Self-consistent theory of the generation of vortex structures in a plasma with an anisotropic pressure under the conditions of the Weibel instability

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A theory of the generation of self-consistent structures is derived for a plasma with an anisotropic pressure. The theory is derived from the equations of rotational anisotropic electron hydrodynamics. Corresponding to this model is a qualitatively new type of vector nonlinearity. Conditions for the applicability of this model are formulated; its consequences in the case of large-scale vortex flows are analyzed. For the nonlinear analog of the Weibel instability, the equations of this model reduce to a system of equations for the magnetic field vector and the vector representing the anisotropy of the electron pressure. The degree of pressure anisotropy is conserved as the vortex motions evolve, and there is a local rotation of the anisotropy vector. The existence of toroidal vortex structures corresponding to θ-pinches, z-pinches, and magnetic-helix structures is pointed out.

1. The dynamic theory of vortex electron structures in plasmas plays an important role in current research on the nonlinear dynamics of continuous media and on synergetics.

Rotational electron flows and magnetic fields are known to arise as the result of an electromagnetic instability of a temperature-anisotropic electron energy distribution.[1] An analytic study of the nonlinear regime of the vortex structures which arise in the course of the Weibel instability has become possible only recently, however, thanks to a new model for the hydrodynamic description of anisotropic electron plasmas which was proposed in Refs. 2–5. This model is based on equations for the first ten moments of the distribution function. Corresponding to these equations of rotational anisotropic electron hydrodynamics is a qualitatively new type of vector nonlinearity [see Eqs. (22) and (39) below].

Our purposes in the present paper are, first, to determine the conditions under which this new model is applicable and, second, to analyze its consequences in the case (of practical importance) of large-scale flows, with a length scale greater than the electromagnetic length c/ω_{Le}, where c is the velocity of light, and ω_{Le} is the electron plasma frequency.

As we will see below, electron plasma flows simplify considerably when the irrotational components of the electric field and the electron velocity are small in comparison with the rotational components. The equations derived to relate the irrotational components to the rotational components in this case make it possible to write out explicit conditions for the applicability of rotational anisotropic electron hydrodynamics.

We will identify a class of rotational motions which correspond to a nonlinear analog of the Weibel instability. For these motions, the equations of this new model reduce to a system of equations for two vectors: the magnetic field and the anisotropy of the electron pressure. As these rotational motions evolve, the degree of anisotropy of the electron pressure is conserved, while there is a local rotational of the anisotropy vector as time elapses.

Spatially localized toroidal θ-pinch or z-pinch vortex structures may arise. For such states, the pressure-anisotropy vector always remains perpendicular to the magnetic field as the two evolve in time. Magnetic-helix structures can develop.

We will show that self-similar solutions of the equations of rotational anisotropic electron hydrodynamics describing large-scale flows lead to explosive growth of the components of the magnetic field and the velocity in a certain bounded spatial region. This behavior is universal and is essentially independent of the magnetic field structure and the flow geometry. The explosive singularity corresponds to a decrease in the length scale of the vortices and to a transition to a short-wave regime, in which collisionless dissipation may render the particle distribution isotropic.

2. Restricting the discussion to the plasma dynamics in the case in which the ion motion can be ignored, we start from a hydrodynamic description of the electrons in the ten-moment approximation. We use the continuity equation for the electron density n, the equation of motion for the average electron velocity $\mathbf{u}$, and the equation for the component P of the stress tensor:

$$\frac{\partial n}{\partial t}+\operatorname{div} n\mathbf{u}=0,\quad \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}[\mathbf{uB}]\right)-\frac{\nabla P}{mn},$$
$$\frac{\partial P}{\partial t}+(\mathbf{u}\nabla)P+\{(P\nabla)\mathbf{u}\}=\frac{e}{mc}\{[P\mathbf{B}]\},\qquad (1)$$

where e and m are the charge and mass of an electron, the braces mean that the corresponding tensor are symmetrized ($\{A_{ij}\}=A_{ij}+A_{ji}$), $[P\mathbf{B}]$ is a tensor with the components $[P\mathbf{B}]_{ij}=e_{ikl}P_{jk}B_l$, and ∇ is a vector with the components $(\nabla P)_i=\partial P_{ij}/\partial x_j$. Equations (1) must be supplemented with Maxwell's equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$:

$$\frac{\partial \mathbf{B}}{\partial t}=-c\operatorname{rot}\mathbf{E},\quad \operatorname{rot}\mathbf{B}=\frac{4\pi en}{c}\mathbf{u}+\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$
$$\operatorname{div}\mathbf{E}=4\pi e(n-n_0),\qquad (2)$$

where n_0 is the ion density. Equations (1) and (2) describe both irrotational and rotational motions. It is the rotational flow in which we will be interested below in connection with our study of the nonlinear stage of the Weibel instability.

