Freezing-in integrals and stability of 3*D* flows

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Using the Lyapunov method, stability of 3*D* flows of an ideal incompressible fluid is studied. The integrals expressing the 3*D* "vector" freezing of the curl of the velocity of a 3*D* flow into the flow itself are stated in terms of locally conserved scalars; this allows us to use them in Lyapunov functionals by means of the method of Lagrange undetermined multipliers. The concept of "dynamic" variations is introduced, equivalent to accounting for supplemental "motion integrals" involving invariance of the system (only those variations of the system that are compatible with its equations of motion are admissible). It is shown that for such perturbations, planar flow that is stable with respect to 2*D* perturbations remains stable with respect to 3*D* perturbations as well. A sufficient condition for stability of toroidal vortices is obtained.

1. INTRODUCTION

Stability of a system is closely associated with the existence of "integrals of motion." This fact is explicitly and directly used in the case of one of the most efficient methods for investigation of stability of motions—the Lyapunov method in which the Lyapunov function is constructed with the aid of a set of motion integrals (cf., e.g., Refs. 1,2). Arnol'd generalized the Lyapunov method to the the stability of 2*D* (Ref. 3) and 3*D* (Ref. 4) incompressible flows. In the 2*D* case, Arnol'd's variational method has been quite fruitful and popular. Using this method, a stability criterion for planar flows of an ideal fluid in the form of a generalized Rayleigh theorem[4] was obtained, as well as a number of results concerning the stability of flows in the atmospheres and oceans of planets[5–8] and also concerning the stability of plasm configurations[9–14] (the references cited are far from complete and presented solely for illustration). However, in the 3*D* case, Arnol'd did not succeed in obtaining the velocity field V of a flow for which his quadratic form $\delta^2 E$ would be of definite sign or 3*D* perturbations[4] which would then signify stability of a flow in the Lyapunov sense. In the subsequent literature such flows have not been considered; in any event, they are unknown to the author.

Since the Lyapunov method provides only a sufficient condition for stability, this does not mean that 3*D* flows or perturbations are necessarily unstable. On the contrary, flows are known, for example Coulette flow,[15] whose 3*D* stability has been established by the spectral method. One would expect in such a situation that allowing for supplementary integrals of motion, i.e. selecting another set of integrals or another Lyapunov functional would permit us to make some progress in the problem of 3*D* stability of motions. This is demonstrated in the present paper.

When discussing supplementary integrals of motion, it is useful to note that fixing an integral of motion amounts in essence to stratification of the phase space of a fluid (the "points" in this space are complete sets of the independent hydrodynamic fields characterizing the instantaneous state of a moving fluid) into "sheets" within which the motion of a point is restricted. The manner in which this stratification is imposed, is, however, another problem. For example, conservation of vorticity of a 2*D* flow in the *xy* plane, $\Omega = (\text{curl }\mathbf{V})_z$, is usually formulated in a form of an undetermined Lagrange "multiplier" in the problem of conditional extrema. Here the integral of motion

$$\mathcal{F} = \int_D F(\Omega)\,dx dy \tag{1.1}$$

[where $F(\Omega)$ is an arbitrary function of its argument and D is the region of the flow] associated with freezing of Ω in the flow is explicitly included into the Lyapunov functional and arbitrary variations $\delta\Omega$ are allowed. In the 3*D* case, however, the freezing of the curl $\mathbf{V}$ in the flow has been taken into account[4] through restrictions on the form of the variations $\delta\mathbf{V}$ in the velocity when only "equivorticity" variations are admitted which do not take the "point" outside the corresponding "sheet". Both methods of imposing "stratifications" are equally valid, although the first is preferable since it allows us to extend without difficulties a set of motion integrals, provided of course that one succeeds in formulating integrals of the form (1.1). In this paper, unlike that of Arnol'd,[4] the freezing-in of the vector curl $\mathbf{V}$ will be stated in the form of an infinite series of invariants of the form (1.1) (cf. Sec. 2). This would allow us to account for supplementary "sheets" in the phase space of the fluid in the form of restrictions on the variation of velocity (Sec. 3) and to obtain a criterion for the 3*D* stability of flows (Sec. 4).

