Electrodynamics of a slowly varying nonuniform plasma

V. S. Krivitskiĭ

Institute of General Physics, USSR Academy of Sciences
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The propagation of an electrostatic wave in a plasma whose density is varied slowly in time by some external source is investigated. It is shown that in such a system with variable composition the number of quanta is not an adiabatic invariant and is not conserved in time, so that wave damping (or amplification) is possible. The cause of nonconservation of the number of quanta in a nonstationary plasma is analyzed.

1. THE PROBLEM

We consider below a collisionless plasma described by the kinetic equation

$$\partial f_{\mathbf{p}}/\partial t+\mathbf{v}\partial f_{\mathbf{p}}/\partial\mathbf{r}-e\mathbf{E}\partial f_{\mathbf{p}}/\partial\mathbf{p}=Q(t,\ \mathbf{r},\ \mathbf{p}), \qquad (1)$$

where $f_{\mathbf{p}}$ is the electron distribution function and Q is an external source. We consider hereafter longitudinal electric fields, without a magnetic field in Eq. (1).

The number of particles in the source Q is assumed to vary with time:

$$\partial n/\partial t+\operatorname{div} n\mathbf{V}=q(t,\mathbf{r})\equiv\int d^3p\, Q(t,\mathbf{r},\mathbf{p})\neq 0, \qquad (2)$$

where the electron density is

$$n(t,\mathbf{r})=\int d^3p\, f_{\mathbf{p}}(t,\mathbf{r}), \qquad (3)$$

and the macroscopic velocity is

$$\mathbf{V}=n^{-1}\int d^3p\, \mathbf{v}f_{\mathbf{p}}. \qquad (4)$$

The source Q with time-dependent particle density can describe processes occurring in a plasma, such as ionization, recombination, and others.

For simplicity we confine ourselves below to a nonstationary but spatially homogeneous situation. We describe the source Q by

$$Q(t,\ \mathbf{r},\ \mathbf{p})=q(t)\delta(\mathbf{p}), \qquad (5)$$

i.e., we assume that the particles are created with zero momentum and at an identical rate at each point of the medium. To meet the condition that the plasma as a whole be electrically neutral we assume for the ion component a source identical with (5) in the right-hand side of the kinetic equation (whose only difference from (1) is that the ion charge $+e$ is positive).

The parameters describing our nonstationary medium with variable $n(t)$ are functions of the time t. In particular, the dielectric constant of the medium ε becomes time-dependent (we assume hereafter this dependence to be slow compared with the characteristic period of the wave propagating in the plasma, i.e., the source Q in (5) is in a certain sense a small quantity; more accurately speaking, we assume $\eta\ll 1$, where $\eta=\max\{1/\omega T,\ 1/\Delta\omega T\}$, ω is the frequency of the propagating wave, T is the characteristic time of variation of the parameters of our nonstationary system, and $\Delta\omega$ is the characteristic scale of the dispersion dependence of the dielectric constant of the medium). As first noted in Ref. 1, the time dependence of ε gives rise to an effective "supplementary" imaginary increment to the dielectric constant of the medium even if $\operatorname{Im}\varepsilon=0$. The appearance of such an imaginary additional contribution to the dielectric constant of a nonstationary medium leads to amplification (or damping) of the wave propagating in it. An investigation of this phenomenon (with an external source (5) that changes the number of particles) is in fact the subject of the present paper.

2. TIME DEPENDENCE OF THE DIELECTRIC CONSTANT

The general equation for the (linear) dependence of the induction of a longitudinal electric field $\mathbf{D}(t,\mathbf{r})$ on the intensity $\mathbf{E}(t,\mathbf{r})$ is

$$\mathbf{D}(t,\mathbf{r})=\int\frac{dt'\,d^3r'}{(2\pi)^4}\varepsilon(t,t';\mathbf{r},\mathbf{r}')\mathbf{E}(t',\mathbf{r}'); \qquad (6)$$

in a stationary spatially homogeneous medium, ε depends only on the differences $t-t'$ and $\mathbf{r}-\mathbf{r}'$:

$$\mathbf{D}(t,\mathbf{r})=\int\frac{dt'\,d^3r'}{(2\pi)^4}\varepsilon(t-t';\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}'), \qquad (7)$$

from which we have for the Fourier components

$$\mathbf{D}_{\omega\mathbf{k}}=\varepsilon_{\omega\mathbf{k}}\mathbf{E}_{\omega\mathbf{k}}, \qquad (8)$$

where $\mathbf{E}_{\omega\mathbf{k}}$ are the Fourier components of the field

$$\mathbf{E}(t,\mathbf{r})=\int d\omega\, d^3k\, \mathbf{E}_{\omega\mathbf{k}}\exp(-i\omega t+i\mathbf{k}\mathbf{r}) \qquad (9)$$

(and similarly for $\mathbf{D}$). The factor $(2\pi)^4$ was added in (6) and (7) for convenience—to eliminate "extra" factors 2π from (8).

