the present paper). The minimum excitation threshold of amplitude perturbations ($\varkappa_{\rm bif} \sim 2\pi$) is less than for phase perturbations, but amplitude perturbations can be stabilized by ensuring that the transmission coefficient of a feedback loop of a system is sufficiently high: $r^2 > 1/3$. These transmission values are realized experimentally and have the additional advantage that the excitation threshold of a stationary nonlinear state decreases on increase in r^2.

Although phase perturbations have a higher excitation threshold, they can be more dangerous because they exist for any value of r^2 and γ. The bifurcation threshold of the ground state in the case of excitation of phase perturbations coincides approximately with the onset of excitation of an ordinary convective STBS process, which is $\varkappa \sim 20$.

However, our model does not describe satisfactorily the system under discussion because of the possible excitation of other spatial gratings of the refractive index and of wide-angle parasitic stimulated scattering. In the range from $\varkappa \sim 1$ (excitation threshold of a stationary state) to $\varkappa \sim 20$ a ring parametric STBS oscillator is stable against the two classes of excitations considered by us.

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