2. INTEGRALS OF "VECTOR" FREEZING-IN

In a general case the equation for the freezing of a vector a into a flow with the velocity field $\mathbf{V}$ is of the form

$$\frac{\partial \mathbf{a}}{\partial t} = \text{rot}[\mathbf{V}\mathbf{a}]. \tag{2.1}$$

In particular the freezing-in equation (2.1) with $a = \text{curl }\mathbf{V}$ is a corollary of the Euler equation describing 3*D* flows of an ideal fluid,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\nabla\left(\frac{P}{\rho}\right), \quad \nabla\mathbf{V} = 0, \tag{2.2}$$

(here P and ρ are respectively the pressure and the density of the fluid).

As in the 2*D* case, an infinite series of integrals of motion in the form of conservation of the velocity circulation along an arbitrary contour γ moving with the fluid:

$$C_\gamma = \oint_\gamma (\mathbf{V} d\mathbf{l}) \tag{2.3}$$

is associated with the freezing-in equation of the curl. However, these integrals are practically useless as regards their use in the problem of stability of flows, in the spirit of the approach described above, as was done with the series of invariants (1.1). The required series of invariants can easily be formulated provided one succeeds to obtain a locally conserved scalar h for which the equation of a "scalar freezing-in" is valid:

$$\frac{\partial h}{\partial t}+(\mathbf{V}\nabla)h=0, \tag{2.4}$$

i.e., h is a function of the Lagrangian coordinates.

The velocity circulation C_γ (2.3) which is closely associated with the Clebsch variables as shown by Eckart,[16] satisfies the above requirement, and in the general case, the vector freezing-in equation (2.1) expresses conservation of the contravariant component of the frozen-in vector in Lagrangian coordinates. Indeed, let x_m be the current Euler coordinates of the fluid elements, x^j the corresponding Lagrangian coordinates, and $g_{ik}=\partial x_i/\partial x^k$ the metric tensor. The Lagrangian property of the system is assured by the condition $\partial x_i/\partial t = V_i$ which implies $\partial g_{ik}/\partial t=\partial V_i/\partial x^k$. Since the fields $\mathbf{V}$ and $\mathbf{a}$ are divergence-free, equations (2.1) can be written in the form

$$\frac{\partial a_\alpha}{\partial t}=a_k g^{kj}\frac{\partial V_\alpha}{\partial x^j} \tag{2.5}$$

(we use here the standard notation; see, e.g.,). Multiplying (2.5) by $g^{\alpha m}$ and summing up over the repeated indices, we arrive, after simple transformations, at

$$\frac{\partial}{\partial t}(a_k g^{km})=a_i\left(\frac{\partial g^{im}}{\partial t}+g^{ij}g^{km}\frac{\partial g_{kj}}{\partial t}\right)=0. \tag{2.6}$$

Since g_{ik}, g^{ik} are symmetric, we have $g^{ij}g_{jm}=\delta^i_m$.

Thus, the contravariant components a^m of the frozen-in vector can be used to construct an infinite series of invariants. However, the variation of these invariants becomes complicated due to the nonlinear dependence of the metric coefficients on the components of the vector displacement of a fluid. It thus makes sense to stipulate the frozen-in property of vector $\mathbf{a}$ in a somewhat different but a more transparent manner. We take the scalar product of Eq. (2.1) with an arbitrary (for the time being) vector $\mathbf{b}$ and transform it according to the rules of vector analysis:

$$\frac{\partial}{\partial t}(\mathbf{ab})+\nabla(\mathbf{V}(\mathbf{ab}))=\mathbf{a}\left(\frac{\partial \mathbf{b}}{\partial t}+\nabla(\mathbf{bV})+[\operatorname{rot}\mathbf{b},\mathbf{V}]\right).$$

This implies that the quantity $\mathbf{a}\cdot\mathbf{b}$ is locally conserved in the sense of (2.4) if the vector $\mathbf{b}$ satisfies the equation

$$\frac{\partial \mathbf{b}}{\partial t}+\nabla(\mathbf{Vb})+[\operatorname{rot}\mathbf{b},\mathbf{V}]=0. \tag{2.7}$$

