Modulational instability of two coupled waves

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The modulational interaction of two modes is analyzed in the particular case of plasma waves. It is shown that this problem can be formulated rigorously only in certain special cases. The modulational instability of two coupled waves is distinguished from that of a single wave in that interference terms arise even in zeroth order and cause a time variation of the spectrum. The appearance of an instability threshold for broad wave spectra is ascribed to interference terms.

1. INTRODUCTION

The modulational instability or, more precisely, modulation interactions play an important role in plasma physics. Although this instability was originally described more than 25 years ago[1] (at about the same time that Gaĭlitis showed on the basis of energy principles that the occurrence of the modulational instability is "advantageous" under certain conditions[2]), and although the topic has since then been the subject of several reviews[3-5] (and, more generally, an extremely large number of papers), several aspects of interactions of this type have yet to be finally resolved, in our opinion. In particular, there is the question of the appearance of a threshold for the instability when the initial waves have broad spectra.

It is well known that a monochromatic wave is *always* unstable from the modulational standpoint, while broad wave spectra introduce a threshold for the occurrence of this instability. In the WKB approximation, this threshold is determined by the following relation (Ref. 3, for example) in the case of plasma waves:

$$\int \frac{W_k\, dk}{k^2 r_{de}^2} > 12 n_e T_e, \tag{1}$$

where W_k is the spectral energy density of the waves, $W = \int dk W_k$, $r_{de} = (T_e/4\pi n_e e^2)^{1/2}$ is the electron Debye length (with the unperturbed density n_e), and T_e is the electron temperature. A similar expression for the threshold was derived by Gaĭlitis[2] on the basis of energy considerations. On the other hand, a different expression has been given for this threshold in several papers (e.g., Ref. 6):

$$\frac{W}{\Delta k^2 r_{de}^2} > 12 n_e T_e. \tag{2}$$

This expression is obviously different from (1) (Δk is the width of the spectrum of plasma waves).

One way to take a step away from the case of the modulational instability of a monochromatic wave in the direction of the instability of broad wave spectra might be to examine a model of the modulational instability of two monochromatic waves. For definiteness, we will discuss plasma waves. [In general, the harmonics making up the spectra in the case of broad wave spectra would have random phases, since expressions (1) and (2) were derived for weak-turbulence conditions.] We would like to analyze the appearance of the threshold in this case and compare the resulting expression with (1) and (2). This formulation of the problem is also of interest in its own right, because of possible applications to beat-wave acceleration (BWA), since the plasma wave in BWA is excited by two electromagnetic waves, and a modulational instability of the latter would obviously impede attempts to achieve maximum acceleration rates. McKinstrie and Bingham[7] offered the first analysis of this problem for the BWA application. In the present paper we will not touch on those aspects of modulational interactions, but we will bear in mind that results on the modulational instability of plasma waves may be applicable, with certain changes, to situations pertinent to BWA.

2. FORMULATION OF THE PROBLEM

The very formulation of the problem of the modulational instability of two waves requires some refinement (for definiteness, we will discuss plasma waves in a homogeneous and isotropic plasma). Before we begin the stability analysis of some state of a physical system, we need to "have" this state, and it must be a steady state (in a certain sense of the word). A stability analysis can also be carried out for so-called quasisteady states, in which the (expected) rise time of some possible instability is much shorter than the time scale of variations in the state of the system without this instability. In this situation, the interaction of the waves is described by an equation which is cubic in the fields,[3]

$$\varepsilon_k E_k^+ = \int \Sigma^{eff}_{1,2,3} E_{k_1}^+ E_{k_2}^+ E_{k_3}^- d_{1,2,3}, \tag{3}$$

where

$$d_{1,2,3} = dk_1\, dk_2\, dk_3\, \delta(k - k_1 - k_2 - k_3),$$

$$k = (\mathbf{k}, \omega), \qquad E_k^{\pm} = \frac{(\mathbf{k}\mathbf{E}_k)}{|\mathbf{k}|}$$

is the Fourier component of the positive-frequency (or negative-frequency) component of the plasma-wave field

$$\mathbf{E}_k^{\pm} = \int \frac{d\mathbf{r}\, dt}{(2\pi)^4} \mathbf{E}(\mathbf{r}, t) \exp(\mp i\omega_{pe} t + i\omega t - i\mathbf{k}\mathbf{r}), \tag{4}$$

and

$$\Sigma^{eff}_{1,2,3} = \Sigma^{eff}(k, k_1, k_2, k_3)$$

is the effective third-order plasma response, which is symmetric under the interchange $k_2 \leftrightarrow k_3$ and which incorporates effects of the interaction through virtual waves (longitudinal and, in general, transverse) at a low frequency $\omega - \omega_1$ and at the doubled frequency $\omega - \omega_3$. Finally, ε_k is the linear longitudinal dielectric constant of the plasma.

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As is well known,[3-5] Eq. (4) has an "equilibrium" solution in the case of a single monochromatic plasma wave (in this case we will not take up the natural questions of the history of the "turning on" of the wave fields and so forth[3]), whose amplitude is spatially uniform and oscillates in time:

$$E^{+}(t)=E_0 \exp\left(i\frac{\omega_{pe}t}{2}\Sigma_0^{eff}\,|E_0|^2\right). \tag{5}$$

[For simplicity, expression (5) has been written for $\mathbf{k}_0=0$, but it is clear that the expression for the equilibrium solution for $\mathbf{k}_0\neq 0$ will not differ in any fundamental way.] In a study of the stability of small deviations from this solution, we obtain the usual dispersion relations, which yield the growth rates of the modulational instability. In the particular case (but a case which holds for plasma waves)

$$\Sigma_0^{eff}=\text{const}\,(\mathbf{k},\omega)\,|_{\omega=\omega_{pe},\mathbf{k}=\mathbf{k}_0},$$

the change caused in the wave frequency by the nonlinearity of the plasma can be "assigned to" a renormalized density (the same throughout the plasma volume) $\tilde{n}_e = n_e(1+\delta n/n_e)$:

$$\frac{\delta n}{n_e}=-\Sigma_0^{eff}\,|E_0|^2. \tag{6}$$

The steady-state nature (or quasi-steady-state nature) of the density perturbation $\delta n/n_e$ justifies our calling the solution (5) a "steady state" or "quasisteady state" of the system.

Let us examine the modulational instability of two monochromatic plasma waves (with frequencies ω_0 and ω_1, $\omega_0\neq\omega_1$). In this case, so-called interference terms arise in the expression for $\delta n/n_e$ [since the superposition principle does not hold for the nonlinear equation (3)]:

$$\frac{\delta n}{n_e}=-\Sigma_0^{eff}|E_0|^2-\Sigma_1^{eff}|E_1|^2-\Sigma_{0,1}^{eff}E_0E_1^{*}e^{-i\delta\omega t}$$
$$-\Sigma_{1,0}^{eff}E_1E_0^{*}e^{i\delta\omega t}, \tag{7}$$

where $\delta\omega\equiv\omega_0-\omega_1$. It is thus generally not possible even to correctly formulate the problem of the stability of two monochromatic waves with respect to modulational perturbations, since the spectrum specified in this manner is not steady, with a restructuring time scale on the order of $\delta\omega^{-1}$. In certain special cases, such an analysis can nevertheless be carried out.