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We write the velocity and the electric field as sums of irrotational components $\mathbf{u}_l$, $\mathbf{E}_l$ (curl $\mathbf{u}_l = 0$, curl $\mathbf{E}_l = 0$), and rotational components $\mathbf{u}_t$, $\mathbf{E}_t$ (div $\mathbf{u}_t = 0$, div $\mathbf{E}_t = 0$). Assuming $|\mathbf{u}_l| \ll |\mathbf{u}_t|$, ignoring the contribution fo $\mathbf{u}_l$ to the equations of motion for $\mathbf{u}_t$ and the equations for P, and also ignoring the change in the density ($|\delta n/n| = |(n-n_0)/n_0| \ll 1$), we find the following system of equations in the case in which the displacement current is negligible:

$$\frac{\partial}{\partial t}\left(\mathbf{\Omega} - \frac{c^2}{\omega_{Le}^2}\Delta\mathbf{\Omega}\right) = \frac{c^2}{\omega_{Le}^2}\text{rot}\left[(\mathbf{\Omega}\nabla)\mathbf{\Omega} - \frac{c^2}{\omega_{Le}^2}(\text{rot}\,\mathbf{\Omega}, \nabla)\,\text{rot}\,\mathbf{\Omega}\right] - \frac{1}{mn}\text{rot}(\nabla P),$$
$$\frac{\partial P}{\partial t} + \frac{c^2}{\omega_{Le}^2}(\text{rot}\,\mathbf{\Omega}, \nabla)P + \frac{c^2}{\omega_{Le}^2}\{(P\nabla)\,\text{rot}\,\mathbf{\Omega}\} = \{[P\mathbf{\Omega}]\}, \quad (3)$$

where $\Omega = e\mathbf{B}/mc$ is the electron gyrofrequency, and $\omega_{Le} = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency. Linearizing Eqs. (3), we easily see that they describe the Weibel instability for wavelengths $\gtrsim c/\omega_{Le}$ with a growth rate corresponding to the kinetic theory of Ref. 6.

In a previous formulation of the model of rotational anisotropic electron hydrodynamics,[1,2] its range of applicability was not discussed in detail. We will accordingly first determine the conditions under which the irrotational components of the field and the velocity have no substantial effect of the rotational motions which we will be discussing. For this purpose we denote by l and τ characteristic length and time scales of the variation in the rotational motions. To ignore the displacement current $\partial \mathbf{E}_t/\partial t$ (in comparison with the conduction current) is to restrict our discussion to fairly slow motions, with a velocity scale $v_p = l/\tau$ which is much lower than the velocity of light. By ignoring the convection $u_l\nabla$ (in comparison with $\partial/\partial t$), we are assuming that the irrotational component of the velocity is small in comparison with v_p. By ignoring the component of the Lorentz force $(e/c)[\mathbf{u}_l\mathbf{B}]$ (in comparison with the electric force $e\mathbf{E}_l$) we are assuming that u_l is small in comparison with $v_p\,(c/\omega_{Le} l)^2$. Putting all this together, we have the following restrictions

$$u_l \ll \min\{v_p,\ v_p c^2/\omega_{Le}^2 l^2\},\quad v_p \ll c. \quad (4)$$

Limiting the discussion to fairly slow processes, which take place over time scales greater than the period of plasma oscillations, so that the inequality

$$\tau \gg \omega_{Le}^{-1} \quad (5)$$

holds, and applying the operator div to the equation of motion for the component $\mathbf{u}_l$, we find the following expression:

$$\delta n = (m\omega_{Le}^2)^{-1}[\Delta B^2/8\pi + \text{div}(\nabla(P - BB/4\pi))]. \quad (6)$$

