Effect of dissipation on the structure of a stationary wave turbulence spectrum

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The development of wave turbulence is considered in which the wave excitation and attenuation regions are separated by an extended intertial range. The conditions which must be satisfied by the function that describes the dissipation in order for a nonequilibrium stationary distribution to exist are found. The effect of dissipation on the structure of the stationary turbulence spectrum is described for both the inertial range (in which the effect is small) and the region of strong dissipation. The general theory is verified in numerical experiments for three physical systems: capillary waves in deep water, gravitational-capillary waves in shallow water and three-dimensional sound with positive dispersion.

The theory of fully developed wave turbulence for the most part has already passed the stage of finding stationary Kolmogorov solutions[1–5] and studying their stability[6–8] in the presence of an infinite inertial range. It remains to investigate in whatever ways are possible how turbulence behaves in more realistic situations, taking into account the fact that the pumping or the wave source is usually spectrally narrow and that the ratio of the wavelength of the excited waves to the wavelength of the efficiently decaying waves is finite; even the finiteness of the total number of modes in the system can be important. The problem of matching the Kolmogorov spectrum with sources of different types was discussed in Ref. 9. In the present paper we study the structure of the stationary spectrum of wave turbulence in the presence of dissipation distributed in k-space. Three questions naturally arise:

1. What should the function look like that describes the behavior of the damping rate of the waves in k-space in order that a stationary distribution should exist?

2. What sort of distortions of the Kolmogorov distribution in the inertial range does the presence of a remote dissipative region introduce?

3. What is the structure of the stationary spectrum of wave turbulence like in the strong-dissipation region?

We will consider weak wave turbulence. In this case the evolution of the pair correlations—the wave occupation numbers $n_\mathbf{k}$—obey a kinetic equation:

$$\frac{\partial n_\mathbf{k}}{\partial t}=I_\mathbf{k}\{n_\mathbf{k}\}+\gamma_\mathbf{k}n_\mathbf{k}. \tag{1}$$

Here $\gamma_\mathbf{k}$ is a function that describes the interaction of the waves with the external environment. In those regions where $\gamma_\mathbf{k}>0$ holds it corresponds to a source, and where we have $\gamma_\mathbf{k}>0$, to a sink, i.e., dissipation. The function $I_\mathbf{k}$ is the collision integral, which describes the interaction of the waves with each other. For compactness of the presentation, we will restrict the discussion to wave turbulence with a decay-type dispersion law (the nondecay case can be treated analogously). In this case three-wave processes play the main role, and the collision integral has the form

$$I_\mathbf{k}\{n_\mathbf{k}\}=\int|V_{k12}|^2\delta(\omega_k-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\times[n_1n_2-n_k(n_1+n_2)]\,d\mathbf{k}_1\,d\mathbf{k}_2$$
$$-2\int|V_{1k2}|^2\delta(\omega_1-\omega_k-\omega_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)\times[n_kn_2-n_1(n_k+n_2)]\,d\mathbf{k}_1d\mathbf{k}_2. \tag{2}$$

Here $V_{k\,12}$ is the matrix element of the interaction, and ω_k is the wave frequency.

When the waves interact the entropy of the wave system

$$s=\int\ln n_\mathbf{k}d\mathbf{k}$$

should grow. In fact

$$\frac{ds}{dt}=\int\frac{\partial n_\mathbf{k}}{\partial t}\frac{d\mathbf{k}}{n_k}=\int\frac{I_\mathbf{k}}{n_\mathbf{k}}d\mathbf{k}+\int\gamma_\mathbf{k}d\mathbf{k}$$
$$=\int|V_{k12}|^2\delta(\omega_k-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{(n_1n_2-n_kn_1-n_kn_2)^2}{n_1n_2n_k}\cdot d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2+\int\gamma_kd\mathbf{k}. \tag{3}$$

As can be seen, the first term on the right-hand side of Eq. (3), which describes the variation of the entropy due to the interaction of the waves with each other, is nonnegative. Consequently, it is necessary that the condition

$$\int\gamma_\mathbf{k}d\mathbf{k}\leqslant 0 \tag{4}$$

be satisfied in order that a stationary distribution exist. The physical meaning of this requirement is obvious: in order that a nonequilibrium stationary state may exist, the external environment must provide a steady outflow of entropy from the system.