For example, since the nonlinear responses $\Sigma_{0,1}^{eff}$ and $\Sigma_{1,0}^{eff}$ depend on the angle between the propagation directions of the plasma waves with ω_0 and ω_1 (for electromagnetic waves, this would be a dependence on the polarizations of the waves with ω_0 and ω_1), in the case $E_0\cdot E_1=0$ we have $\Sigma_{0,1}^{eff}=\Sigma_{1,0}^{eff}=0$. In this case, the interference terms drop out of (7), and the formulation of the problem of the modulational instability of two monochromatic waves is not fundamentally different from that for a single monochromatic wave.

We might also assume that the phases of the waves ω_0 and ω_1 are random and that only the density variation averaged over phase, $\langle\delta n/n_e\rangle$, influences the interaction of these waves. In this case, however, we could no longer regard each of the high-frequency waves as monochromatic, because each would be smeared over a frequency interval τ_{corr}^{-1}, where τ_{corr} is the time scale of the phase disruption of the wave. In this case the problem essentially reduces to a study of the interaction of two wave packets, but under the condition $\tau_{corr}^{-1}\ll\delta\omega$, γ, where γ is the instability growth rate, the waves can be regarded as "nearly monochromatic." Finally, the exact expression (7) might be replaced by some approximate expression in a situation in which the instability develops rapidly, i.e., with $\gamma\gg\delta\omega$. Assuming then that we are interested in the behavior of the system at times $\tau\sim\gamma^{-1}$, we might replace the exponential function in (7) by unity and study the instability of the quasisteady spectrum specified in this manner. It is clear on the basis of general considerations that the nature of the modulational instability would be analogous to that of the instability of a single "slightly smeared" mode.

All the examples listed above are valid for nondecay situations, in which $\delta\omega$ does not coincide with the frequency of a natural low-frequency mode of the medium (in the case at hand, these would be ion acoustic waves; for two electromagnetic waves they might be plasma waves in addition to ion acoustic waves). It is well known[8] that two plasma waves excite ion sound if

$$|\delta\omega|>\omega_s\sim|\delta\mathbf{k}|v_s;\quad \delta\mathbf{k}=\mathbf{k}_0-\mathbf{k}_1, \tag{8}$$

where $v_s=(T_e/m_i)^{1/2}$ is the ion acoustic velocity, and m_i is the ion mass. In this case, it is also necessary to introduce the fields of the resonantly excited sound. In this formulation, the original state is not a steady state simply because, at resonance, the amplitude of the low-frequency (ion acoustic) wave which is excited depends on the time by virtue of the effect of the high-frequency (plasma) waves. If we set $\gamma\tau_{dec}\gg 1$, however, where the time scale of the decay $l\to l+s$ is[8]

$$\tau_{dec}\sim\omega_{pe}^{-1}\frac{n_eT_e}{W}\frac{m_i}{m_e}\mathbf{k}_0^2r_{de}^2 \tag{9}$$

(we are assuming $|\mathbf{k}_0|\sim|\mathbf{k}_1|$, $|E_0|\sim|E_1|$), then the initial state can be regarded as quasisteady, at least from this point of view. In this case it is necessary either to derive a theory of modulational interactions involving preexisting ion sound or, under the assumption that the ion acoustic waves have not yet had time to appear, go over to a nonresonant analysis.

In this paper we examine the modulational interactions of two monochromatic plasma waves in the absence of ion sound (taking into account the comments above regarding the formulation of the problem). In the three situations which we examine, the focus is on the conditions under which an instability threshold arises. In the Conclusion we present some arguments which we believe cast light on the reason for the appearance of a threshold in the case of "smeared" wave spectra.

3. ZEROTH APPROXIMATION AND GENERAL FORM OF THE EQUATION FOR MODULATIONAL PERTURBATIONS

Under the assumption that decays are forbidden, let us examine the modulational interaction of two monochromatic waves which is described by Eq. (3). In the zeroth approximation we set

$$E_k^{\pm}=\pm E_0^{(*)}\delta(k\mp k_0)\pm E_1^{(*)}\delta(k\mp k_1). \tag{10}$$

Substituting (10) into (3), we find

$$\varepsilon_0 E_0\delta(k-k_0)+\varepsilon_1 E_1\delta(k-k_1)=-\Sigma_{0,0,0,-0}|E_0|^2E_0\delta(k-k_0)$$
$$-\Sigma_{0,0,1,-1}|E_1|^2E_0\delta(k-k_0)-\Sigma_{0,1,0,-1}|E_1|^2E_0\delta(k-k_0)$$
$$-\Sigma_{2\cdot 0-1,0,0,-1}(E_0)^2E_1^*\delta(k-(2k_0-k_1))+(0\leftrightarrow 1), \tag{11}$$

where $\Sigma_{i,j,l,m}\equiv\Sigma^{eff}(k_i,k_j,k_l,k_m)$, $\varepsilon_i\equiv\varepsilon_{k_i}$, $i, j, l, m=0,1$. The last term on the right side of (10) (and also the term found from it through the interchange $0\leftrightarrow 1$) is an interference term and corresponds to beats in the density variation (7). In the case

$$\Sigma_{2\cdot 0-1,0,0,-1}=\Sigma_{2\cdot 1-0,1,1,-0}=0, \tag{12}$$

there are no such beats, and from (11) we find an equation for the steady state of the wave E_0,

$$\varepsilon_0+\Sigma_{0,0,0,-0}|E_0|^2+\Sigma_{0,1,0,-1}|E_1|^2+\Sigma_{0,0,1,-1}|E_1|^2=0, \tag{13}$$

and also, via the interchange $0\leftrightarrow 1$, the corresponding equation for E_1.