In a stationary (and a spatially homogeneous, as before) medium the function ε in (7) should have besides the argument $\tau\equiv t-t'$ also a "slow" temporal argument describing the time dependence of the dielectric constant of the medium. Beginning with Ref. 1, this second argument is usually written in the form of the symmetric combination $(t+t')/2=t-\tau/2$; in the case of a slow dependence of ε on this argument, when expansion in the "short" time τ is possible in the second argument of the function $\varepsilon(\tau,t-\tau/2)$, the role of $\varepsilon_{\omega\mathbf{k}}$ is assumed by the quantity $\varepsilon_{\omega\mathbf{k}}(t)+(i/2)\partial^2\varepsilon_{\omega\mathbf{k}}(t)/\partial\omega\partial t$ (see Ref. 1), where

$$\varepsilon_{\omega\mathbf{k}}(t)=\int\frac{d\tau\,d^3\Delta r}{(2\pi)^4}\varepsilon(\tau,t;\Delta\mathbf{r})\exp(i\omega\tau-i\mathbf{k}\Delta\mathbf{r}). \qquad (10)$$

The second argument ε was chosen in Ref. 1 in the form

$(t+t')/2$ to maintain the number of quanta as an adiabatic invariant in time (more below).

Another approach to the problem however is possible. We forgo the phenomenological description of the electrodynamics of the medium using the function ε whose argument is determined by postulating the familiar quantum-number conservation theorem[2] in a weakly stationary medium with a variable number of particles. Instead, we can ascertain, by a detailed microscopic analysis of the dynamics of the medium (1), the actual time dependence of the dielectric constant ε, and then investigate the evolution of the number of quanta with time. The substantial influence of the specific type of microprocess producing polarization currents and charges in the medium on the character of the dependence of ε on the "slow" time was pointed out in Ref. 3. This analysis for a collisionless weakly turbulent plasma described by (1), performed in Ref. 4, yielded

$$\mathbf{D}(t,\mathbf{r})=\int\frac{dt'\,d^3r'}{(2\pi)^4}\varepsilon(t-t',t';\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}'), \tag{11}$$

where

$$\varepsilon_{\omega\mathbf{k}}(t)=1+\frac{4\pi e^2}{\mathbf{k}^2}\int d^3p\frac{1}{\omega-\mathbf{kv}+i0}\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Phi_{\mathbf{p}}(t), \tag{12}$$

where Φ_{p} is the unperturbed time-dependent electron distribution function.

Let us describe briefly the essence of the investigation,[4] on the basis of the kinetic equation (1), of the time dependence of ε. We resolve the distribution function $f_{\mathrm{p}}(t,\mathbf{r})$ into an unperturbed part $\Phi_{\mathrm{p}}(t)$ and a small perturbation $\delta f_{\mathrm{p}}(t,\mathbf{r})=f_{\mathrm{p}}-\Phi_{\mathrm{p}}$. The dependence of Φ_{p} is defined, according to (1), by the equation

$$\partial\Phi_{\mathbf{p}}/\partial t=Q, \tag{13}$$

and for δf_{p} we have, linearizing (1),

$$\partial\delta f_{\mathbf{p}}/\partial t+\mathbf{v}\partial\delta f_{\mathbf{p}}/\partial\mathbf{r}-e\mathbf{E}\partial\Phi_{\mathbf{p}}/\partial\mathbf{p}=0. \tag{14}$$

When resolving (1) into Eqs. (13 and (14) we have assumed that the external source Q has a "purely regular character," i.e., the source Q alters the unperturbed regular distribution function Φ_{p}, but has no random (fluctuating) component and does not act on δf_{p}. Of course, not all sources satisfy this requirement; under certain conditions, however, ionization can be such a regular source (in the case of ionization Q does not depend at all on the electron distribution function).

The solution of (14) is (following a Fourier transform and then its inverse)

$$\delta f_{\mathbf{p}}(t,\mathbf{r})=ie\int d\omega\,d^3k\frac{dt'\,d^3r'}{(2\pi)^4}\frac{\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')-i\omega(t-t')]}{\omega-\mathbf{kv}+i0}$$
$$\times\mathbf{E}(t',\mathbf{r}')\frac{\partial}{\partial\mathbf{p}}\Phi_{\mathbf{p}}(t'). \tag{15}$$

The pole in (15) is bypassed in accordance with the casualty principle.

Finding next the polarization $\mathbf{P}$ of the medium (from $\operatorname{div}\mathbf{P}=e\int d^3p\,\delta f_{\mathrm{p}}$) and the induction $\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$, we arrive at relations (11) and (12).

The second argument of ε in (11) is $t'=t-\tau$ and is not equal to the one chosen in Ref. 1. The resultant "effective" imaginary increment $i\partial^2\varepsilon/\partial\omega\partial t$ to the dielectric constant differs therefore by a factor of two.