The next fundamental property of the collision integral is that the energy and momentum conservation laws be obeyed. Formally, this is connected with the presence in Eq. (2) of δ-functions of the frequencies and the wave vectors, which express the energy and momentum conservation laws

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during each elementary interaction event of the waves. We will consider isotropic distributions. The total momentum of such distributions is zero. Conservation of energy

$$E=\int \omega_k n_{\mathbf{k}} d\mathbf{k}=\int \varepsilon_{\mathbf{k}} d\mathbf{k}$$

when the waves interact is expressed by the following equality:

$$\int I_{\mathbf{k}} \omega_k d\mathbf{k}=0. \tag{5}$$

This relation is valid if the integrals that enter into it converge. Assuming the absence of singularities in $n_{\mathbf{k}}$ at $k=0$ and the power-law behavior of the functions ω_k and V_{k12} as $k\to 0$, it can be shown that the integrals in Eq. (5) converge if in the region of large k the occupation numbers n_k fall off faster than k^{-m-d}. Here m is the homogeneity index of V_{k12} in the limit $k\to\infty$, and d is the dimensionality of **k**-space. In order to clarify the physical meaning of this condition it is necessary to introduce the concept of energy flow in **k**-space. Relation (5) allows one to write the collision integral in divergence form:

$$\operatorname{div} \mathbf{p}_{\mathbf{k}}=-\omega_k I_{\mathbf{k}}, \tag{6}$$

where $\mathbf{p}_{\mathbf{k}}$ is the energy flux in **k**-space. The rate of change of the total wave energy dE/dt as a result of their interaction with each other is equal to the integral over the surface of a sphere of infinite radius of the normal component of the vector $\mathbf{p}_{\mathbf{k}}$. In order that this integral vanish as $k\to\infty$, as may be seen from Eq. (6), it is necessary that n_k fall off faster than k^{-m-d} (the distribution $n_k\sim k^{-m-d}$ is the Kolmogorov spectrum corresponding to a constant energy flux in **k**-space).

For the stationary distribution it follows from Eqs. (1) and (5) that γ_k should satisfy the condition

$$\int \gamma_k \omega_k n_k dk=0. \tag{7}$$

From Eq. (7) it is clear that if the function γ_k is to ensure stationarity it must pass through zero somewhere, i.e., it must describe sinks as well as sources of wave energy. The relative positions of the sources and sinks in **k**-space can in no way be arbitrary. Let us consider the isotropic situation. We define the energy density in wave number space

$$E_k=(2k)^{d-1}\pi\varepsilon_k=(2k)^{d-1}\pi\omega_k n_k$$

and the corresponding flux P_k with the spherical normalization $\partial P_k/\partial k=(2k)^{d-1}\cdot\pi\omega_k I_k$. The stationary kinetic equation in this case can be written in the form

$$\partial P_k/\partial k=\gamma_k E_k. \tag{8}$$

We integrate it from some wave number k_m to infinity:

$$P(\infty)-P(k_m)=\int_{k_m}^{\infty}\gamma_k E_k dk. \tag{9}$$

Assuming that the occupation numbers decay rapidly enough (faster than k^{-m-d}), we have $P(\infty)=0$. Note that for all the cases of wave turbulence encountered (see Ref. 4) we have $m+d\geqslant\alpha$, where m and α are indices that characterize the power-law behavior of the matrix element and the dispersion law as $k\to\infty$. This inequality expresses the fact that for large k the nonequilibrium stationary distributions should decay as a function of k at least as rapidly as the equilibrium Rayleigh-Jeans distribution $n_k=T/\omega_k$. This means that at large k_m the energy flux is positive, $P(k_m)>0$. Indeed, in the equilibrium solution we have $\varepsilon_k=\text{const}$ and $P=0$; for more rapidly decaying distributions the flux is directed toward regions where the energy density is less, i.e., towards larger k. Returning to Eq. (9), we see that for the stationary solution there exists k_m such that for arbitrary $k>k_m$

$$\int_k^{\infty}\gamma_{k'}E_{k'}dk'=\int_k^{\infty}(2k')^{d-1}\pi\gamma_{k'}\omega_{k'}n_{k'}dk'<0. \tag{10}$$

Thus, a necessary condition for the existence of the nonequilibrium stationary distribution is the presence of an energy sink in the region of large k.