We now set $(\mathbf{E}_0\mathbf{E}_1)=0$. In other words, we assume that the waves $\mathbf{E}_0$ and $\mathbf{E}_1$ are propagating perpendicularly to one another. Using the fairly good approximation

$$\Sigma^{eff}_{1,2,3}=\frac{-1}{4\pi n_eT_e}\frac{1}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}_3|}$$
$$\times\Big\{(\mathbf{k}\mathbf{k}_1)(\mathbf{k}_2\mathbf{k}_3)\Big[\frac{(\mathbf{k}-\mathbf{k}_1)^2v_s^2}{(\omega-\omega_1)^2-(\mathbf{k}-\mathbf{k}_1)^2v_s^2}$$
$$+(\mathbf{k}-\mathbf{k}_1)^2r_{de}^2+((\mathbf{k}+\mathbf{k}_1)(\mathbf{k}_2-\mathbf{k}_3))r_{de}^2\Big]$$
$$+\frac{1}{3}r_{de}^2[(\mathbf{k}\mathbf{k}_3)(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}_1+\mathbf{k}_2)^2$$
$$-2(\mathbf{k}_1+\mathbf{k}_2)^{-2}(\mathbf{k}(\mathbf{k}_1+\mathbf{k}_2))(\mathbf{k}_1(\mathbf{k}_1+\mathbf{k}_2))$$
$$\times(\mathbf{k}_2(\mathbf{k}_1+\mathbf{k}_2))(\mathbf{k}_3(\mathbf{k}_1+\mathbf{k}_2))$$
$$-(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}(\mathbf{k}_1+\mathbf{k}_2))(\mathbf{k}_3(\mathbf{k}_1+\mathbf{k}_2))$$
$$-(\mathbf{k}\mathbf{k}_3)(\mathbf{k}_1(\mathbf{k}_1+\mathbf{k}_2))(\mathbf{k}_2(\mathbf{k}_1+\mathbf{k}_2))]\Big\} \tag{14}$$

for Σ (we are incorporating the electron nonlinearities,[9] $|\omega-\omega_1|\ll|\mathbf{k}-\mathbf{k}_1|v_{Te}, v_{Te}=(T_e/m_e)^{1/2}$), we can easily show that under the condition $\mathbf{k}_1\|\mathbf{k}_2\perp\mathbf{k}_3$ the value of $\Sigma^{eff}_{1,2,3}$ is zero. However, it would hardly be possible to make expressions (12) exactly equal to zero. In particular, the small terms of the next higher order in the electron nonlinearities, which are proportional to[10]

$$\mathbf{k}_0^4r_{de}^4\frac{|E_0|^2}{4\pi n_eT_e}, \tag{15}$$

may turn out to be nonzero. Their contribution, however, *can* be assumed small enough to be ignored (for example, at the level of the terms of fifth order in the field amplitude). Furthermore, in our model we might in principle have limited the discussion to, say, the Zakharov system, which (Refs. 3–5, for example) corresponds to a term

$$\frac{(\mathbf{k}-\mathbf{k}_1)^2v_s^2}{(\omega-\omega_1)^2-(\mathbf{k}-\mathbf{k}_1)^2v_s^2}$$

on the right side of (14), when (12) is satisfied exactly with $k_2\perp k_3$.

With $\mathbf{k}_0\perp\mathbf{k}_1$ we have [along with (12)]

$$\Sigma_{0,1,0,-1}\approx\Sigma_{1,0,1,-0}\approx 0 \tag{16}$$

within electron nonlinearities. Consequently, the contribution from, for example, the response $\Sigma_{0,1,0,-1}|E_1|^2$, which is on the order of

$$\mathbf{k}_1^2r_{de}^2\frac{|E_1|^2}{4\pi n_eT_e}\ll\frac{|E_1|^2}{4\pi n_eT_e}, \tag{17}$$

can definitely be ignored in comparison with, say, $\Sigma_{0,0,1,-1}|E_1|^2\sim|E_1|^2/4\pi n_eT_e$.

In the zeroth approximation, the solutions are thus

$$\varepsilon_0+\Sigma_{0,0,0,-0}|E_0|^2+\Sigma_{0,0,1,-1}|E_1|^2=0, \tag{18}$$

$$\varepsilon_1+\Sigma_{1,1,1,-1}|E_1|^2+\Sigma_{1,1,0,-0}|E_0|^2=0. \tag{19}$$

By virtue of the structure of the responses we can assume

$$\Sigma_{0,0,0,-0}=\Sigma_{0,0,1,-1}=\Sigma_{1,1,0,-0}=\Sigma_{1,1,1,-1}. \tag{20}$$

In general, Eqs. (20) depend on the way in which the fields E_0 and E_1 are "turned on," holding if the two fields are turned on in the same way (quasistatically, for example). Under conditions (20), the influence of nonlinear effects reduces in the zeroth approximation to simply a density renormalization of the form (6) (two waves are involved, of course), which is the same through the plasma volume and the same for each of the waves.

Let us examine the stability of a steady-state solution of Eqs. (18) and (19) with respect to small perturbations of the field. In place of (10) we adopt

$$E_k^{\pm}=\pm E_0^{(*)}\delta(k\mp k_0)\pm E_1^{(*)}\delta(k\mp k_1)+\delta E_k^{\pm}. \tag{21}$$

Substituting (21) into (3) and linearizing, we find the following equation (perturbations of the frequency, $\Delta\omega$, and of the wave vector, $\Delta\mathbf{k}$, associated with the modulational interaction have been introduced in the standard way[3]):

$$[\varepsilon_{\Delta+0}+2\Sigma^{symm}_{\Delta+0,\Delta+0,0,-0}|E_0|^2+2\Sigma^{symm}_{\Delta+0,\Delta+0,1,-1}|E_1|^2]\delta E^+_{\Delta+0}$$
$$+\Sigma_{\Delta+0,0,\Delta+1,-1}E_0E_1^*\delta E^+_{\Delta+1}+\Sigma_{\Delta+0,1,\Delta+0+\delta,-0}E_0^*E_1\delta E^+_{\Delta+0+\delta}$$
$$-\Sigma_{\Delta+0,0,0,\Delta-0}(E_0)^2\delta E^-_{\Delta-0}-2\Sigma^{symm}_{\Delta+0,0,1,\Delta-1}\delta E^-_{\Delta-1}E_0E_1$$
$$-\Sigma_{\Delta+0,1,1,\Delta-(1-\delta)}(E_1)^2\delta E^-_{\Delta-(1-\delta)}=0, \tag{22}$$

where

$$\Sigma^{symm}_{i,j,k,l}={}^1/_2(\Sigma_{i,j,k,l}+\Sigma_{i,k,j,l}),\qquad \delta E^+_{\Delta+0+\delta}\equiv\delta E^+_{\Delta k+k_0+\delta k},$$

etc.

To close the system of equations we need [in addition to (22)] equations containing $\varepsilon_{\Delta-0}$, $\varepsilon_{\Delta+1}$, $\varepsilon_{\Delta-1}$ and also $\varepsilon_{\Delta+0+\delta}$ and $\varepsilon_{\Delta-(1-\delta)}$. If it suddenly became necessary to add only equations which do not contain a dielectric constant with frequencies (and wave vectors) $\Delta k+(k_0+\delta k)$, $\Delta k-(k_1-\delta k)$, then the system would be closed. The dispersion relation for the modulational interaction would follow from the vanishing of the determinant of the (4×4) matrix of the equation coupling four perturbations: at the frequencies $\Delta\omega\pm\omega_0$ and $\Delta\omega\pm\omega_1$.