One possibility is $\mathbf{b}=\nabla\varphi$ where the function φ satisfies in turn the equation

$$\frac{\partial \varphi}{\partial t}+(\mathbf{V}\nabla)\varphi=\psi(t), \tag{2.8}$$

and $\psi(t)$ is arbitrary so that, at each instant of time, φ is determined up to a constant. This is indeed not surprising since only $\nabla\varphi$ enters into the resulting part. For simplicity, we shall assume below that $\psi=0$. In the end, we have for the Euler equation (2.2) a series of invariants:

$$\mathscr{F}=\int_D F(h)\,dx\,dy\,dz, \tag{2.9}$$

$$h=(\operatorname{rot}\mathbf{V},\ \nabla\varphi)=\nabla(\varphi\operatorname{rot}\mathbf{V}), \tag{2.10}$$

where φ is advected together with the fluid, so that some set of three functions φ_1, φ_2, and φ_3 satisfying (2.8) and the condition $\det(\partial\varphi_k/\partial x_i)\neq 0$ form a system of Lagrangian coordinates and the corresponding h_j are analogous to the contravariant components of the curl $\mathbf{V}$.

We will not discuss the possibility here of some other choice of the vector $\mathbf{b}$ in (2.7), although this problem is of interest. Note that in the case of the equations of ideal magnetohydrodynamics with the magnetic field vector frozen into a plasma, invariants of the form (2.10) were studied by Gordin and Petviashvili[11-13] in the course of an analysis of the stability of plasma configurations.

3. VARIATION OF VELOCITY AND OF THE LAGRANGE COORDINATES AND THE SUPPLEMENTARY INVARIANTS

In the Lyapunov method, the stability of a motion follows from the extremization of a single integral of the motion of the system under the condition that one or several other invariants are conserved. It is important to show this for all the possible variations which conserve the required invariants. Since h_j in (2.10) depends on the choice of φ_k, a problem arises as to the meaning of "all" variations in the application to (2.9). In the $2D$ case (1.1) there is no problem here since it is clear that only the argument of the function $\delta\Omega$ varies arbitrarily, while the function itself does not change (although it is arbitrary). What, however, is meant by "the function itself"? If, for example, we take $F=\Omega^\alpha$ then the exponent α enters into the definition of the function and remains unchanged as Ω varies. Analogously, we include the Lagrangian coordinates φ_k in the definition of F in (2.9) and, in the course of variation, we ought to concern ourselves with the invariance of the functional dependence $F(h_j,\ \varphi_k)$ as well as of the invariance of the chosen system of Lagrange coordinates φ_k although the φ_k vary in time. In other words, variations of φ_k and $\mathbf{V}$ cannot be viewed as independent, since otherwise it would be impossible to know whether we are tracing the invariance of values of the very same invariant or comparing values of different invariants, which does not make sense.

To answer the question on the relation between variations of φ_j and $\mathbf{V}$ it is necessary to analyze how, in general, a perturbation of the velocity field $\mathbf{V}$ arises. Arnol'd[4] considered fields of equal vorticity obtained from a given field $\mathbf{V}$ by means of a displacement of the fields such that curl $\mathbf{V}$ is frozen into the "displaced" flow. Here the displacement vector ξ is not connected with the field $\mathbf{V}$; it is essential only that in the displacement process the equation

$$\frac{\partial}{\partial\tau}\operatorname{rot}\mathbf{V}=\operatorname{rot}[u,\operatorname{rot}\mathbf{V}]$$

be valid (where $\mathbf{u}=\partial\xi/\partial\tau$ and τ is a fictitous time). In our approach it means that the φ_j should also be frozen into this "hypothetical" flow u, which generates the perturbation $\delta\mathbf{V}$. Assuming that $\mathbf{u}$ is an arbitrarily assigned divergence-free vector field, we obtain the first two terms of the expansions of $\delta\mathbf{V}$, curl $\delta\mathbf{V}$, and $\delta\varphi_j$ in the small fictitious time τ (i.e., the first and second variations):