The asymptotic limit of the function γ_k as $k\to\infty$ should also satisfy some condition. To find it, let us consider the Kolmogorov situation in which the regions of pumping ($\gamma_k>0$) and dissipation ($\gamma_k<0$) are separated by an inertial range, where $\gamma_k\approx 0$ (or more precisely $\gamma_k n_k\leqslant I_k$). The Kolmogorov spectrum $n_k=\lambda P^{1/2}k^{-m-d}$ should be realized in the inertial range. Here λ is a dimensional constant, and

$$P=\int_0^k\gamma_k\omega_k n_k(2k)^{d-1}\pi dk$$

is the energy flux associated with the distribution. In the dissipative region the function γ_k should be such as to ensure the absorption of the flux transported in the inertial interval. In the region where $\gamma_k\approx 0$ the flux is constant. Substituting the Kolmogorov distribution $n_k\sim k^{-m-d}$ in the relation

$$\partial P/\partial k=(2k)^{d-1}\pi\gamma_k\omega_k n_k$$

we see that if the function γ_k grows more slowly as $k\to\infty$ than the power function $k^{m-\alpha}$, then the dissipation is not able to ensure the absorption of all the flux.

Of course, the requirement that the flux P vanish as $k\to\infty$ has meaning only when there are an infinite number of modes in the system. If there exists a maximum wave number $k_M<\infty$ and a maximum frequency ω_M corresponding to it, then $n(\omega_M)$ does not, generally speaking, have to vanish. Let us consider, for example, the case of a finite and discrete ω-space with an equally spaced spectrum: $\omega_i=i\omega_0$, $i=1,\dots,M$. Interest in such a system is due, in particular, to the fact that just such a set of frequencies is excited by a spectrally narrow source in a continuous medium in which the waves have a decay-like dispersion law (see Ref. 9). Defining the energy flux by the formula

$$P_i=\sum_{l=1}^{i}lI_l,$$

we see that $P_1=0$ holds by definition, and $P_M=0$ holds as a consequence of energy conservation. Indeed, the condition $P_M=0$ is relation (5) written in discrete form. Consequently, in this case the function γ_i can be quite arbitrary. Only the requirement that condition (4) be satisfied remains in force, or, in discrete form,

$$\sum_{i=1}^{M} \gamma_i \leqslant 0. \qquad (11)$$

In the case of the model system with three spherical harmonics (an ω-space consisting of three points) for which we have according to Eqs. (1) and (2)

$$\frac{\partial n_1}{\partial t} = -2V_1^2(n_1n_2 - n_3n_1 - n_3n_2) - 2V_2^2(n_1^2 - 2n_1n_2) + \gamma_1 n_1,$$

$$\frac{\partial n_2}{\partial t} = V_2^2(n_1^2 - 2n_1n_3) - 2V_1^2(n_1n_2 - n_3n_1 - n_3n_2) + \gamma_2 n_2,$$

$$\frac{\partial n_3}{\partial t} = 2V_1^2(n_1n_2 - n_3n_1 - n_3n_2) + \gamma_3 n_3,$$

it is possible to convince oneself that

$$\sum_{i=1}^{3} \gamma_i < 0$$

is a necessary and sufficient condition for the existence of at least one (and there may be several) stationary state with positive n_1, n_2, n_3.

When the system has an arbitrary number of modes, condition (11) can be proved sufficient only in the limit $\Sigma\gamma_i \to -0$. In this case the system is almost in equilibrium even though individual γ_i can be very large. Indeed, relation (3) in the discrete case takes the following form:

$$\sum_{i,l=1}^{M} U(i,l) \frac{(n_l n_{i-l} - n_i n_l - n_i n_{i-l})^2}{n_i n_l n_{i-l}} + \sum_{i=1}^{M} \gamma_i = 0, \qquad (12)$$

where $U(i,l)$ is a positive function expressed in terms of the square of the matrix element and the wave frequencies. It is evident from Eq. (12) that as $\Sigma\gamma_i \to -0$ each of the expressions in parentheses in the first sum should approach zero. This is possible only for the Rayleigh-Jeans distribution $n_i = A/i$ (here i is the coordinate in ω-space, and n_i is the wave density in k-space taken as a function of frequency). Recall that we are considering the isotropic situation. The stationary distribution can be constructed by perturbation theory: $n_i = A/i + \varphi_i + \ldots$. Substituting such a distribution into the discrete analog of the kinetic equation, we can show that

$$\psi_i \propto \gamma_i, \quad A \propto \frac{(\Sigma\gamma_i)^2}{(\Sigma\gamma_i)},$$

and the small parameter on which the expansion is based is $(\Sigma\gamma_i)^2/(\Sigma\gamma_i^2)$. Thus, as $\Sigma\gamma_i \to 0$ the effective "temperature" of the stationary distribution tends to infinity, and the characteristic time required to establish it grows.