The real situation is more complicated, however. Even in Eq. (22) there are perturbations of the fields at the frequencies $\Delta\omega+(\omega_0+\delta\omega)$ and $\Delta\omega-(\omega_1-\delta\omega)$. Writing four equations containing $\varepsilon_{\Delta\pm 0}$ and $\varepsilon_{\Delta\pm 1}$, we see that perturbations arise in them at the frequencies $\Delta\omega\pm(\omega_0+\delta\omega)$

and $\Delta\omega \pm (\omega_1 - \delta\omega)$, for which we must also write corresponding equations. It ultimately turns out that in the course of the modulational interaction satellites are generated not only for modes with frequencies ω_0 and ω_1 but also for waves not present in the zeroth approximation (in our case with $\mathbf{k}_0 \perp \mathbf{k}_1$): at frequencies $\omega_0 + n\delta\omega$, $n = 1, \pm 2, \pm 3, \ldots$ (or, equivalently, $\omega_1 - n\delta\omega$, $n = 1, \pm 2, \pm 3, \ldots$, since by definition we have $\omega_0 - 2\delta\omega = \omega_1 - \delta\omega$, etc.)

We thus obtain a system of a (generally infinite) number of equations which describe an infinite number of coupled amplitudes (physically, however, it is clear that such a system will be finite, if only because the condition $\delta\omega \ll \omega_{0,1}$ holds by virtue of the dispersion relation of the plasma waves). The matrix of this system is block-diagonal:

$$A = \begin{pmatrix} \cdot\ \cdot\ \cdot\ \cdot & \cdot\ 0\ 0\ \cdot & & \\ \cdot\ \cdot\ \cdot\ \cdot & 0\ \cdot\ \cdot\ 0 & 0 & \cdots \\ \cdot\ \cdot\ \cdot\ \cdot & 0\ \cdot\ \cdot\ 0 & & \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ 0\ 0\ \cdot & & \\ \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \\ 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & \\ 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & \cdots \\ \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \\ & \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \\ & 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & \\ 0 & 0\ \cdot\ \cdot\ 0 & 0\ \cdot\ \cdot\ 0 & \cdots \\ & \cdot\ 0\ 0\ \cdot & \cdot\ 0\ 0\ \cdot & \\ \cdots & & \cdots & \cdots \end{pmatrix}. \tag{23}$$

The dots represent coefficients which are not zero. This matrix acts on the column

$$(\delta E^{+}_{\Delta+0}, \delta E^{-}_{\Delta-0}, \delta E^{+}_{\Delta+1}, \delta E^{-}_{\Delta-1}, \delta E^{+}_{\Delta+0+\delta}, \delta E^{-}_{\Delta-(0+\delta)}, \ldots)^T.$$

Finding the determinant of such a matrix in its general form is a rather difficult problem.

4. EQUATIONS OF THE MODULATIONAL INSTABILITY IN THE CASE OF PERPENDICULAR PROPAGATION OF THE PUMP WAVES

We first consider only those modulational perturbations which are directed perpendicular to both the wave with ω_0 and the wave with ω_1: $\Delta\mathbf{k} \perp \mathbf{k}_{0,1}$. Ignoring the electron nonlinearities and other corrections to the Zakharov response, i.e., assuming

$$|\Delta\mathbf{k}| v_{T_i} \ll |\Delta\omega| \ll |\Delta\mathbf{k}| v_{Te},$$

$$v_{T_i} = (T_i/m_i)^{1/2},$$

$$|\Delta\omega| \ll \omega_{pi} \equiv (4\pi n_e e^2/m_i)^{1/2},$$

by virtue of the presence of angular dependences of the form

$$\Sigma^{eff}_{1,2,3} = \frac{(\mathbf{k}\mathbf{k}_1)(\mathbf{k}_2\mathbf{k}_3)}{|\mathbf{k}|\,|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}_3|}$$

$$\times\ \Sigma_\Delta \equiv \frac{(\mathbf{k}\mathbf{k}_1)(\mathbf{k}_2\mathbf{k}_3)}{|\mathbf{k}|\,|\mathbf{k}_1|\,|\mathbf{k}_2|\,|\mathbf{k}_3|}\frac{1}{4\pi n_e T_e}\frac{\Delta k^2 v_s^2}{\Delta\omega^2 - \Delta k^2 v_s^2}, \tag{24}$$

in the responses we find (as in the zeroth approximation) that all the coefficients which contain field perturbations at frequencies $\omega_0 + n\delta\omega$, $n = 1, \pm 2, \pm 3, \ldots$, are zero. As a result, the modulational interactions "tie" only the satellites of the fields at the frequencies ω_0 and ω_1, and the matrix of the system becomes

$$A = \begin{pmatrix} \tilde{\varepsilon}_0^{+} & (\tilde{E}_0)^2, & -\tilde{E}_0\tilde{E}_1^* & \tilde{E}_0\tilde{E}_1 \\ (\tilde{E}_0^*)^2 & \tilde{\varepsilon}_0^{-} & \tilde{E}_0^*\tilde{E}_1^* & -\tilde{E}_0^*\tilde{E}_1 \\ -\tilde{E}_1\tilde{E}_0^* & \tilde{E}_0\tilde{E}_1 & \tilde{\varepsilon}_1^{+} & (\tilde{E}_1)^2 \\ \tilde{E}_0^*\tilde{E}_1^* & -\tilde{E}_0\tilde{E}_1^* & (\tilde{E}_1^*)^2 & \tilde{\varepsilon}_1^{-} \end{pmatrix}, \tag{25}$$

where

$$\tilde{E}^{(*)}_{0(1)} \equiv \left(\frac{\mathbf{k}^2_{0(1)}}{\Delta\mathbf{k}^2 + \mathbf{k}^2_{0(1)}}\Sigma_\Delta\right)^{1/2} E^{(*)}_{0(1)}, \tag{26}$$

$$\tilde{\varepsilon}_{0(1)} \equiv \varepsilon_{\Delta\pm0(1)} - |\tilde{E}_{0(1)}|^2. \tag{27}$$

The dispersion relation $\det A = 0$ is

$$1 = \Sigma_\Delta\left[\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2}\left(\frac{1}{\varepsilon_{\Delta+0}} + \frac{1}{\varepsilon_{\Delta-0}}\right) + (0 \leftrightarrow 1)\right], \tag{28}$$

which is a natural generalization of the dispersion relation which describes the modulational instability of a single monochromatic plasma wave:[3]

$$1 = \Sigma_\Delta\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2}\left(\frac{1}{\varepsilon_{\Delta+0}} + \frac{1}{\varepsilon_{\Delta-0}}\right). \tag{29}$$