$$\operatorname{rot}\delta\mathbf{V}=\operatorname{rot}[\xi\operatorname{rot}\mathbf{V}]+{}^1/_2\operatorname{rot}[\xi\operatorname{rot}[\xi\operatorname{rot}\mathbf{V}]], \tag{3.1a}$$

$$\delta\mathbf{V}=[\xi\operatorname{rot}\mathbf{V}]+{}^1/_2[\xi\operatorname{rot}[\xi\operatorname{rot}\mathbf{V}]]+\nabla\alpha, \tag{3.1b}$$

$$\delta\varphi_j=-(\xi\nabla)(\varphi_j-{}^1/_2(\xi\nabla)\varphi_j). \tag{3.1c}$$

Here $\xi=\tau\mathbf{u}$ is a small displacement and vector a is an arbitrary function. The variations (3.1) by themselves assure the freezing-in of vorticity and in this sense are redundant with respect to utilization of the integral of motion (2.9) in the Lyapunov functional. The invariant (2.9) is useful if some other variations are considered which do not have the same vorticity, but conserve some supplementary invariant or property of the system (which defines the supplementary "sheet"); however, it is necessary here to retain also the frozen-in property of curl **V**.

It would seem expedient to consider physical variations of the velocity field corresponding to preservation of the system as such, i.e., consistent with the equations of motion of the system. We shall assume that only those perturbations of the velocity field that arise as a result of the action of arbitrary but not nonsingular time-dependent external forces **f** applied to the fluid are admissible. The difference between these variations and (3.1) can be clarified to some extent by analyzing the stability of an equilibrium of a simple pendulum on an inclined plane (see the figure). An analog of variation (3.1) would be the admissibility of all possible deviation of the pendulum from equilibrium, including those under the inclined plane. If one confines oneself to the deviations that are consistent with the equations of motion under nonsingular forces which do not destroy the plane, only the upward deviations will be admissible. The question is: Which criterion of stability is closer to the truth?

The connection between the first and second variations of the field velocity and the Lagrangian coordinates φ_k and the force **f** can be obtained by solving, on a short time interval $(t, t+\tau)$, $\tau\to 0$ the system of equations

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{V}=\operatorname{rot}[\mathbf{V}\operatorname{rot}\mathbf{V}]+\operatorname{rot}\mathbf{f}, \qquad (3.2a)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{V}\nabla\right)\varphi_j=0. \qquad (3.2b)$$

Hence we have

$$\operatorname{rot}\delta\mathbf{V}=\tau\operatorname{rot}\mathbf{U}_1+\tau^2\operatorname{rot}\mathbf{U}_2+\ldots, \qquad (3.3a)$$

$$\delta\mathbf{V}=\tau\mathbf{U}_1+\tau^2\mathbf{U}_2+\ldots, \qquad (3.3b)$$

$$\delta\varphi_j=-\tau(\mathbf{V}\nabla\varphi_j)-\frac{\tau^2}{2}(\mathbf{U}_1\nabla\varphi_j-(\mathbf{V}\nabla)(\mathbf{V}\nabla)\varphi_j), \qquad (3.3c)$$

where the notation

$$\mathbf{U}_1=\mathbf{f}+[\mathbf{V}\operatorname{rot}\mathbf{V}]+\nabla\alpha_1,$$
$$\mathbf{U}_2={}^1/_2[\mathbf{U}_1\operatorname{rot}\mathbf{V}]+{}^1/_2[\mathbf{V}\operatorname{rot}\mathbf{U}_1]+\nabla\alpha_2 \qquad (3.3d)$$

is used. Here along with variations of the velocity field under the action of the force **f**, variations in **V** arising in the unperturbed flow during the time τ are also taken into account. These $\delta\mathbf{V}$ ought to be excluded; we shall retain them, however, since they are irrelevant. Moreover, in the case of stationary flows whose stability is usually investigated, Eq. (3.3d) can be simplified by using the Bernoulli equation