Formally, of course, any function ε of two arguments can be written in the form of another new function of an arbitrary combination of these two arguments, but the choice of the arguments $t-t'\equiv\tau$ and t' for ε in (12) is not simply arbitrary: we see that according to (15) this choice is mathematically correct in the sense that it ensures factorization of (15) into a product of functions that depend only on the fast time τ and a slow function of $t'=t-\tau$. It is this natural separation of the "fast" and "slow" time dependences which makes possible the expansion in terms of small τ in a function that depends on the "slow" argument $t'=t-\tau$.

The result (15), the obvious difference notwithstanding, does not by itself contradict, generally speaking, the phenomenological theory of Ref. 1: after specifying in Eq. (1) the nature of the quantity Q that is responsible for the nonstationary behavior of the medium, we must take into account, besides the ensuing increment $i\partial^2\varepsilon/\partial\omega\partial t$, also the "direct" contribution of Q to $\operatorname{Im}\varepsilon$, i.e., that not due to the nonstationary behavior (if this contribution exists). Under the conditions of Ref. 1, when the dynamics of the system is described by a Hamiltonian $H(\lambda)$ and with a slowly varying parameter $\lambda=\lambda(t)$, the simultaneous allowance for these two contributions to the total "effective" dielectric constant can lead to conservation of the number of quanta (as was the case, for example, in Ref. 4, where Q corresponded to quasilinear interaction). However, by no means do all types of source correspond to the conditions of Ref. 1 (and conserves correspondingly the number of quanta); in particular, the source (5) which we consider here does not conserve the number of quanta (more below).

In this and following sections we consider only the electron contribution to the dielectric constant, assuming the ions to be infinitely heavy and to make no contribution to ε. The role of the ions reduces here to neutralization of the electron charge that increases in accordance with (13), so as to keep the medium as a whole electrically neutral. On the other hand, we do not consider at all an equation similar to (14) for the perturbed part of the ion distribution function, and assume that $m_i\gg m_e$; the subscript e labeling the quantities f_{p}, n, $\mathbf{V}$, etc., is omitted for simplicity. (In principle, the analysis of the ion motion is perfectly analogous to that of the electron component and can be easily carried out.)

3. CHANGE OF THE AMPLITUDE

Consider a longitudinal Langmuir wave propagating in a nonstationary plasma having the dielectric constant (12). Let the unperturbed distribution function Φ_{p} be isotropic, and let the characteristic velocity $\mathbf{v}$ of the particles of the distribution Φ_{p} be much lower than the phase velocity of the wave, $kv\ll\omega$. There is then no Landau damping, i.e., no imaginary part of ε connected with the pole of $\omega-\mathbf{k}\cdot\mathbf{v}=0$ in (12), and the evolution of the wave amplitude is determined only by the effects connected with the time-dependent character of the medium.

We express the wave field, in the geometric-optics approximation, in the form

$$\mathbf{E}(t,\mathbf{r})=\mathbf{E}_0\exp\left\{i\mathbf{kr}-i\int^t[\omega(t')+i\gamma(t')]dt'\right\}$$
$$=\mathbf{E}_0(t)\exp\left[i\mathbf{kr}-i\int^t\omega(t')dt'\right], \tag{16}$$

where the field **E** of the longitudinal wave is parallel to **k**. The wave amplitude $\mathbf{E}_a(t) \equiv \mathbf{E}_0 \exp(\int \gamma dt)$, which can be chosen to be real, and the frequency $\omega(t)$ both vary slowly with time. We substitute (16) in (11) and write down the wave propagation condition: $\mathbf{D} \equiv 0$ (there are no external charges or currents). We represent $\omega(t)$ and $\gamma(t)$ by expansions in powers of the small parameter η indicative of the time-dependent character of the system:

$$\omega = \omega^{(0)} + \omega^{(1)} + \ldots, \quad \gamma = \gamma^{(1)} + \gamma^{(2)} + \ldots, \quad \omega^{(j)} \sim \eta^j, \quad \gamma^{(j)} \sim \eta^j.$$

In the zeroth approximation in η, the equation $\mathbf{D} = 0$ (11) yields a dispersion equation relating the wave frequency to the wave vector:

$$\varepsilon_{\omega^{(0)}, \mathbf{k}}(t) = 0. \tag{17}$$

We expand the quantities $\mathbf{E}_a$, ω, and ε that depend on the slow time argument $t - \tau$ in powers of the small quantity τ and gather terms of order η^1. We find then that the change in the frequency is $\omega^{(1)} = 0$ and the rate of change of the wave amplitude is

$$\gamma^{(1)}(t) = \left[\frac{1}{2} \frac{\partial^2 \varepsilon_{\omega\mathbf{k}}(t)}{\partial \omega^2} \frac{\partial \varepsilon_{\omega\mathbf{k}}(t)}{\partial t} \left(\frac{\partial \varepsilon_{\omega\mathbf{k}}(t)}{\partial \omega} \right)^{-2} - \frac{\partial^2 \varepsilon_{\omega\mathbf{k}}(t)}{\partial \omega \partial t} \left(\frac{\partial \varepsilon_{\omega\mathbf{k}}(t)}{\partial \omega} \right)^{-1} \right]_{\omega = \omega^{(0)}(t)} \tag{18}$$

(see Ref. 4 for details).