The tensor BB has been introduced in accordance with the definition of the components of the tensor AA used below. This tensor is constructed from two vectors $\mathbf{A}$: $(AA)_{ij} = A_j A_j$. The irrotational components $\mathbf{E}_l$ and $\mathbf{u}_l$ can be determined from

$$\text{div}\,\mathbf{E}_l = 4\pi e\delta n,\quad \text{div}\,\mathbf{u}_l = -\frac{1}{n_0}\frac{\partial \delta n}{\partial t}. \quad (7)$$

According to (6) the condition under which the density perturbations are small, $|\delta n/n_0| \ll 1$ imposes the following restrictions:

$$B^2/8\pi m n_0 c^2 \ll (\omega_{Le} l/c)^2, \quad (8)$$

$$l \gg r_D, \quad (9)$$

where $r_D = (P/mn_0\omega_{Le}^2)^{1/2}$ is a characteristic electron "Debye length."

Going back to inequality (9), we note that there is actually a more restrictive lower limit on the quantity l. The reason is that our hydrodynamic mode ignores kinetic effects, which may dominate if the characteristic "thermal" velocity $V_T = r_D\omega_{Le}$ is not small in comparison with the characteristic "phase" velocity v_p. For this reason, we should introduce $v_p \gg v_T$ as a condition for the applicability of the hydrodynamic model. This condition takes the form

$$l \gg v_T\tau \gg r_D. \quad (10)$$

According to (7) the first of inequalities (4) can be rewritten as a inequality for the magnetic field:

$$B^2/8\pi m n_0 c^2 \ll \min(1,\ l^2\omega_{Le}^2/c^2), \quad (11)$$

which includes condition (8). In the case $l \gtrsim c/\omega_{Le}$, condition (11) is somewhat trivial, since it does not allow relativistic values of the magnetic field. Such values could of course not be dealt with on the basis of our original nonrelativistic equations, (1). For this reason, the conditions for the applicability of the hydrodynamic model take the following simple form for sufficiently long-wave perturbations, $l \gtrsim c/\omega_{Le}$, which arise, for example, in the initial stage of the Weibel instability:

$$c \gg v_p \gg v_T. \quad (12)$$

For small-scale motions, $l < c/\omega_{Le}$, the corresponding conditions for applicability are

$$l\omega_{Le} \gg v_p \gg v_T,\ B^2/8\pi \ll m n_0 \omega_{Le}^2 l^2. \quad (13)$$

Having obtained relations between the irrotational fields and currents and the rotational fields, we have thus explicitly formulated the conditions under which the irrotational motions which arise from the nonlinearity do not have any substantial effect on the rotational motions which we will be discussing. Along with the limitations which follow from our ignoring kinetic effects, we therefore have conditions (12) and (13) for the applicability of the dynamic theory of the Weibel instability. These conditions are also important for analyzing the results of numerical simulations of the Weibel instability, in which the physically real irrotational fields may be accompanied by effects of a numerical buildup of irrotational perturbations as a result of fluctuations. The appearance of electrostatic components has been mentioned in some numerical calculations.[7,8]

3. In the case $l \gg c/\omega_{Le}$, the equations of rotational anisotropic electron hydrodynamics, (3), simplify, reducing to

$$\frac{\partial \mathbf{\Omega}}{\partial t} = -\frac{1}{mn_0}\text{rot}\,\nabla P,\quad \frac{\partial P}{\partial t} = \{[P\mathbf{\Omega}]\}. \quad (14)$$

We will use this system of equations to study self-consistent large-scale electromagnetic structures under the conditions for the Weibel instability.

At any time, and at each point in space, the symmetric stress tensor P can be diagonalized by the standard method, with the help of the rotation tensor

$$D=R^{-1}PR. \tag{15}$$

Here D is the diagonalized stress tensor and R is a rotation tensor, characterized by, e.g., three angular variables $RR^{-1}=I$, where I is the unit tensor $[(I)_{ij}=\delta_{ij}]$. Using (15), we easily see that system (14) has the three integrals

$$\begin{aligned} D_{11}(\mathbf{r},\ t)&=P_1(\mathbf{r}),\\ D_{22}(\mathbf{r},\ t)&=P_2(\mathbf{r}),\\ D_{33}(\mathbf{r},\ t)&=P_3(\mathbf{r}). \end{aligned} \tag{16}$$