$$[\mathbf{V}\operatorname{rot}\mathbf{V}]=\nabla(P/\rho+V^2/2).$$

The dynamic variations (3.3), unlike (3.1), do not make the vorticities of the fields **V** and $\mathbf{V}+\delta\mathbf{V}$ different; however, they assure as before that the variations $\delta\varphi_j$ do not result in a substitution of one invariant by another. The essential difference between the "dynamic" variations (3.3) and the "kinematic" ones (3.1) is that in the "dynamic" case the first variation of φ_j is actually zero, while in (3.1) this is not so. This is of importance in studying the stability of flows. The nature of this difference is connected with a difference in the dependence of the path on time under equally accelerated motions with zero and non-zero initial velocities.

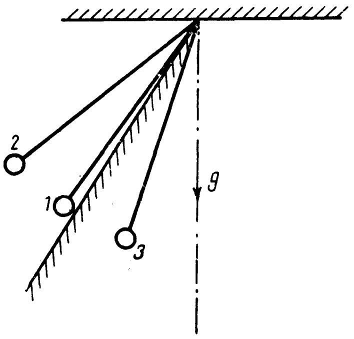

FIG. 1. Simple pendulum on an inclined plane: 1—equilibrium; 2—deviations admissible in terms of the equations of motion; 3—deviations distorting the system, i.e. inadmissible in terms of the equations of motion.

Finally, we note that along with the freezing-in integrals (2.9) the energy integral

$$E=\int_D {}^1/_2\mathbf{V}\mathbf{V}\,dxdydz \qquad (3.4)$$

and the helicity integral[18]

$$H=\int_D {}^1/_2\mathbf{V}\operatorname{rot}\mathbf{V}\,dx\,dy\,dz \qquad (3.5)$$

play an important role in the analysis of stability. Helicity is closely connected with self-organization of flows of the form $\mathbf{V}=\lambda\operatorname{curl}\mathbf{V}$, where λ is a numerical factor.

4. STABILITY OF FLOWS

Consider the functional consisting of the freezing-in, energy and helicity integrals:

$$L=\int_D\left(\frac{1}{2}\mathbf{V}\mathbf{V}+\frac{\lambda}{2}\mathbf{V}\operatorname{rot}\mathbf{V}+F(h_1,h_2,h_3,\varphi_1,\varphi_2,\varphi_3)\right)dx\,dy\,dz, \qquad (4.1)$$

where F is an arbitrary function of its arguments and λ is an undetermined Lagrange multiplier whose value is found by equating the helicity integral to its initial value. As was already mentioned, the functions φ_j can be viewed as Lagrangian coordinates which initially are chosen arbitrarily provided only that $\det(\partial\varphi_i/\partial x_m)\neq 0$ holds.

We calculate the first variation of the functional L in (4.1), taking into account the definition of h_j (cf [2.10]) which connects δh_j with the variations of curl **V** and φ_j, but formally $\delta\mathbf{V}$ and φ_k are considered for the time being independent:

$$\delta L=\int_D\left\{\delta\mathbf{V}\left(\mathbf{V}+\lambda\operatorname{rot}\mathbf{V}-\left[\nabla\varphi_j,\nabla\frac{\partial F}{\partial h_j}\right]\right)+\delta\varphi_j\left[\frac{\partial F}{\partial\varphi_j}-\left(\operatorname{rot}\mathbf{V},\nabla\frac{\partial F}{\partial h_j}\right)\right]\right\}dxdydz. \qquad (4.2)$$

Here summation over repeated indices is assumed and also the velosity is assumed to vanish on the boundary of the flow region. Although in principle φ_k depends on $\delta\mathbf{V}$, one could make the first variation δL vanish by requiring that each one of the terms in (4.2) vanish. As the result, we arrive at the Euler equations

$$\mathbf{V}+\lambda\,\mathrm{rot}\,\mathbf{V}=\left[\nabla\varphi_j,\nabla\frac{\partial F}{\partial h_j}\right], \tag{4.3}$$

$$\frac{\partial F}{\partial\varphi_j}-\left(\mathrm{rot}\,\mathbf{V},\nabla\frac{\partial F}{\partial h_j}\right)=0, \tag{4.4}$$