In the case $\Delta\mathbf{k} \perp \mathbf{k}_{0,1}$ we have

$$\frac{1}{\varepsilon_{\Delta+0}} + \frac{1}{\varepsilon_{\Delta-0}} = \frac{6\Delta k^2 r_{de}^2}{4\dfrac{\Delta\omega^2}{\omega_{pe}^2} - 9\Delta k^4 r_{de}^4} = \frac{1}{\varepsilon_{\Delta-1}} + \frac{1}{\varepsilon_{\Delta+1}}, \tag{30}$$

and also, according to (24),

$$\Sigma_\Delta = \frac{1}{4\pi n_e T_e}\frac{\Delta k^2 v_s^2}{\Delta\omega^2 - \Delta k^2 v_s^2}. \tag{31}$$

From (28) we then find

$$(\Delta\omega^2 - \Delta k^2 v_s^2)\left(4\frac{\Delta\omega^2}{\omega_{pe}^2} - 9\Delta k^4 r_{de}^4\right)$$

$$= \frac{6\Delta k^4 r_{de}^2 v_s^2}{4\pi n_e T_e}\left(\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2} + \frac{\mathbf{k}_1^2|E_1|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_1^2}\right). \tag{32}$$

At the same time, we find from (29) (with $\Delta\mathbf{k} \perp \mathbf{k}_0$)

$$(\Delta\omega^2 - \Delta k^2 v_s^2)\left(4\frac{\Delta\omega^2}{\omega_{pe}^2} - 9\Delta k^4 v_{de}^4\right) = \frac{6\Delta k^4 r_{de}^2 v_s^2}{4\pi n_e T_e}\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2}. \tag{33}$$

In the very simple case $\Delta\mathbf{k} \perp \mathbf{k}_{0,1}$ (in which all the interference terms are zero), the modulational instability of two plasma modes thus differs from that of a single mode only in that the following change occurs in the dispersion relation:

$$\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2} \to \frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2} + \frac{\mathbf{k}_1^2|E_1|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_1^2}.$$

We see that the spectra $|E_0|^2$ and $|E_1|^2$ are effectively (in a sense) smeared out in k space (the same thing happens in the case of a single mode). Note that (as in the case of a single mode) no instability threshold arises, although the form of the solution of (33) [or, with a minor substitution, that of (32)],

$$\Delta\omega^2 = \frac{1}{2}\Delta k^2 v_s^2\left(1 + \frac{9}{4}\Delta k^2 r_{de}^2\frac{m_i}{m_e}\right)\cdot$$

$$\left\{1 \pm \left[1 + \frac{6m_i/m_e}{1 + \frac{9}{4}\Delta k^2 r_{de}^2\frac{m_i}{m_e}}\left(\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2 + \mathbf{k}_0^2} - \frac{3}{2}\Delta k^2 r_{de}^2\right)\right]^{1/2}\right\} \tag{34}$$

might lead one to believe that the condition

$$\frac{\mathbf{k}_0^2|E_0|^2}{\Delta\mathbf{k}^2+\mathbf{k}_0^2}\frac{1}{4\pi n_e T_e}=\frac{3}{2}\Delta\mathbf{k}^2 r_{de}^2 \tag{35}$$

determines a threshold. Actually, (35) is simply an equation from which we find the critical values of the wave vectors of the modulational perturbations (one for each pump level), i.e., the values above which there is no instability.

We turn now to the case in which the wave vector of the modulational perturbations, $\Delta\mathbf{k}$, is directed along the wave vector of one of the pump waves, e.g., $\mathbf{k}_0$. In this case we have a matrix of an infinite system of the form (23) (with minor simplifications). However, we will again use a 4×4 matrix to estimate the instability growth rates in this case. We should bear in mind that the modulational interactions studied in the preceding section of this paper are described exactly (within the range of applicability of this approach) by the matrix (25), while a similar analysis for the case $\Delta\mathbf{k}\|\mathbf{k}_0$ can be no more than an estimate, since in that case we are ignoring the satellites of the waves at the frequencies $\omega_0+n\delta\omega$, $n=1, \pm2, \pm3,\ldots$.

In the case $\Delta\mathbf{k}\|\mathbf{k}_0$, the matrix (25) is replaced by the following matrix (taking into account the above comments):

$$A=\begin{pmatrix} \tilde{\varepsilon}_0^+ & -(+\cdot-)(\tilde{E}_0)^2 & -(+)\tilde{E}_0\tilde{E}_1^* & (+)\tilde{E}_0\tilde{E}_1 \\ -(+\cdot-)(\tilde{E}_0^*)^2 & \tilde{\varepsilon}_0^- & -(-)\tilde{E}_0^*\tilde{E}_1^* & (-)\tilde{E}_0^*\tilde{E}_1 \\ -(+)(\tilde{E}_0^*\tilde{E}_1) & -(-)\tilde{E}_0\tilde{E}_1 & \varepsilon_1^+ & (\tilde{E}_1)^2 \\ (+)\tilde{E}_0^*\tilde{E}_1^* & (-)\tilde{E}_0\tilde{E}_1^* & (\tilde{E}_1^*)^2 & \tilde{\varepsilon}_1^- \end{pmatrix} \tag{36}$$

where for $\tilde{\varepsilon}_0^\pm$ and $E_1^{(0)}$ we have used the notation in (26), (27), and for $\tilde{\varepsilon}_1^\pm$ and $\widetilde{E}_0$ we have

$$\tilde{E}_0^{(*)}=(\Sigma_\Delta)^{1/2}E_0^{(*)}, \tag{37}$$

$$\tilde{\varepsilon}_1^\pm=\varepsilon_{\Delta\pm1}-|E_1|^2-\frac{\Delta\mathbf{k}^2|E_0|^2}{\Delta\mathbf{k}^2+\mathbf{k}_1^2}\Sigma_{\Delta\mp\delta},$$
$$\Sigma_{\Delta\mp\delta}=\frac{1}{4\pi n_e T_e}\frac{(\Delta\mathbf{k}\mp\delta\mathbf{k})^2 v_s^2}{(\Delta\omega\mp\delta\omega)^2-(\Delta\mathbf{k}\mp\delta\mathbf{k})^2 v_s^2}. \tag{38}$$

In addition, we have adopted the following notation in (36):

$$(\pm)=\frac{(\mathbf{k}^0(\Delta\mathbf{k}\pm\mathbf{k}_0))}{|\mathbf{k}_0|\,|\Delta\mathbf{k}\pm\mathbf{k}_0|},\qquad (+\cdot-)=(+)\cdot(-). \tag{39}$$

We see that a distinction from the case $\Delta\mathbf{k}\perp\mathbf{k}_0$ is that now the quantities $\varepsilon_1^\pm$ acquire an additional nonlinear frequency shift [see the last term on the right side of (38)]. Moreover, there is no smearing in $\mathbf{k}$ space for E_0 [see (37)].