In the case of interest to us, of small $\mathbf{k}$ ($kv \ll \omega$), we obtain from (12), (17), and (10) the dielectric constant

$$\varepsilon_{\omega\mathbf{k}}(t) = 1 - \omega_p^2(t)/\omega^2, \tag{19}$$

$$\omega_p^2(t) = 4\pi e^2 n_0(t)/m, \tag{20}$$

$$n_0(t) = \int d^3p \, \Phi_\mathbf{p}, \tag{21}$$

where $m \equiv m_e$ is the electron mass. Thus, the wave propagation frequency is equal in first order to the electron plasma frequency, $\omega(t) = \omega_p(t)$, and the rate of change of the amplitude $E_a(t) \equiv |\mathbf{E}_a(t)|$ is

$$\dot{E}_a(t)/E_a(t) = \gamma(t) = -\dot{\omega}(t)/2\omega(t) = -\dot{n}_0(t)/4n_0(t). \tag{22}$$

These relations can be simply verified without any use of the concept of dielectric constant of a nonstationary medium. In the limit of small **k** the wave propagation can be described with the aid of the hydrodynamics equations, the first of which is Eq. (2), which takes the following form after resolving n into a sum of an unperturbed density n_0 and a small perturbation δn in the wave field:

$$dn_0(t)/dt = q(t), \tag{23}$$

$$\partial \delta n/\partial t + n_0 \operatorname{div} \mathbf{V} = 0. \tag{24}$$

The source $q(t)$ enters only in (23) but not in (24) (see above).

Multiplying the kinetic equation (1) by **v** and integrating over d^3p, we obtain a second hydrodynamic equation:

$$\partial(nV_i)/\partial t + \partial \Pi_{ij}/\partial r_j = -(e/m) n E_i, \tag{25}$$

where $\Pi_{ij} = \int v_i v_j f_\mathbf{p} d^3p$. Linearizing (25), we can replace n in the left- and right-hand sides by n_0; as for $\partial \Pi_{ij}/\partial r_j$, in the limit when the (thermal) spread of the particle velocities in the unperturbed distribution $\Phi_\mathbf{p}$ this term is much less than the perturbation phase velocity ω/k and can be left out of (25) (this can be verified by obtaining Π_{ij} from the solution $\delta f_\mathbf{p}$ of the kinetic equation). Equation (25) takes thus the form

$$\partial(n_0 \mathbf{V})/\partial t = -(e/m) n_0 \mathbf{E}. \tag{26}$$

Note that the second hydrodynamic equation we use must be just (26) and not a linearized Euler equation (without a term describing the pressure, since $kv \ll \omega$): $\partial \mathbf{V}/\partial = -(e/m)\mathbf{E}$, since n_0 depends on t. In fact, Eq. (25) is equivalent to the Euler equation only when the continuity equation holds; in our case, however, the continuity equation contains in the right-hand side the source $q(t)$, and when (25) is reduced to the Euler form it will likewise contain an additional term with $q(t)$; the form (26) is therefore more suitable.

The set (24) and (26) of equations of cold collisionless hydrodynamics, together with the equation

$$\operatorname{div} \mathbf{E} = -4\pi e \delta n \tag{27}$$

describes the propagation of a longitudinal perturbation in a time-dependent plasma. On the basis of this system, in analogy with the previous kinetic treatment, we can obtain Eq. (11) with the dielectric constant (19). We shall derive (22) disregarding the concept of dielectric constant altogether.

We differentiate (24) with respect (22) and expressing δn in terms of **E** according to (27) and $\partial(n_0 \mathbf{V})/\partial t$ according to (26), obtain an equation for **E**:

$$\partial^2 \mathbf{E}/\partial t^2 = -\omega_p^2(t) \mathbf{E} \tag{28}$$

(the same equation holds for δn but not for **V**).

Substitution of (16) reduces (28) to

$$-(\omega + i\gamma)^2 - i(\dot{\omega} + i\dot{\gamma}) = -\omega_p^2(t), \tag{29}$$

whence, in the zeroth approximation in η, we obtain $\omega(t) = \omega_p(t)$, and in the next approximation $\gamma(t) = -\dot{\omega}(t)/2\omega(t)$, which corresponds to (22).