We have numerically modeled the establishment of the stationary state for the discrete kinetic equation which describes capillary waves on deep water. In this case we have $\alpha = 3/2$, $m = 9/4$, $d = 2$, and the kinetic equation can be written in the form[4]

$$\frac{\partial n_k}{\partial t} = \sum_{l=1}^{k-1} U(k,l)[n_l n_{k-l} - n_k(n_l + n_{k-l})]$$
$$+ \sum_{l=k+1}^{M} U(l,k)[n_k n_{l-k} - n_l(n_k + n_{l-k})] + \gamma_l n_k, \qquad (13)$$

where

$$U(k,l) = \frac{k^{8/3} x^{2/3} (1-x)^{2/3}}{[4x^{4/3}(1-x)^{4/3} - [1 - x^{4/3} - (1-x)^{4/3}]^2]^{1/2}}$$
$$\cdot \left\{ \frac{(1-x^{2/3})^2}{(1-x)^{1/3}} + \frac{[1-(1-x)^{2/3}]^2}{x^{1/3}} - [x^{2/3} - (1-x)^{2/3}]^2 \right\}^2,$$

$x = k/l$, and the function γ_k has been chosen in the form

$$\gamma_k = A\Delta_{k_1} - k^{1/2}. \qquad (14)$$

Figure 1a depicts the time dependence of the total energy of the distribution for $M = 100$, $A = 100$, $\Sigma\gamma_k = -571.4$. Figure 1b shows the logarithmic derivative $\partial \lg E/\partial t$. It can be seen that for $t \gtrsim 3.8$ the evolution enters the exponential regime

$$E(t) = E_0 - E_1 \exp\left(-\frac{t}{\Delta t}\right).$$

By determining the slope of the curve in Fig. 1b we can find the characteristic time Δt. It is interesting to trace out the variation of Δt with the growth of the number of modes M. Function (14) grows slower than $k^{-h} = k^{3/4}$ with increasing k, therefore in an infinite system the stationary distribution should be absent. In a finite system, however, the characteristic time Δt falls as a function of M as a result of the growth of $|\Sigma\gamma_k|$ (see Fig. 1c). It is clear that in the limit $\Sigma\gamma_k \to 0$ we have $\Delta t^{-1} \sim \Sigma\gamma_k$.

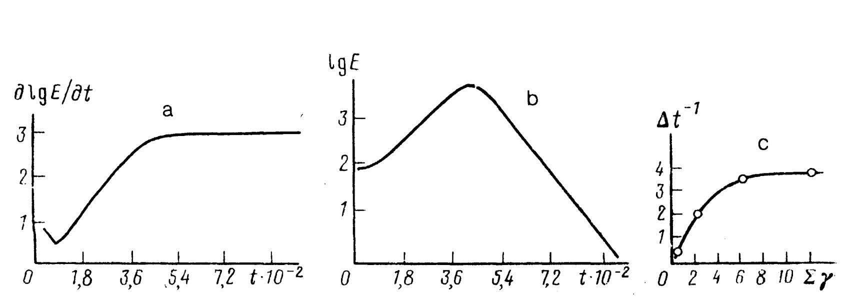

FIG. 1.

Two other common wave systems with a decay-like dispersion law—gravitational-capillary waves on shallow water (two-dimensional sound), for which we have $d=2$, $\alpha=1$, $m=1$, $m_1=1$, $U(k,l)=k(k-l)$, and three-dimensional sound, for which $d=3$, $\alpha=1$, $m=3/2$, $m_1=1$, $U(k,l)=k^2(k-l)^2$—demonstrate similar behavior when numerically modeled.