The integrals (16) correspond to the conservation of the pressure anisotropy in the course of the nonlinear relaxation. This pressure anisotropy is generally determined by three electron temperatures. Consequently, only the directions (or direction) of the pressure anisotropy undergoes a change in large-scale rotational flows. Consequently, the mathematical problem reduces to one of finding six functions: the three components of the vector Ω and three independent components of the rotation tensor R. The corresponding equations are

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}=-\frac{1}{mn_0}\operatorname{rot}\nabla(RDR^{-1}),\quad \frac{\partial}{\partial t}R=[R\boldsymbol{\Omega}]. \tag{17}$$

The second of equations (17) describe a local rotation of the rotation tensor R with respect to the magnetic field, while the first describes the generation of a magnetic field by the rotational component of the pressure force. The diagonal tensor D is specified at the initial time $t=0$ by the pressure P_1, P_2, P_3, and it is conserved [see (16)] in the course of the relaxation. In the long-wave limit, the generation of a spatially nonuniform magnetic field leads only to "mixing" of the pressure anisotropy, in the course of which the local temperatures (the pressures $P_{1,2,3}$) are conserved.

To pursue this analysis, it is convenient to replace the tensor formulation (17) by a vector formulation. For this purpose we introduce two mutually orthogonal unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, which are functions of the coordinates and the time, and we write the tensor P as

$$P=P_3I+(P_1-P_3)n_1n_1+(P_2-P_3)n_2n_2. \tag{18}$$

Here P_3 is the pressure in the direction across the plane formed by the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, while P_1 and P_2 are the pressures along the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. Using the conditions

$$|\mathbf{n}_1|=|\mathbf{n}_2|=1,\ \mathbf{n}_1\mathbf{n}_2=0, \tag{19}$$

we see that the tensor P is determined by six quantities (as is necessarily the case for a symmetric tensor): three pressures and three independent components of the unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. By virtue of (16), the quantities P_1, P_2, and P_3 remain constant over time (i.e., they are integrals of motion).

We restrict the discussion below to the case $P_1(\mathbf{r})=P_1^{(0)}=\text{const}$, $P_2(\mathbf{r})=P_2^{(0)}=\text{const}$, $P_3(\mathbf{r})=P_3^{(0)}=\text{const}$. This case corresponds to the standard formulation of the problem of the excitation of the Weibel instability in a homogeneous plasma with an anisotropic pressure.[3] We can thus replace (14) by

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}=\operatorname{rot}\left\{\frac{P_1^{(0)}-P_3^{(0)}}{mn_0}([\mathbf{n}_1\operatorname{rot}\mathbf{n}_1]-\mathbf{n}_1\operatorname{div}\mathbf{n}_1)+\frac{P_2^{(0)}-P_3^{(0)}}{mn_0}([\mathbf{n}_2\operatorname{rot}\mathbf{n}_2]-\mathbf{n}_2\operatorname{div}\mathbf{n}_2)\right\},\quad \frac{\partial\mathbf{n}_{1,2}}{\partial t}=[\mathbf{n}_{1,2}\boldsymbol{\Omega}]. \tag{20}$$

Equations (20), along with conditions (19), make it possible to describe large-scale rotational flows in the case of three-dimensional (biaxial) anisotropy. The last two equations describe local rotations of the anisotropy vectors in the nonuniform magnetic field.

We restrict the discussion below to the usual case for the Weibel instability: uniaxial anisotropy of the plasma pressure. At the initial time, the plasma pressure $P_1^{(0)}=P_\parallel$ along some selected axis $\mathbf{n}_1(\mathbf{r},t=0)=\mathbf{n}_0$ differs from the pressures in the plane perpendicular to this axis, $P_2^{(0)}=P_3^{(0)}=P_\perp$.