4. WAVE ENERGY

The energy W of the electric field in a dispersive medium averaged over the period of the oscillations, is given by the well known expression (see, e.g., Ref. 5)

$$\overline{W} = (1/16\pi) E_a^2(t) \partial(\omega \varepsilon_{\omega\mathbf{k}})/\partial\omega.$$

A natural generalization of this equation to the case of a slowly varying medium can be the expression frequently used by many workers

$$\overline{W} = \left. \frac{\partial(\omega \varepsilon_{\omega\mathbf{k}}(t))}{\partial \omega} \right|_{\omega = \omega(t)} \frac{E_a^2(t)}{16\pi} \tag{30}$$

(to determine the energy we must change over to the real field $\mathbf{E}_{\text{real}}$; we assume that $\mathbf{E}_{\text{real}}$ is given by the real part of (9): $\mathbf{E}_{\text{real}} = \operatorname{Re}\mathbf{E} = (\mathbf{E} + \mathbf{E}^*)/2$). Actually, however, it is not so simple to define the concept of wave energy in the general case of a nonstationary and/or dissipative dispersive medium (more below). Strictly speaking, relation (30) is valid only in the case of a stationary medium with $\operatorname{Im}\varepsilon = 0$, when it yields little information because it does not describe the dynamics (growth or damping) of the wave energy.

Let us consider this question in greater detail. For the dielectric constant (19), the energy (30) is equal to

$$\overline{W}=E_a^2(t)/8\pi. \tag{31}$$

The energy obtained above by means of the dynamic description n developed above

$$W=mn\mathbf{V}^2/2+\mathbf{E}^2/8\pi \tag{32}$$

(to the required accuracy we must put $n = n_0$, in the first term since $\delta n\mathbf{V}^2$ is proportional to the third power of the perturbation) agrees with (31) when averaged over the oscillation period $2\pi/\omega$. In fact, in the set of equations (24), (26), (27) only two, (24) and (26), are dynamic; the third, (27), contains no differentiation with respect to time and describes a connection. Eliminating it with the aid of δn, we obtain a set of two equations, (26) and

$$\partial\mathbf{E}/\partial t=4\pi e n_0\mathbf{V} \tag{33}$$

for the two quantities $\mathbf{E}$ and $\mathbf{V}$. Expressing $\mathbf{V}$ in terms of $\mathbf{E}$ with the aid of (33) and substituting to the required accuracy in (32), we get

$$\overline{W}=\frac{mn_0}{2}\overline{\left[\frac{1}{2}\left(\frac{-i\omega\mathbf{E}}{4\pi e n_0}+\text{c. c.}\right)\right]^2}+\frac{1}{8\pi}\overline{\left(\frac{\mathbf{E}+\mathbf{E}^*}{2}\right)^2}=\frac{|\mathbf{E}|^2}{16\pi}\left(\frac{\omega^2 m}{4\pi e^2 n_0}+1\right)=\frac{|\mathbf{E}|^2}{8\pi}=\frac{|\mathbf{E}_a|^2}{8\pi}=\frac{E_a^2}{8\pi} \tag{34}$$

[we use the equality $\omega(t) = \omega_p(t)$].

We note in passing that to this accuracy ($\sim\eta^0$) the energy (32) is equal to $E_a^2/8\pi$ even without averaging over the time: it is easily seen that to this accuracy the oscillating terms from the first and second components of (32) are mutually canceled, just as the nonoscillating contributions from these two components add up in (34).

We define the number N of the quanta (in units of $\hbar$) as

$$N=\frac{\overline{W(t)}}{\omega(t)}=\frac{\partial\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega}\bigg|_{\omega=\omega(t)}\frac{E_a^2(t)}{16\pi} \tag{35}$$

[we have used (17)].

The energy (30) [or (31)] is not conserved in time. Differentiating (31) we have, with allowance for (22),

$$\overline{W}^{-1}d\overline{W}/dt=2\gamma=-\dot{n}_0/2n_0. \tag{36}$$

The number N of the quanta is likewise not conserved in time. Differentiating the general expression (30) [with account taken of (17) and (18)], we find that

$$\frac{1}{N}\frac{dN}{dt}=\left[-\frac{\partial^2\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega^2}\frac{\partial\varepsilon_{\omega\mathbf{k}}(t)}{\partial t}\left(\frac{\partial\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega}\right)^{-2}+\frac{\partial^2\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega\partial t}\times\left(\frac{\partial\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega}\right)^{-1}\right]_{\omega=\omega(t)}+2\gamma(t)=-\frac{\partial^2\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega\partial t}\left(\frac{\partial\varepsilon_{\omega\mathbf{k}}(t)}{\partial\omega}\right)^{-1}\bigg|_{\omega=\omega(t)}. \tag{37}$$

If the second argument of ε were not t' but the average $(t+t')/2$, as in Ref. 1, the second term in (18) would be twice as large, and in place of (37) we would get $dN/dt = 0$, i.e., the number of quanta as defined by (35) would be constant in time.

In the limit $kv\ll\omega$, expression (37) reduces to

$$2\Gamma\equiv N^{-1}dN/dt=-2\dot{\omega}/\omega=-\dot{n}_0/n_0, \tag{38}$$

which coincides, naturally, with the result of differentiating (35) with the energy (31). (The factor 2 has been introduced into the definition (38) of Γ to agree with the results of Refs. 4 and 6, and also because of Γ and γ are equal in the case of a stationary medium, when the growth of the amplitude is due to the nonzero Im ε).