Let us turn now to the second of the questions formulated in the beginning of the article. Assume that in k-space, starting with some k_m, there is a strong wave damping ($\gamma_k<0$), leading to the rapid falloff of the occupation number n_k for $k>k_m$. In the region $k_0\ll k\ll k_m$, where k_0 is the scale of the source, the stationary distribution should be close to the Kolmogorov distribution $n_k^0\sim k^{-m-d}$ if the condition for the interaction to be local holds, i.e., if the collision integral in the Kolmogorov solution converges. It is easy to convince oneself that if for $k_1\ll k$ the asymptotic limit of the matrix element has the form

$$|V_{k12}|^2=V^2k_1^{m_1}k^{2m-m_1}, \tag{15}$$

the collision integral (2) converges for the power-law solutions $n_k=k^{-S}$ if the index S falls within the localization range:[5]

$$m_1+d-1+2\alpha=S_1>S>S_2=2m-m_1+d+1-2\alpha. \tag{16}$$

As is obvious, the Kolmogorov index lies right in the middle of the localization range, which exists if

$$m_1-m-1+2\alpha>0. \tag{17}$$

If the localization condition (17) is fulfilled, then the deviation of the stationary solution from the power-law distribution caused by the effect of the distant sink can be found with the help of perturbation theory in the small parameter k/k_m.

For isotropic distributions the angle-averaged three-wave collision integral is given to within some unimportant constant factors by[4]

$$I_k\propto\iint_0^\infty dk_1dk_2[R(k,k_1,k_2)-R(k_1,k,k_2)-R(k_2,k,k_1)],$$
$$R(k,k_1,k_2)=|V_{k12}|^2(k_1k_2)^{d-1}\Delta_d^{-1}\delta(k^\alpha-k_1^\alpha-k_2^\alpha)\cdot\theta(k-k_1)(n_1n_2-n_kn_1-n_kn_2). \tag{18}$$

Here Δ_d^{-1} is (to within a factor of 2π) the result of the angle-averaging of the d-dimensional δ-function of the wave vectors,

$$\Delta_2={}^1/_2[2(k^2k_1^2+k^2k_2^2+k_1^2k_2^2)-k^4-k_1^4-k_2^4]^{1/2},$$
$$\Delta_3=kk_1k_2.$$

If the Kolmogorov power-law solution $n_k^0=Ak^{-m-d}$ holds in the infinite interval $k\in(0,\infty)$, then the collision integral is identically equal to zero. The absence of waves for $k>k$ (we assume that $n=0$ holds for $k>k$) causes the collision integral to differ from zero by a small amount at $k\ll k_m$:

$$\delta I_1=2A^2\int_{k_m}^\infty dk_1|V_{1k2}|^2(k_1k_2)^{d-1}\Delta_d^{-1}[(kk_2)^{-m-d}-(kk_1)^{-m-d}-(k_1k_2)^{-m-d}]. \tag{19}$$

Here $k_2^\alpha=k_1^\alpha-k^\alpha$. In order that the distribution n_k be stationary, the additive term δI_1, which arises as a result of the boundedness of the inertial range, should be balanced by the additive term δI_2 due to the small deviation of the solution from the power-law form ($n_k=n_k^0+\delta n_k$, $\delta n_k\ll n_k^0$ for $k\ll k_m$):

$$\delta I_2=\hat{L}_k\delta n_k=2\iint_0^\infty dk_1\,dk_2\,(k_1k_2)^{d-1}\Delta_d^{-1}\{|V_{k12}|^2\cdot\delta(k^\alpha-k_1^\alpha-k_2^\alpha)\theta(k-k_1)[\delta n_1(n_2^0-n_k^0)-\delta n_kn_1^0]-|V_{1k2}|^2\delta(k_1^\alpha-k^\alpha-k_2^\alpha)\theta(k_1-k)[\delta n_k(n_2^0-n_1^0)+\delta n_1[n_k^0+n_2^0]+\delta n_2(n_k^0-n_1^0)]. \tag{20}$$

Here $\hat{L}$ is the operator of the kinetic equation linearized about the background n_k^0. This integral operator is scale-invariant $L_{\lambda k}=\lambda^{m-\alpha}\hat{L}_k$, with index $m-\alpha=-h$. Thus, in order to determine δn_k, it is necessary to solve the linear integral inhomogeneous equation

$$\delta I_2+\delta I_1=\hat{L}\delta n_k+\delta I_1=0. \tag{21}$$

First we calculate δI_1. Since we have $k_1>k_m\gg k$, we make use of the asymptotic behavior of the matrix element $\lim|V_{1k2}|=V^2k^{m_1}k^{2m-m_1}$ and expand the expression in brackets in Eq. (19) out to the first nonvanishing terms in $(k/k_1)^\alpha$. As a result we obtain

$$\delta I_1=2A^2\frac{(m+d)V^2}{\alpha(m_1+2\alpha-1-m)}k^{m+1-2\alpha-m_1}k_m^{m_1-m+\alpha-d-1}. \tag{22}$$

In accordance with the localization condition (17) we have $\delta I_1>0$.