The dispersion relation determined by the condition $\det A=0$, where A is given by (36), is

$$1=\Sigma_\Delta\left[|E_0|^2\left(\frac{1}{\varepsilon_{\Delta+0}}+\frac{1}{\varepsilon_{\Delta-0}}\right)+\frac{\mathbf{k}_1^2|E_1|^2}{\Delta\mathbf{k}^2+\mathbf{k}_1^2}\left(\frac{1}{\bar{\varepsilon}_{\Delta+1}}+\frac{1}{\bar{\varepsilon}_{\Delta-1}}\right)\right], \tag{40}$$

where

$$\bar{\varepsilon}_{\Delta\pm1}=\varepsilon_{\Delta\pm1}-\frac{\Delta\mathbf{k}^2}{\Delta\mathbf{k}^2+\mathbf{k}_1^2}\Sigma_{\Delta\mp\delta}|E_0|^2. \tag{41}$$

The nonlinear frequency shift due to the presence of a mode with a frequency ω_0 remains in the dielectric constant in the second term on the right side of (40). This equation is now considerably more complicated than (28). To study the (possible) near-threshold behavior, we assume that $|\Delta\omega|$ is smaller than all the other characteristic frequencies of the problem. From (40), (41) we then find

$$\Delta\omega^2=\Delta\mathbf{k}^2v_s^2\left\{1+\frac{1}{4\pi n_e T_e}\left[\frac{2|E_0|^2}{r_{de}^2(4\mathbf{k}_0^2-3\Delta\mathbf{k}^2)}-\frac{\mathbf{k}_1^2|E_1|^2}{\Delta\mathbf{k}^2+\mathbf{k}_1^2}\left(\frac{1}{3\Delta\mathbf{k}^2(r_{de}^+)^2}+\frac{1}{3\Delta\mathbf{k}^2(r_{de}^-)^2}\right)\right]\right\}, \tag{42}$$

where

$$(r_{de}^\pm)^2=r_{de}^2+\frac{1}{3}\frac{|E_0|^2}{\Delta\mathbf{k}^2+\mathbf{k}_1^2}\Sigma_{\Delta\pm\delta}. \tag{43}$$

It can be concluded from the form of (42) that at small values of $|\Delta\mathbf{k}|$, such that the second term is dominant in the expression in square brackets, this equation has imaginary solutions. We also note that the upper limit $|\Delta\mathbf{k}|_{\max}$ on the instability is of course set by the pump level, but actually one can always specify some $|\Delta\mathbf{k}|_{\max}$ so that an instability is possible under the condition $|\Delta\mathbf{k}|<|\Delta\mathbf{k}|_{\max}$, regardless of the pump level. We thus conclude that Eq. (42) is also free of an instability threshold.

5. EQUATION OF THE MODULATIONAL INSTABILITY IN THE CASE OF PARALLEL PROPAGATION OF THE PUMP WAVES

Consider the situation in which the two waves E_0 and E_1 are parallel: $\mathbf{k}_0\|\mathbf{k}_1$. In this case, as was mentioned above, interference terms arise even in the zeroth approximation, so we will restrict the discussion to the quasisteady case, $|\Delta\omega|\gg|\delta\omega|$. The "steady" state is now the following state, instead of the state defined by Eqs. (18) and (19):

$$\varepsilon_0+\Sigma_{0,0,0,-0}|E_0|^2+\Sigma_{0,1,0,-1}|E_1|^2+\Sigma_{0,0,1,1}|E_1|^2+\Sigma_{2\cdot0-1,0,0,-1}E_0E_1^*=0, \tag{44}$$

$$\varepsilon_1+\Sigma_{1,1,1,-1}|E_1|^2+\Sigma_{1,0,1,-0}|E_0|^2+\Sigma_{1,1,0,-0}|E_0|^2+\Sigma_{2\cdot1-0,1,1,-0}E_0^*E_1=0. \tag{45}$$

In this case it is generally no longer possible to write the nonlinear frequency shift in the zeroth approximation as a renormalization of the density, since there are terms

$$\Sigma_{0(1),1(0),0(1),-1(0)}|E_{1(0)}|^2=-\Sigma_\delta|E_{1(0)}|^2,$$

$$\Sigma_{2\cdot0(1)-1(0),0(1),-1(0)}E_{0(1)}E^*_{1(0)}=-\Sigma_\delta\frac{(\mathbf{k}_0\mathbf{k}_1)(\mathbf{k}_{0(1)}(2\mathbf{k}_{0(1)}-\mathbf{k}_{1(0)}))E_{0(1)}E^*_{1(0)}}{|\mathbf{k}_{0(1)}|^2|\mathbf{k}_{1(0)}|\,|2\mathbf{k}_{0(1)}-\mathbf{k}_{1(0)}|},$$

$$\Sigma_\delta=\frac{1}{4\pi n_e T_e}\frac{\delta\mathbf{k}^2v_s^2}{\delta\omega^2-\delta\mathbf{k}^2v_s^2}. \tag{46}$$

To simplify the discussion below, we assume $|\delta\omega|\ll|\delta\mathbf{k}|v_s$ (this condition is, generally speaking, the same as the requirement that ion sound not be excited) and also $|\mathbf{k}_0|>|\mathbf{k}_1|$. The vectors $\mathbf{k}_0$ and $\mathbf{k}_1$ are in the same direction (hence $|\delta\mathbf{k}|<|\mathbf{k}_1|$). Finally, we set $E_0=E_1$. In this case the nonlinear frequency shift can be assigned to a density renormalization (the same throughout the plasma volume):

$$\frac{\delta n}{n_e}\approx-4\Sigma_{0,0,0,-0}|E_0|^2. \tag{47}$$

Now examining perturbations of the zeroth-order fields, (21), and linearizing, we find an infinite system with a matrix whose structure is again that in (23). In this case, however, there is no need to ignore (as we did in the case $\Delta\mathbf{k}\|\mathbf{k}_0\perp\mathbf{k}_1$) the satellites of the beat waves. Using $|\Delta\omega|\gg|\delta\omega|$, and limiting the discussion to the first four rows in a matrix A of the form (23), we assume that these satellites correspond to the fundamental modes ω_0 and ω_1. In this case the satellite lying at a distance $\Delta\omega\pm\delta\omega$ from the fundamental wave is coupled specifically with the latter wave. We are thus in a sense "not distinguishing" frequencies which differ by $\delta\omega$, but we are distinguishing those which differ by $2\delta\omega$. We thus finally find the matrix