We interpret the nonconservation of the number N of the quanta (35) as the presence in the plasma of a parametric instability capable of amplifying (damping) the Langmuir wave. This nonconservation of N does not contradict the general theorem of Ref. 2. In fact, let us rewrite the set of hydrodynamic equations (26) and (33) with new symbols:

$$\mathbf{J}\equiv(4\pi)^{1/2}e n_0\mathbf{V},\quad \mathbf{G}\equiv\mathbf{E}/(4\pi)^{1/2}. \tag{39}$$

In this notation, the system (26) and (33) takes the form

$$\dot{\mathbf{J}}=-\omega_p^2(t)\mathbf{G},\quad \dot{\mathbf{G}}=\mathbf{J}, \tag{40}$$

and the energy W (32) is equal to

$$W=[\mathbf{J}^2/\omega_p^2(t)+\mathbf{G}^2]/2. \tag{41}$$

The theorem[2] on the adiabatic invariance of the number N of quanta is not valid in this case because the energy W (in a system with variable particle composition) is not the Hamiltonian for Eqs. (40); in other words, $\mathbf{G}$ and $\mathbf{J}$ are not canonically conjugate variables (coordinate and momentum). In fact, if (41) is taken to be the Hamiltonian for the "coordinate" $\mathbf{G}$ and the "momentum" $\mathbf{J}$ (or vice versa) the equations of motion will be not (40) but $\mathbf{J} = \pm\mathbf{G}$ and $\mathbf{G} = \pm\mathbf{J}/\omega_p^2(t)$ (the + or − sign depends on which is the coordinate and which is the momentum).

The system (40) conserves the "phase volume" $d^3J d^3G$ and a Hamiltonian can be chosen for it (more below); it will not, however, coincide with the energy (41).

The wave energy in a time-dependent system, generally speaking, is a rather difficult concept and cannot be unambiguously defined (at least in a linear theory). We shall not analyze here the various aspects of this question and the various approaches to the definition of the wave energy (see, e.g., Ref. 7 and the literature therein). If the wave energy W is defined by (31) as above, then relations (36) and (38) describe correctly the time dependence of the wave energy W and of the number of quanta $N = W/\omega$.

The following arguments favor the choice of (31) as the quantity corresponding to the wave energy: first, the wave energy (30) is then defined by a natural generalization of the expression for the energy in a stationary medium; second, one can reduce to the same expression the energy (32) written in the dynamic description of the medium in terms of the variables $\mathbf{E}$ and $\mathbf{V}$.

The total energy U of the system is the sum of the kinetic energy of the particle motion and the electric-field energy:

$$U=\int d^3p\, f_p m\mathbf{v}^2/2+\mathbf{E}^2/8\pi, \tag{42}$$

where f_p should be the solution of the kinetic equation (1).

As the unperturbed-distribution temperature vanishes, $T_e \to 0$, the first nonvanishing contribution is made to (42) by the unperturbed-distribution-function correction $\delta f_{\mathbf{p}}^{(1)} \equiv \delta f_{\mathbf{p}}$ (15) that follows the first correction linear in the field and is proportional to the square of the field $\mathbf{E}$. This correction $\delta f_{\mathbf{p}}^2$ can be calculated by a standard iterative solution of the kinetic equation (1) by perturbation theory. $\delta f_{\mathbf{p}}^{(2)}$ is expressed in terms of $\delta f_{\mathbf{p}}^{(1)}$ of (15) just as the latter is expressed in terms of $\Phi_{\mathbf{p}}$, using the same equation (14); the Fourier component of $\delta f_{\mathbf{p}}^{(2)}$ is given by

$$\delta f_{\mathbf{p},\omega\mathbf{k}}^{(2)}$$

$$= \frac{ie}{\omega - \mathbf{kv}} \int d\omega_1 \, d^3k_1 \, d\omega_2 \, d^3k_2 \left(\mathbf{E}_{\omega_2\mathbf{k}_2}, \frac{\partial}{\partial \mathbf{p}} \right) \frac{ie}{\omega - \omega_2 - (\mathbf{k} - \mathbf{k}_2)\mathbf{v}} \times \left(\mathbf{E}_{\omega_1\mathbf{k}_1}, \frac{\partial}{\partial p} \right) \Phi_{\mathbf{p},\omega-\omega_1-\omega_2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}. \qquad (43)$$

Substitution (to the required accuracy) of the solution of the kinetic equation in expression (42) for the energy does not yield the result (32). The difference Δ between the kinetic energies "in kinetics" and "in hydrodynamics"

$$\Delta \equiv U - W = \int d^3p \, f_{\mathbf{p}} m\mathbf{v}^2/2 - mn\mathbf{V}^2/2, \qquad (44)$$

differs from zero if the medium is not stationary (i.e., if we have $n_0 \neq 0$). Substitution of the solution (43) of the kinetic equation (1) in (44) leads in the case of a "cold" $\Phi_{\mathbf{p}}$ to the following result for the time variation of $\Delta(t)$:

$$\frac{d\Delta(t)}{dt} = \frac{m}{2} \frac{dn_0(t)}{dt} \mathbf{V}^2. \qquad (45)$$