An important consequence of Eqs. (17) and (20) is the conservation of the local anisotropy of the plasma pressure as the magnetic field evolves. In the case at hand, this conservation means that the anisotropy of the local pressure remains uniaxial in the nonlinear stage of the evolution of the magnetic field, i.e.,

$$P=P_\parallel nn+P_\perp(I-nn), \tag{21}$$

and Eqs. (20) can be written

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}=\frac{P_\parallel-P_\perp}{mn_0}\operatorname{rot}([\mathbf{n}\operatorname{rot}\mathbf{n}]-\mathbf{n}\operatorname{div}\mathbf{n}),\quad \frac{\partial\mathbf{n}}{\partial t}=[\mathbf{n}\boldsymbol{\Omega}]. \tag{22}$$

A new type of vector nonlinearity, $\operatorname{curl}([\mathbf{n}\operatorname{curl}\mathbf{n}]-\mathbf{n}\operatorname{div}\mathbf{n})$, corresponds to the dynamic equations (22). This type of nonlinearity has not been seen previously in the theory of the nonlinear dynamics of plasmas and continuous media. Since we have $|\mathbf{n}|=1$ for any coordinate system which can be characterized by a unit basis $(\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$, we can use the following parametrization for the vector $\mathbf{n}$:

$$\mathbf{n}=(n_1,\ n_2,\ n_3)=(\cos\Phi\cdot\sin\Psi,\ \sin\Phi\cdot\sin\Psi,\ \cos\Psi). \tag{23}$$

The problem thus reduces to one of finding five functions: three components of the magnetic field and the two angles Φ and Ψ of the local rotation of the anisotropy vector with respect to the original direction of the anisotropy vector, $\mathbf{n}_0$.

Converting to dimensionless quantities in Eqs. (21), we find it convenient to transform to the following universal system:

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}=\operatorname{rot}([\mathbf{n}\operatorname{rot}\mathbf{n}]-\mathbf{n}\operatorname{div}\mathbf{n}), \tag{24}$$

$$\frac{\partial\mathbf{n}}{\partial t}=[\mathbf{n}\boldsymbol{\Omega}]. \tag{25}$$

Here we have assumed $P_\parallel>P_\perp$. This assumption corresponds to a buildup of the Weibel instability in the direction transverse with respect to the original anisotropy direction $\mathbf{n}_0$. The coordinates in (24) and (25) have been nondimensionalized with respect to the length scale of the Weibel instability, c/ω_{Le}, while Ω and t^{-1} have been scaled by the characteristic growth rate $(\omega_{Le}/c)((P_\parallel-P_\perp)/mn_0)^{1/2}$. Linearization of Eqs. (24) and (25) shows that these equations describe the long-wave Weibel instability with a growth rate which depends linearly on the wave number. The primary nonlinear effect described by Eqs. (24) and

(25) is the inverse effect of the magnetic field, increasing with time, on the electrons. This effect consists of a local rotation of the axis of the electron-pressure anisotropy, without a change in the degree of anisotropy.

4. Let us illustrate Eqs. (24) and (25) in a specific example, which describes rotational structures of a magnetic-helix type. Introducing a cylindrical coordinate system, we have the following equations for the magnetic field, which depends on only the radial coordinate and which does not have a radial component:

$$\begin{aligned}
&\frac{\partial}{\partial t}\Omega_\varphi=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rn_rn_z),\\
&\frac{\partial}{\partial t}\Omega_z=-\frac{1}{r}\frac{\partial^2}{\partial r^2}(rn_rn_\varphi),\\
&\frac{\partial n_\varphi}{\partial t}=-n_r\Omega_z,\quad \frac{\partial n_z}{\partial t}=n_r\Omega_\varphi,\\
&\qquad n_z^2+n_r^2+n_\varphi^2=1.
\end{aligned}\tag{26}$$

For $n_\varphi=0$ (and thus $\Omega_z=0$), Eqs. (26) are an analog of the corresponding equations describing a self-consistent z-pinch. With $n_z=0$ (i.e., $\Omega_\varphi=0$), they become an analog of the equations for a θ-pinch.[3]

It was shown in Refs. 3 and 5 that for the self-similar variable

$$\xi=Vt/r\tag{27}$$

the solutions for z-pinch and θ-pinch structures are singular. According to those solutions, the magnetic field can become infinite over a finite time interval (a peaking regime[9]). We will show that the solutions are again of this nature in the more general case described by system (26) with a radial dependence $\Omega(r,t)$ and $\mathbf{n}(r,t)$.