which describe arbitrary stationary flows. Indeed carrying out the vector multiplication of (4.3) by curl **V**, we obtain after some simple manipulations

$$[\mathbf{V}\,\mathrm{rot}\,\mathbf{V}]=-\nabla\left(F-h_j\frac{\partial F}{\partial h_j}\right),$$

which is equivalent to the stationary Bernoulli equation

$$[\mathbf{V}\,\mathrm{rot}\,\mathbf{V}]=\nabla\left(\frac{P}{\rho}+\frac{V^2}{2}\right).$$

Equations (4.3) and (4.4) together with the definition (2.10) of h describe a stationary flow of an ideal fluid whose form depends on a specific choice of the function φ_j and the form of the function F. Also, Eq. (4.3) does not contradict the condition of freezing-in of φ_j and h_j since even in a stationary flow the Lagrangian coordinates, when comoving with the fluid in general vary in time. Note that (4.3) and (4.4) admit a transition from $3D$ to the $2D$ case of flow in the xy plane in the usual notation:

$$\varphi_1=z,\quad h_1=(\mathrm{rot}\,\mathbf{V})_z,\quad \lambda=0,\quad \mathbf{V}=\left[\mathbf{e}_z,\nabla\frac{\partial F}{\partial h_1}\right],$$

(here $\mathbf{e}_z$ is a unit vector perpendicular to the xy plane) and the coordinates φ_2 and φ_3 can arbitrarily be assigned in the xy plane connecting them, for example, with the streamlines of planar flow. Here $h_2=h_3=0$.

We have thus shown that for any stationary flow, the first variation of the kinetic energy δE vanishes assuming conservation of the helicity of H and the freezing-in integral (2.4) under arbitrary variations of the velocity and Lagrangian coordinates. We now calculate the second variation δ^2L, as before without utilizing as yet the connection between variations of **V** and of φ_k, i.e., formally for arbitrary $\delta\mathbf{V}$ and $\delta\varphi_k$:

$$\delta^2L=\int_D\Bigg\{\frac{1}{2}\delta\mathbf{V}_1\delta\mathbf{V}_1+\delta\mathbf{V}_2(\mathbf{V}+\lambda\,\mathrm{rot}\,\mathbf{V})+\frac{\lambda}{2}\delta\mathbf{V}_1\,\mathrm{rot}\,\delta\mathbf{V}_1$$
$$+\sum_{k=1}^{3}\frac{1}{2}\left[\frac{\partial^2F}{\partial\varphi_k^2}(\delta\varphi_k)_1^2+\frac{\partial^2F}{\partial h_k^2}(\delta h_k)_1^2\right]$$
$$+\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2F}{\partial\varphi_j\,\partial h_k}(\delta\varphi_j)_1(\delta h_k)_1$$
$$+\sum\sum_{j\neq k}\Bigg(\frac{\partial^2F}{\partial\varphi_k\,\partial\varphi_j}(\delta\varphi_k)_1(\delta\varphi_j)_1$$
$$+\frac{\partial^2F}{\partial h_k\,\partial h_j}(\delta h_k)_1(\delta h_j)_1\Bigg)$$
$$+\sum_{k=1}^{3}\left[\frac{\partial F}{\partial\varphi_k}(\delta\varphi_k)_2+\frac{\partial F}{\partial h_k}(\delta h_k)_2\right]\Bigg\}dxdydz. \tag{4.5}$$

Here δ_{V1}, $(\delta\varphi_j)_1$ denote the parts of the variations which are linear in the small parameter τ and $\delta\mathbf{V}_2$ $(\delta\varphi_j)_2$ and $(\delta h_j)_2$ are the corresponding quadratic parts. Since the variations h_j and φ_j represent are divergences of some vectors in both orders of expansion [for h_j this follows from (2.10) while as far as φ_k is concerned we utilize here the relations (3.1c) and (3.3c)], the last sum in the (4.5) can be transformed, integrating by parts, which will result in substantial cancellations when we take (4.3) and (4.4) into account. Indeed, it follows from the definition of h that for both the "kinematic" (3.1) and the "dynamic" (3.3) variations

$$(\delta h_j)_2=\nabla([\delta\mathbf{V}_2,\nabla\varphi_j]+(\delta\varphi_j)_2\,\mathrm{rot}\,\mathbf{V}+(\delta\varphi_j)_1\,\mathrm{rot}\,\delta\mathbf{V}_1).$$