Recognizing that $mn_0 \overline{\mathbf{V}^2}/2$ is half the wave energy $\overline{W}$ [see (34)], and that dn_0/dt can be expressed in terms of γ in accordance with (22), we can rewrite (45) in the form

$$d\bar{\Delta}/dt = 2\gamma \overline{W}. \qquad (46)$$

As seen from (36) and (46), the total energy $\overline{U} = \overline{W} + \overline{\Delta}$ is conserved, $d\overline{U}/dt = 0$, a perfectly natural result because $d(\int d^3rU)/dt$ (the integration with respect to r is over the final normalization volume of the system) is equal to the work A performed (per unit time) by the external source. In turn, it can be shown by differentiating (42) with respect to time that the work A is equal to $m\mathbf{v}^2/2 - e\varphi) \cdot Qd^3pd^3r$ plus an analogous expression for ions, with opposite sign of the charge and with omission of the kinetic-energy term, since we have $m_i \gg m_e \equiv m$; the ion kinetic energy was left out of (42) for the same reasons (φ is the potential for the field $\mathbf{E}$:$\mathbf{E} = -\operatorname{grad}\varphi$). The terms with the potential cancel out in the expression for A, since the charges of the electron and ion are opposite, and the external source generates equal numbers of electrons and ions; on the other hand, the term with the kinetic energy, $\int (m\mathbf{v}^2/2) Qd^3pd^3r$, is equal to zero for a source (5) that generates particles with zero momentum.

Let us find also the electromagnetic-field energy in a nondissipative dispersive nonstationary medium described phenomenologically with the aid of the dielectric constant (11). We determine the field energy in the medium by starting with the general equation

$$\partial U/\partial t = (1/4\pi) \mathbf{E} \partial \mathbf{D}/\partial t, \qquad (47)$$

in which the connection between $\mathbf{D}$ and $\mathbf{E}$ is given by (11). It is important that the expression for the field energy U obtained from (47) also depends on the character of the relation (11), i.e., on the specific dynamics of the microprocesses that polarize the medium in an electric field. It can be found that Eq. (47) for the "slow" time argument of the function ε in the form t' [Eq. (11)] gives rise to an expression that differs from the "extrapolation of the equation for the stationary case" (30):

$$\overline{U} = \overline{W} + \overline{\Delta}, \quad \frac{d\overline{\Delta(t)}}{dt} = \frac{E_a^2(t)}{16\pi} \frac{\partial^2(\omega\varepsilon_{\omega\mathbf{k}}(t))}{\partial\omega\,\partial t} \bigg|_{\omega=\omega(t)} \qquad (48)$$

(the intermediate steps have been left out). The quantity $\overline{W}$ in (48) is given by (30); (48) was derived by substituting in (47) the electric field in the form of the real part of (16) (but not necessarily corresponding to the natural-mode field satisfying the equation $\mathbf{D} = 0$).

The energy U (48) of the field in the medium actually agrees with the energy of the system (42), also designated by U and calculated in the kinetic description. This can be easily verified by substituting (19) and (48) and comparing with (45). Incidentally, the conservation of U (48) in the case of the wave field is obvious at any rate, since $\mathbf{D}(t) \equiv 0$ for the wave and (47) reduces to the identity $\partial U/\partial t \equiv 0$. (We note in passing that this statement is valid for any form of the time dependence of ε in (6), so that for the choice (in Ref. 1) of the slow argument in the form $(t + t')/2$ the expression (47) likewise does not yield (30), since the latter depends on the time and is proportional to $\omega(t)$ under the conditions of Ref. 1).

We emphasize once more in this connection that the choice of any expression to describe the wave energy in a dispersive nonstationary and/or dissipative medium is somewhat arbitrary. Indeed, in the context of (47) we shall always have for the wave energy $\dot{U} = 0$ and hence $U = \text{const}$. Thus, for example, for Landau damping (or for any nonlinear wave interaction etc.) the right-hand side of (47) will contain two terms that add up to zero: one proportional to $\dot{U} = 0$ and the other to $U = \text{const}$. At the same time, it is natural to regard as the wave energy not the (constant) sum U the quantity W (30) (regarding their difference $\Delta = U - W$ as part of the plasma-particle energy). Equation (47) will mean then that the rate of change $d|\mathbf{E}_a|^2/dt$ of the wave energy is equal to the rate, proportional to $\operatorname{Im} \varepsilon \cdot |\mathbf{E}_a|^2$, of energy transfer from the wave to the plasma particles—it is this which constitutes the natural interpretation of Landau damping.