Introducing the variable ξ as in (27), the rotation angles Φ and Ψ as in (23), the anisotropy vectors $n_1=n_r$, $n_2=n_\varphi$, $n_3=n_z$, and the functions $b_\varphi=r\Omega_\varphi$, $b_z=r\Omega_z$, we can rewrite (26) as the following system of equations:

$$V\frac{d\Psi}{d\xi}=-\cos\Phi b_\varphi,\quad V\frac{d\Phi}{d\xi}=-b_z+\sin\Phi\,\mathrm{ctg}\,\Psi\, b_\varphi,\tag{28}$$

$$V\frac{db_z}{d\xi}=\xi\frac{d}{d\xi}\Big[\sin2\Phi\sin^2\Psi+\frac{\xi}{V}\Big(\frac{1}{2}\sin\Phi\sin2\Psi b_\varphi+\cos2\Phi\sin^2\Psi b_z\Big)\Big],\tag{29}$$

$$[V+\xi^2(\cos^2\Psi-\sin^2\Psi\cos^2\Phi)]b_\varphi-\frac{\xi^2}{2}\sin2\Psi\sin\Phi b_z=C_1-\frac{\xi}{2}\sin2\Psi\cos\Phi,\tag{30}$$

where C_1 is an integration constant. The functions b_z and b_φ have unbounded-growth singularities at a certain value $\xi=\xi_0$. Near these singularities, where we have

$$\xi^{-1}\frac{db_z}{d\xi}\approx\frac{d}{d\xi}\xi^{-1}b_z,$$

we find the following expressions with the help of Eqs. (29) and (30):

$$b_z\approx C_2\Big[\frac{V^2}{\xi}-\xi\cos2\Phi\sin^2\Psi-\frac{(\xi^3/4)\sin^2 2\Psi\sin^2\Phi}{V+\xi^2(\cos^2\Psi-\sin^2\Psi\cos^2\Phi)}\Big]^{-1},\tag{31}$$

$$b_\varphi\approx\frac{b_z}{2}\frac{\xi^2\sin2\Psi\sin\Phi}{V+\xi^2(\cos^2\Psi-\sin^2\Psi\cos^2\Phi)}.\tag{32}$$

Here C_2 is an integration constant. From the vanishing of the denominator in (31) we find the value of ξ_0. Introducing the deviations $\delta\Phi$ and $\delta\Psi$ of the functions Φ and Ψ from their values at $\xi=\xi_0$, and using Eqs. (28)–(32), we find the nature of the solutions near ξ_0:

$$\delta\Phi,\delta\Psi\propto(\xi-\xi_0)^{1/2},b_z,b_\varphi\propto\frac{1}{(\xi-\xi_0)^{1/2})}.$$

There is a singularity only for finite $\xi_0\neq0$. As time elapses, the magnetic-field thus develops a square-root singularity, as in the case of z-pinches and θ-pinches.[3] The solution found here differs from that in Ref. 3 in that it describes a two-component magnetic field. It has been shown that the peaking is accompanied by a simultaneous explosive growth of both components of the magnetic field.

5. Equations (24) and (25) have a class of solutions describing self-consistent vortex structures in which the magnetic field vector is orthogonal to the anisotropy vector: $\mathbf{n}\boldsymbol{\Omega}=0$. In this case the problem reduces to one of solving a single nonlinear dynamic equation for the vector $\mathbf{n}$

$$\frac{\partial^2\mathbf{n}}{\partial t^2}+\mathbf{n}\Big(\frac{\partial\mathbf{n}}{\partial t}\Big)^2=[\mathbf{n}\,\mathrm{rot}([\mathbf{n}\,\mathrm{rot}\,\mathbf{n}]-\mathbf{n}\,\mathrm{div}\,\mathbf{n})].\tag{33}$$

Knowing $\mathbf{n}$, we can easily find the magnetic field of a vortex:

$$\boldsymbol{\Omega}=\Big[\frac{\partial\mathbf{n}}{\partial t}\mathbf{n}\Big].\tag{34}$$

Linearizing (33), we see that this equation also describes the Weibel instability in the linear approximation. Toroidal structures serve to illustrate possible solutions of Eq. (33).