Then

$$\int_D\sum_{k=1}^{3}\frac{\partial F}{\partial h_k}(\delta h_k)_2dxdydz$$
$$=\int_D\Bigg[-\delta\mathbf{V}_2(\mathbf{V}+\lambda\,\mathrm{rot}\,\mathbf{V})-\sum_{k=1}^{3}\frac{\partial F}{\partial\varphi_k}(\delta\varphi_k)_2$$
$$-\sum_{k=1}^{3}(\delta\varphi_k)_1\left(\mathrm{rot}\,\delta\mathbf{V}_1,\nabla\frac{\partial F}{\partial h_k}\right)\Bigg]dx\,dy\,dz.$$

From this and (4.5) we obtain an expression for δ^2L equally valid for the variations (3.1) and (3.3):

$$\delta^2L=\delta^2\mathcal{F}+\int_D\Bigg[\frac{1}{2}(\delta\mathbf{V}_1\delta\mathbf{V}_1)+\frac{\lambda}{2}(\delta\mathbf{V}_1\,\mathrm{rot}\,\delta\mathbf{V}_1)$$
$$-\sum_{k=1}^{3}(\delta\varphi_k)_1\left(\mathrm{rot}\,\delta\mathbf{V}_1,\nabla\frac{\partial F}{\partial h_k}\right)\Bigg]dxdydz, \tag{4.6}$$

where the notation

$$\delta^2\mathcal{F}=\int_D\Bigg\{\sum_{k=1}^{3}\frac{1}{2}\left[\frac{\partial^2F}{\partial\varphi_k^2}(\delta\varphi_k)_1^2+\frac{\partial^2F}{\partial h_k^2}(\delta h_k)_1^2\right]$$
$$+\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2F}{\partial\varphi_j\,\partial h_k}(\delta\varphi_j)_1(\delta h_k)_1$$
$$+\sum\sum_{j\neq k}\Bigg[\frac{\partial^2F}{\partial\varphi_k\,\partial\varphi_j}(\delta\varphi_k)_1(\delta\varphi_j)_1$$
$$+\frac{\partial^2F}{\partial h_k\,\partial h_j}(\delta h_k)_1(\delta h_j)_1\Bigg]\Bigg\}dxdydz \tag{4.7}$$

is used. Hence $\delta^2\mathcal{F}$ characterizes the "convexity" of F as a function of its arguments, and in the $2D$ case it is the condition that $\delta^2\mathcal{F}$ be positive which yields the Rayleigh theorem. In the $3D$ case, the quadratic form (4.6) contains additional destabilizing terms which describe the interaction between perturbations of curl V and the basic flow. In particular, the last term in (4.6) describes the destabilizing role of bending of vortex filaments. Since this term is proportional to the first variation $(\delta\varphi_j)_1$, its contribution is quite different in the cases of the "kinematic" variations (3.1) and the "dynamic" ones (3.3). In the case of "kinematic" variations, the expression (4.6) coincides with the expression for δ^2E obtained earlier by Arnold.[4] In the present paper, as in Ref. 4, we have not succeeded in obtaining flows which would ensure the definiteness of $\delta^2 L$. However, taking account of the supplementary "integral of motion" in the form of a "sheet" imposed by constraints (3.3) on the form of the variations, allows us to obtain an interesting result whose validity is verified by direct calculations in (4.6) taking (3.3), (4.3), and (4.4) into account: every stationary flow of the form

$$\mathbf{V}=\left[\nabla\varphi,\nabla\frac{dF}{dh}\right], \tag{4.8}$$

where we have $(\text{curl}\,\mathbf{V},\nabla h)=0$, φ is frozen-in (cf. [2.8]), F is a function of $h=(\text{curl}\,\mathbf{V},\nabla\varphi)$ is stable provided