As a consequence of the homogeneity of the operator L the equation

$$\hat{L}\delta n_k=k^{-x} \tag{23}$$

has a power-law solution

$$\delta n_k=k^{h-x}[g(x-h)AV^2]^{-1}, \tag{24}$$

where $g(S)$ is a dimensionless integral obtained if we substitute $\delta n_k=k^{-S}$ in Eq. (20) and we then take out the factor AV^2k^{-h-S}. This is the case, however, only if the index $S=x-h$ falls within the localization range of the collision integral (20). As can be seen, the localization range (S_2,S_1) of the linearized collision integral (20) is the same as for the total integral (18). In our case, i.e., upon substituting Eq. (22) in Eq. (21), we have $x=m+d+1-m_1-\alpha$, and the quantity $x-h=2m-m_1+d+1-2\alpha$ coincides with the lower boundary of the locality interval S_2 [see Eq. (16)]. Neglecting the slow logarithmic dependence in the integration, we obtain

$$\delta n_k=\frac{2(m+d)}{\alpha(m_1+2\alpha-1-m)}Ak^{-m-d}(k/k_m)^{m_1+2\alpha-1-m}\ln^{-1}(k_m/k). \tag{25}$$

Formula (25) is valid for $k\ll k_m$ and shows that the finiteness of the scale of the sink gives rise to an increase of the occupation numbers in the inertial interval since $\delta n_k>0$. As

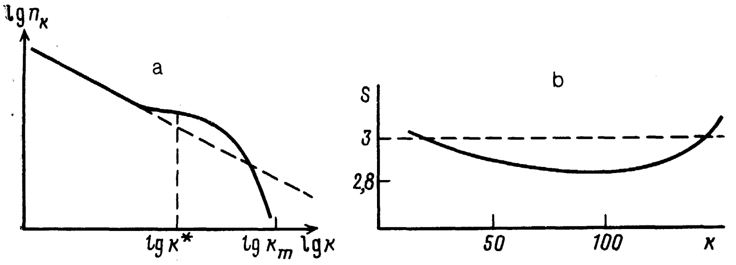

FIG. 2.

k increases the quantity δn_k also grows, i.e., some rising of the distribution takes place. Of course, for $k \approx k_m$ an abrupt falloff of n_k should take place, which can no longer be described within the framework of perturbation theory. Thus, the stationary distribution should look roughly as depicted in Fig. 2a, where the dashed line corresponds to the Kolmogorov power-law solution. The dependence of log n_k on log k should have an inflection point (denoted in Fig. 2 as k^*), in which the current index of the solution

$$S(k) = \partial \lg n_k / \partial \lg k$$

passes through a minimum (see Fig. 2b). The wave slope for $k \lesssim k^*$ is apparently due to an effect of the "bottleneck" type, arising as a consequence of the falloff of the flux at $k \gtrsim k^*$ due to the decrease of the occupation numbers at $k \lesssim k_m$. This picture is confirmed by a numerical experiment that was carried out for capillary waves on the surface of a deep liquid and for sound. Thus, for example, Fig. 3a shows the dependence of the running index of the steady-state solution on the wave vector, obtained by numerically modeling two-dimensional acoustic turbulence. A well-defined maximum of $S(k)$ is clearly visible at $k^* = 77$. By using various k_m in the course of the numerical experiment, one can easily convince oneself that the observed effect is connected with the finiteness of the scale of the sink—the location of the minimum is proportional to k_m: $k^* = Bk_m$; for two-dimensional sound and the attenuation arising as a result of the jump at $k = k_m$, we have $B \approx 1/3$. Figure 3b shows how the index $S(k)$ behaves in the analogous situation for three-dimensional sound. Note that nonmonotonic behavior of the index has been previously observed in a numerical experiment carried out by A. V. Shafarenko and one of the authors of the present paper (G. E. F.).