The energy U can be expressed in terms of the wave amplitude:

$$U = \frac{E_a^2(t)}{8\pi} \frac{\omega(t)}{\omega(t_0)} = \text{const}, \qquad (49)$$

where t_0 is the instant when the external source (5) is turned on: $q(t) = 0$ at $t < t_0$, $q(t) \neq 0$ at $t > t_0$. The energy U is conserved in time and, like W, is not a Hamiltonian for Eqs. (40). Accordingly, the ratio $M \equiv U/\omega(t)$, i.e., "the number of quanta calculated for the energy U," is also not an adiabatic invariant, since $U = \text{const}$ and $\omega(t) = \omega_p(t) \neq \text{const}$; Thus, M varies like $M(t) \propto \omega_p^{-1}(t) \propto n_0^{-1/2}(t)$.

The choice of U as the quantity indicative of the wave energy is less desirable than the choice of W, for a number of reasons, some of which were mentioned above [W is the energy (32) expressed using the dynamic description (26) and (33)]; in addition, the total system energy U does not describe the dynamics of the wave variation in time, since $U=$ const independently of the evolution of the amplitude $E_a(t)$.

Expressing (naturally, quite arbitrarily) U as a sum of the wave energy W and of the quantity Δ (43), we conclude that in the course of variation of $E_a(t)$ the wave exchanges energy with the particles in the medium. Furthermore, since U is constant, a decrease of $W(t)$ leads to an increase of $\Delta(t)$ (and vice versa), where $\Delta(t)$ is the "supplementary" kinetic energy of the medium not included in the energy of the "collective" wave motion (32). This energy exchange $W \rightleftarrows \Delta$ between the wave and the medium is reversible and differs from dissociation in ordinary absorbing media. The quantity Δ (more accurately, $\dot{\Delta}$) is also proportional to the square of the wave amplitude.

Note, finally, that if we assume the wave energy to be

$$H(t)=\frac{E_a^2(t)}{8\pi}\frac{\omega^2(t)}{\omega^2(t_0)}, \tag{50}$$

then H will be, for the "coordinate" **G** and "momentum" **J** [after rescaling the latter by a factor $\omega(t_0)$], the Hamiltonian that gives rise to Eqs. (40). Accordingly, the new "number of quanta" $H(t)/\omega(t)$ will be conserved [this number of quanta is equal, accurate to within a factor $\omega(t_0)$, to the conserved energy U]. It is difficult, however, to advance any arguments in favor of using (50) to describe the wave energy.

5. CONCLUSION

We have considered here the propagation of a Langmuir wave in a slowly varying plasma with a regular external source (5) that alters in the course of time the number of particles in the system. The main result of the investigation are relations (36) and (38), which show the variation of the wave energy W and of the number of quanta N with time.

We have considered above a time-dependent but spatially homogeneous medium. The methods developed can be naturally generalized to include the case of weak spatial inhomogeneity of the medium.

Results similar to (36) and (38) can be obtained also for the propagation of electromagnetic radiation in a nonstationary plasma. Omitting the intermediate calculations, we present the final result for a transverse electromagnetic wave: under the same conditions as above, with an external particle source (5), the change of the number of quanta N_{em} is given in the limit $v \ll \omega/k (\gtrsim c)$ by the relation

$$2\Gamma\equiv\dot{N}_{em}/N_{em}=-4\pi\dot{n}_0e^2/m\omega^2 \tag{51}$$

[the dispersion of the electromagnetic wave is given by $\omega^2(t)=k^2c^2+\omega_p^2(t)$.]

We make one more remark concerning the assumed "regular character" of the external source Q. According to this assumption, Q enters in Eqs. (13) and (23) for the unperturbed part of the distribution function but does not affect $\delta f_{\mathbf{p}}$ (or δn). Such a source can correspond to ionization ($\dot{n}_0>0$), when Q may be altogether independent of $f_{\mathbf{p}}$. As seen from (38), however, great interest attaches to the case of recombination, when $\dot{n}_0<0$ and $\Gamma>0$, i.e., the wave is enhanced. The recombination of charged particles is not described by such a "regular" source, since for recombination the value of $\partial n/\partial t$ depends on the particle density n itself, and accordingly also contains a component that must be taken into account in Eq. (24) for the perturbation. In the simplest case, recombination can be described with the aid of a source

$$dn_{e,i}/dt=-\beta n_e n_i, \tag{52}$$

where β is a constant (see, e.g., Ref. 8). One can verify, however, that by writing down the collisionless cold hydrodynamics equations linearized in small perturbations for the electron and ion components, with the source (52), we obtain for $\delta_\rho\equiv\delta n_e-\delta n_i$ (we assume as above an ion charge $z=1$) and for $\mathbf{u}\equiv\mathbf{V}_e-\mathbf{V}_i$ a system of equations fully analogous to (24), (26), and (27), with the natural replacement of $m_e\equiv m$ by $\mu\equiv m_em_i/(m_e+m_i)\approx m_e$. The main results (36) and (38) can therefore be used also to describe recombination in a plasma.

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