$$\frac{d^2F}{dh^2}>0. \tag{4.9}$$

We emphasize that we are dealing here with stability against $3D$ perturbations. We shall now verify the validity of conditions (4.9). For this purpose, we first note that (4.8) is a particular case of (4.3), (4.4) when F depends only on one of the scalars h_j, for example, $h_1=h$, and does not depend explicitly on h_2, h_3 and the Lagrangian coordinates φ_k. This choice is not contradictory, and it simply restricts the class of the flows under consideration, still leaving it sufficiently broad because $F(h)$ is arbitrary. In accordance with (3.3c)

$$(\delta\varphi)_1=-\tau\mathbf{V}\nabla\varphi=\tau\lambda h=0,$$

since in (4.8) $\lambda=0$ holds. Then (4.6) can be written in the form

$$\delta^2L=\int_D\left[\frac{1}{2}\delta\mathbf{V}_1\delta\mathbf{V}_1+\frac{1}{2}\frac{d^2F}{dh^2}(\delta h)_1{}^2\right]dx\,dy\,dz,$$

from which the sufficient condition (4.9) follows. Toroidal vortices, which in the cylindrical coordinates z, r, and ϑ are written as

$$\mathbf{V}=\left[\frac{1}{r}\mathbf{e}_\vartheta,\nabla\frac{dF}{dh}\right], \tag{4.10}$$

where $h=r^{-1}(\text{curl}\,\mathbf{V})$, may serve as an example of flows of the form (4.8). A specific choice of the function $F(h)$ assigns a map of streamlines in the rz plane and for the spatial distribution of h we have here the nonlinear problem:

$$h=\frac{1}{r^2}\frac{\partial^2}{\partial z^2}\left(\frac{dF}{dh}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{dF}{dh}\right)\right]. \tag{4.11}$$

In accordance with (4.9) the toroidal vortices (4.10) with $d^2F/dh^2>0$ are stable. Leaving open the problem of the existence of solutions of (4.11) localized in all directions with $d^2F/dh^2>0$, we present an example of a stable solution of (4.11) in the form a cylindrical imbedded jet with a Gaussian velocity profile (**a** and **b** are numerical factors)

$$V_z=\frac{a}{b}\exp\left(-\frac{r^2}{2}b\right), \tag{4.12}$$

which corresponds to the choice $F=h^2/2b^2$, i.e., $d^2F/dh^2=1/b^2>0$ and the case $\partial/\partial z=0$.

Another example of flows (4.8) are planar flows. In accordance with the foregoing the following assertion is valid concerning their stability:

Every planar flow of an ideal fluid which is stable with respect to $2D$ perturbations of velocity, is also stable with respect to $3D$ perturbations. A certain connection between this assertion and the Squire theorem is worth noting.

5. CONCLUSION

In this paper we have attempted to answer a number of questions associated with the manifestation of the property of a "vector" freezing of the velocity curl of a $3D$ flow in the flow itself. Firstly, this property was formulated in terms of local freezing-in of certain scalar functions (or functionals) in the form (2.9) closely associated with the conservation of contravariant (in the Lagrange coordinates) components of the frozen-in vector. This allowed us, when searching for a conditional extremum of the kinetic energy of the flow, to account for conservation of vorticity in the $3D$ case by the method of undetermined Lagrange multipliers. To analyze the stability of flows, the notion of "dynamic" variations of the velocity field and the Lagrangian coordinates (3.3) was introduced. This corresponds to making allowance for conservation of one or more invariants which express the invariance of a system as such (it is assumed that the transition of the system from one state into another can be accomplished only in accordance with the motion equations, i.e., can only be evolutionary). Using these variations, we have found a sufficient condition for stability (4.9) of flows of the form (4.8) against $3D$ perturbations. This condition implies the Lyapunov stability of $2D$ flows with respect to $3D$ perturbations, provided their stability with respect to $2D$ perturbations is valid, as well as a sufficient condition for stability of toroidal vortices.

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