Let us turn, finally, to the last of the considered questions. Let us consider the behavior of the stationary turbulence spectrum in the dissipative region for $k \gg k_m$. We assume that the damping rate γ_k grows with k faster (or falls off more slowly) than the inverse interaction time of the waves in the inertial interval (i.e., the function k^{-h}). As a consequence of this, in the dissipative region the occupation numbers should fall off as a function of k faster than the Kolmogorov law. The character of this falloff depends on what kind of interaction is dominant for waves in the dissipative region: between themselves or with waves from the inertial range. The dependence on k of the interaction time of the waves with sharply different wave numbers can be found by substituting the asymptotic solution for the matrix element[15]

$$t_1^{-1}(k) \propto k^{2m-m_1+1-\alpha} \qquad (26)$$

in the collision integral (18). If the damping rate increases faster with k than $t_1^{-1}(k)$, then the asymptotic behavior of the distribution as $k \to \infty$ is determined by the interaction among the waves in the dissipation region. In this case an exponential "quasi-Planck" spectrum

$$n_k = B\omega_k^{-b} \exp(-\omega_k/\omega_m),$$

is formed, analogous to the exponential asymptotic limits arising in the shortwave region when the distribution evolves freely (see Ref. 10). Indeed, assuming that the damping rate

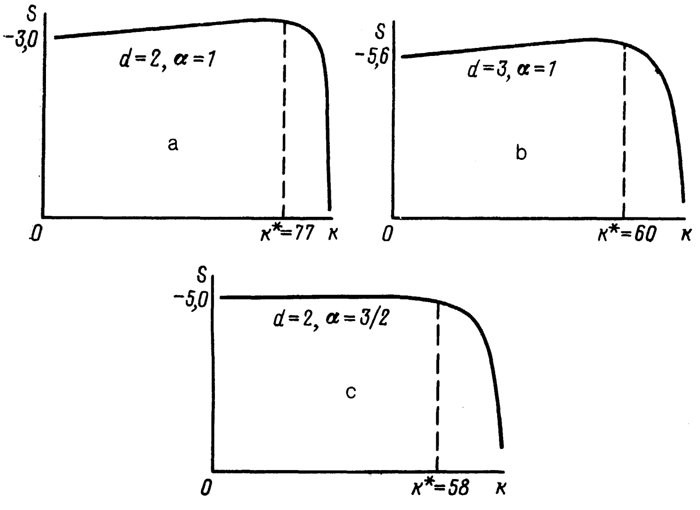

FIG. 3.

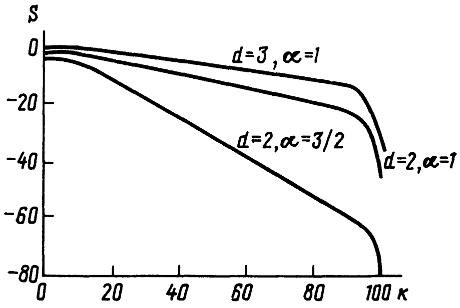

FIG. 4.

γ_k grows according to a power law $\gamma_k = -Ak^\alpha$ (e.g., for the viscosity $\alpha = 2$), but the main contribution to the collision integral comes from integrating over the region $k_1 \ll k$. We obtain the stationary kinetic equation in the form

$$Ak^a n_k = Bk^{2m-m_1+1-\alpha} n_k \int_{k_m}^{k} k_1^{\alpha-1-\alpha+m_1-b}[1-\exp(-k_1^\alpha/k_m^\alpha)]^2\, dk_1. \tag{27}$$

Since the integral standing on the right side of Eq. (27) is a nondecreasing function of k, a solution of such a form can exist only if the inequality

$$a > 2m - m_1 + 1 - \alpha \tag{28}$$

is satisfied. For all three wave systems discussed in this article as well as viscous damping this inequality is fulfilled. Indeed, numerical modeling shows that for the choice

$$\gamma_k = A\Delta_{k_1} - A_1 k^2$$

the occupation numbers in the strong dissipation region fall off exponentially. Figure 4 shows the dependence of the current index of the steady-state solution

$$S(k) = \frac{\lg(n_{k+1}/n_k)}{\lg(k/(k+1))}$$

on the wave number for capillary waves on the surface of a shallow ($\alpha = 1$) and a deep ($\alpha = 3/2$) liquid. The segment on which $S(k)$ decreases linearly corresponds to exponential falloff of n_k.

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