$$A=\begin{pmatrix} \tilde{\varepsilon}_0^+-|E_\Delta|^2 & -|E_\Delta|^2 & -|E_\delta|^2-|E_\Delta|^2 & -|E_+|^2-|E_\Delta|^2 \\ -|E_\Delta|^2 & \tilde{\varepsilon}_0^--|E_\Delta|^2 & -|E_-|^2-|E_\Delta|^2 & -|E_\delta|^2-|E_\Delta|^2 \\ -|E_\delta|^2-|E_\Delta|^2 & -|E_-|^2-|E_\Delta|^2 & \tilde{\varepsilon}_1^+-|E_\Delta|^2 & -|E_\Delta|^2 \\ -|E_+|^2-|E_\Delta|^2 & -|E_\delta|^2-|E_\Delta|^2 & -|E_\Delta|^2 & \tilde{\varepsilon}_1^--|E_\Delta|^2 \end{pmatrix}, \tag{48}$$

where

$$|E_{\Delta(\delta)}|^2=\Sigma_{\Delta(\delta)}|E_0|^2, \tag{49}$$

$$|E_\pm|^2=2\Sigma_{\Delta\pm\delta}|E_0|^2, \tag{50}$$

$$\tilde{\varepsilon}_0^\pm=\varepsilon_{\Delta\pm0}-|E_\pm|^2, \tag{51}$$

$$\tilde{\varepsilon}_1^\pm=\varepsilon_{\Delta\pm1}-|E_\mp|^2.$$

From (48) we find the dispersion relation

$$\begin{aligned}
1&-\frac{|E_-|^4}{\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^+}-\frac{|E_+|^4}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_1^-}-\frac{|E_\delta|^4}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_1^+}-\frac{|E_\delta|^4}{\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^-}
+\frac{(|E_\delta|^4-|E_+|^2|E_-|^2)^2}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^+\tilde{\varepsilon}_1^-}\\
&=|E_\Delta|^2\Big\{\frac{1}{\tilde{\varepsilon}_0^+}+\frac{1}{\tilde{\varepsilon}_0^-}+\frac{1}{\tilde{\varepsilon}_1^+}+\frac{1}{\tilde{\varepsilon}_1^-}
+\frac{2|E_+|^2}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_1^-}+\frac{2|E_-|^2}{\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^+}+\frac{2|E_\delta|^2}{\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^-}\\
&+\frac{2|E_\delta|^2}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_1^+}-\frac{(|E_\delta|^2-|E_-|^2)^2}{\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^-}\Big(\frac{1}{\tilde{\varepsilon}_0^+}+\frac{1}{\tilde{\varepsilon}_1^+}\Big)
-\frac{(|E_\delta|^2-|E_+|^2)^2}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_1^+}\Big(\frac{1}{\tilde{\varepsilon}_0^-}+\frac{1}{\tilde{\varepsilon}_1^-}\Big)\\
&+\frac{2}{\tilde{\varepsilon}_0^+\tilde{\varepsilon}_0^-\tilde{\varepsilon}_1^+\tilde{\varepsilon}_1^-}(2|E_\delta|^2-|E_+|^2-|E_-|^2)
\times(|E_\delta|^4-|E_+|^2|E_-|^2)\Big\}.
\end{aligned} \tag{52}$$

Equation (52) is extremely complex. Examining the evolution of the instability near the threshold, we consider (52) in the case of small values of $\Delta\omega$—smaller than nearly all the characteristic frequencies of the problem (except $|\delta\omega|$). In this case we have $|\Delta\omega|\ll|\Delta\mathbf{k}|v_s$ and

$$2|E_\delta|^2\approx|E_+|^2\approx|E_-|^2. \tag{53}$$

After some rather lengthy calculations we find the dispersion relation

$$\Delta\omega^2=\Delta\mathbf{k}^2v_s^2\Big(1-\frac{|E_0|^2}{4\pi n_eT_e}\frac{2}{3k_0k_1r_{de}^2}\Big), \tag{54}$$

in which we clearly see an instability threshold:

$$\frac{|E_0|^2}{4\pi n_eT_e}\Big|_{thr}=\frac{3}{2}k_0k_1r_{de}^2. \tag{55}$$

Note the similarity between expressions (55) and (35) [to within the "broadening" of the spectrum, $\sim\mathbf{k}_0^2/(\Delta\mathbf{k}^2+\mathbf{k}_0^2)$, which is present in (35) because of $\Delta\mathbf{k}\perp\mathbf{k}_0$].

6. CONCLUSION

Expression (55) for the instability threshold makes it possible to answer the question of which inequality, (1) or (2), gives a better description of the actual situation. To find this answer, we transform expression (1) (Ref. 3). The integral (1) is determined in the case of a flat spectrum W_k by the minimum possible wave number,

$$\int\frac{W_k\,dk}{k^2}\approx\frac{\text{const}}{k_{min}}, \tag{56}$$

while the total energy is determined by the maximum possible wave number of the spectrum,

$$W=\int W_k\,dk\simeq\text{const}\cdot k_{max}. \tag{57}$$

Condition (1) can thus be rewritten as

$$\frac{W}{n_eT_e}\Big|_{thr}\approx12r_{de}^2k_{min}k_{max}. \tag{58}$$

Noting that the total energy of the two plasma waves of amplitude E_0 is

$$\begin{aligned}W&=\frac{1}{4\pi}\int_{-\infty}^{t}\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}dt\\
&=\frac{1}{8\pi}\int dk\,dk'\frac{\partial}{\partial\omega}(\omega\varepsilon)\mathbf{E}_k\mathbf{E}_{k'}\exp(i(\mathbf{k}+\mathbf{k}')\mathbf{r}\\
&-i(\omega+\omega')t)=\frac{1}{2\pi}|2E_0|^2=\frac{2|E_0|^2}{\pi}\end{aligned} \tag{59}$$

in this case, we find an expression for the threshold from (55):

$$\frac{W}{n_eT_e}\Big|_{thr}=12k_0k_1r_{de}^2. \tag{60}$$

From the agreement of (58) and (60), to within a numerical factor, we can conclude that expression (1), which determines the threshold, is more accurate than expression (2). We should of course recall that threshold condition (55) [and, correspondingly, (58)] was derived under some extremely strong assumptions.

From this analysis we can also determine the reasons for the onset of a threshold for the development of the modulational instability. We believe that the presence of the threshold is intimately related to the interference terms which appear in both the zeroth approximation and the dispersion relation. Note, however, that the nonlinear frequency shift is not responsible for the threshold. In the case $\Delta\mathbf{k}\|\mathbf{k}_2\perp\mathbf{k}_1$ this shift is present in the expressions for ε [see (40) and (41)], but no threshold arises. The suppression of the instability in the latter case, which we ascribe to the presence of interference terms, may occur because the low-frequency satellite of one of the pump waves (E_0, say) oscillates out of phase with the high-frequency satellite of the other pump wave (E_1, say). A qualitative picture of this sort was previously suggested by Dendy and ter Haar,[6] but the quantitative analysis there, which seems to us to require some corrections, led to a less accurate expression for the threshold, of the type in (2).

We should also point out that even in the absence of interference terms in the zeroth approximation (in the case, for example, $\Delta\mathbf{k}\|\mathbf{k}_0\perp\mathbf{k}_1$, there is an infinite set of frequencies of the type $\Delta\omega + n\delta\omega$, $n = 1, \pm 2, \pm 3, \ldots$, in the modulational perturbations. In other words, satellites (formally, an infinite number of satellites) still appear on the "missing" frequencies $\omega_0 + n\delta\omega$.

In conclusion, we repeat that the situation which we have studied here can be regarded as no more than an approximate model (more or less adequate) for studying the modulational interactions of broad wave spectra—a topic of much interest in plasma physics and also other branches of physics (nonlinear optics, etc.).

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Electrodynamics of a slowly varying nonuniform plasma

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The propagation of an electrostatic wave in a plasma whose density is varied slowly in time by some external source is investigated. It is shown that in such a system with variable composition the number of quanta is not an adiabatic invariant and is not conserved in time, so that wave damping (or amplification) is possible. The cause of nonconservation of the number of quanta in a nonstationary plasma is analyzed.

1. THE PROBLEM

We consider below a collisionless plasma described by the kinetic equation

$$\partial f_{\mathbf{p}}/\partial t+\mathbf{v}\partial f_{\mathbf{p}}/\partial \mathbf{r}-e\mathbf{E}\partial f_{\mathbf{p}}/\partial \mathbf{p}=Q(t,\ \mathbf{r},\ \mathbf{p}), \tag{1}$$

where $f_{\mathbf{p}}$ is the electron distribution function and Q is an external source. We consider hereafter longitudinal electric fields, without a magnetic field in Eq. (1).

The number of particles in the source Q is assumed to vary with time:

$$\partial n/\partial t+\operatorname{div} n\mathbf{V}=q(t,\mathbf{r})\equiv\int d^3p\, Q(t,\mathbf{r},\mathbf{p})\neq 0, \tag{2}$$

where the electron density is

$$n(t,\mathbf{r})=\int d^3p\, f_{\mathbf{p}}(t,\mathbf{r}), \tag{3}$$

and the macroscopic velocity is

$$\mathbf{V}=n^{-1}\int d^3p\, \mathbf{v}f_{\mathbf{p}}. \tag{4}$$

The source Q with time-dependent particle density can describe processes occurring in a plasma, such as ionization, recombination, and others.

For simplicity we confine ourselves below to a nonstationary but spatially homogeneous situation. We describe the source Q by

$$Q(t,\ \mathbf{r},\ \mathbf{p})=q(t)\delta(\mathbf{p}), \tag{5}$$

i.e., we assume that the particles are created with zero momentum and at an identical rate at each point of the medium. To meet the condition that the plasma as a whole be electrically neutral we assume for the ion component a source identical with (5) in the right-hand side of the kinetic equation (whose only difference from (1) is that the ion charge $+e$ is positive).

The parameters describing our nonstationary medium with variable $n(t)$ are functions of the time t. In particular, the dielectric constant of the medium ε becomes time-dependent (we assume hereafter this dependence to be slow compared with the characteristic period of the wave propagating in the plasma, i.e., the source Q in (5) is in a certain sense a small quantity; more accurately speaking, we assume $\eta \ll 1$, where $\eta=\max\{1/\omega T,\ 1/\Delta\omega T\}$, ω is the frequency of the propagating wave, T is the characteristic time of variation of the parameters of our nonstationary system, and $\Delta\omega$ is the characteristic scale of the dispersion dependence of the dielectric constant of the medium). As first noted in Ref. 1, the time dependence of ε gives rise to an effective "supplementary" imaginary increment to the dielectric constant of the medium even if $\operatorname{Im}\varepsilon=0$. The appearance of such an imaginary additional contribution to the dielectric constant of a nonstationary medium leads to amplification (or damping) of the wave propagating in it. An investigation of this phenomenon (with an external source (5) that changes the number of particles) is in fact the subject of the present paper.

2. TIME DEPENDENCE OF THE DIELECTRIC CONSTANT

The general equation for the (linear) dependence of the induction of a longitudinal electric field $\mathbf{D}(t,\mathbf{r})$ on the intensity $\mathbf{E}(t,\mathbf{r})$ is

$$\mathbf{D}(t,\mathbf{r})=\int\frac{dt'\,d^3r'}{(2\pi)^4}\varepsilon(t,t';\mathbf{r},\mathbf{r}')\mathbf{E}(t',\mathbf{r}'); \tag{6}$$

in a stationary spatially homogeneous medium, ε depends only on the differences $t-t'$ and $\mathbf{r}-\mathbf{r}'$:

$$\mathbf{D}(t,\mathbf{r})=\int\frac{dt'\,d^3r'}{(2\pi)^4}\varepsilon(t-t';\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}'), \tag{7}$$

from which we have for the Fourier components

$$\mathbf{D}_{\omega\mathbf{k}}=\varepsilon_{\omega\mathbf{k}}\mathbf{E}_{\omega\mathbf{k}}, \tag{8}$$

where $\mathbf{E}_{\omega\mathbf{k}}$ are the Fourier components of the field

$$\mathbf{E}(t,\mathbf{r})=\int d\omega\, d^3k\, \mathbf{E}_{\omega\mathbf{k}}\exp(-i\omega t+i\mathbf{k}\mathbf{r}) \tag{9}$$

(and similarly for $\mathbf{D}$). The factor $(2\pi)^4$ was added in (6) and (7) for convenience—to eliminate "extra" factors 2π from (8).

In a stationary (and a spatially homogeneous, as before) medium the function ε in (7) should have besides the argument $\tau\equiv t-t'$ also a "slow" temporal argument describing the time dependence of the dielectric constant of the medium. Beginning with Ref. 1, this second argument is usually written in the form of the symmetric combination $(t+t')/2=t-\tau/2$; in the case of a slow dependence of ε on this argument, when expansion in the "short" time τ is possible in the second argument of the function $\varepsilon(\tau,t-\tau/2)$, the role of $\varepsilon_{\omega\mathbf{k}}$ is assumed by the quantity $\varepsilon_{\omega\mathbf{k}}(t)+(i/2)\partial^2\varepsilon_{\omega\mathbf{k}}(t)/\partial\omega\partial t$ (see Ref. 1), where

$$\varepsilon_{\omega\mathbf{k}}(t)=\int\frac{d\tau\, d^3\Delta r}{(2\pi)^4}\varepsilon(\tau,t;\Delta\mathbf{r})\exp(i\omega\tau-i\mathbf{k}\Delta\mathbf{r}). \tag{10}$$

The second argument ε was chosen in Ref. 1 in the form

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