We introduce toroidal coordinates σ, τ, φ, specifying the z axis and a circle of a radius a—the toroidal axis—which lies in the plane perpendicular to the z axis. The coordinate τ shows the distance from a selected point A to the toroidal axis in the plane passing through the z axis. The angular variable $\sigma(-\pi\leqslant\sigma\leqslant\pi)$ is the angle between the radius vectors (in the same plane) which start from point A and go to two diametrically opposite points on the toroidal axis. The angle φ is the azimuthal angle.

Equations (33) and (34) simplify for two simple geometric configurations, which correspond to toroidal analogs of the z-pinch and θ-pinch. In the former case, there are only a poloidal component of the magnetic field, $\boldsymbol{\Omega}=(\Omega_0,0,0)$, and a toroidal component of the current density. There is no poloidal component of the anisotropy vector: $n_\sigma=0$ (Fig. 1). Using the two-dimensional analog of (23),

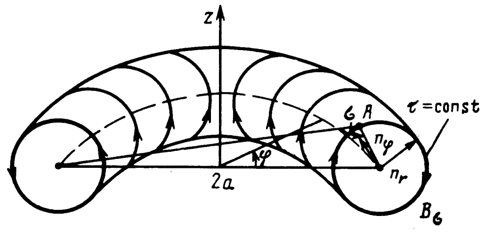

FIG. 1.

$$n_\tau = \cos\Phi, \quad n_\varphi = \sin\Phi, \tag{35}$$

we find $\Omega_\sigma = -\partial\Phi/\partial t$ from Eqs. (33) and (34). We also find the following equation for Φ, the angle through which the anisotropy vector is rotated with respect to the poloidal axis:

$$\frac{\partial^2\Phi}{\partial t^2} = \frac{1}{(g_2 g_3)^{1/2}} \frac{\partial}{\partial\tau}\left[\frac{\cos\Phi}{g_2^{1/2}} \frac{\partial}{\partial\tau} g_3^{1/2} \sin\Phi + \frac{\sin\Phi}{g_2} \frac{\partial}{\partial\tau}(g_2 g_3)^{1/2} \cos\Phi\right]. \tag{36}$$

Here g_2 and g_3 are components ($\tau\tau$ and $\varphi\varphi$) of the metric tensor for the toroidal coordinate system:

$$g_2 = \frac{a^2}{(\operatorname{ch}\tau - \cos\sigma)^2}, \quad g_3 = \frac{a^2 \operatorname{sh}^2\tau}{(\operatorname{ch}\tau - \cos\sigma)^2}. \tag{37}$$

According to (36), the solutions depend parametrically on the angular variable σ. Equation (36) simplifies near the toroidal axis, i.e., under the condition $\tau \gg 1$. In this case we have

$$\frac{\partial^2\Phi}{\partial t^2} = \frac{1}{8a^2} e^\tau \frac{\partial}{\partial\tau} e^\tau \left(\frac{\partial}{\partial\tau} \sin 2\Phi - \sin 2\Phi\right). \tag{38}$$

Since we are interested in the development of the singularity, we retain only the highest derivatives in Eq. (38). For the spatial variable $x = 2ae^{-\tau}$ we then find

$$2\Phi_{tt} - (\sin 2\Phi)_{xx} = 0. \tag{39}$$

Equation (39) resembles one derived in Ref. 4, where it was shown that a self-similar solution of this equation, which depends on the variable $\xi = Vt/x$, leads to a square-root peaking of the poloidal magnetic field, $\propto (\xi - \xi_0)^{-1/2}$.

We turn now to the toroidal analog of the θ-pinch. This has a toroidal magnetic field $\mathbf{\Omega} = (0,0,\Omega_\varphi)$ and a poloidal current, but the anisotropy vector has no toroidal component: $n = (n_\sigma, n_\tau, 0)$. By analogy with (35), we make the substitution $n_\sigma = \cos\Phi$, $n_\tau = \sin\Phi$. Here we have $\Omega_\varphi = -\partial\Phi/\partial t$, and in place of Eq. (36) we have

$$\begin{aligned}\frac{\partial^2\Phi}{\partial t^2} = \frac{1}{g_2}\Big\{&\left(\frac{\partial}{\partial\sigma}\cos\Phi + \frac{\partial}{\partial\tau}\sin\Phi\right)\frac{1}{g_2^{1/2}} \\ &\times\left(\frac{\partial}{\partial\sigma} g_2^{1/2}\sin\Phi - \frac{\partial}{\partial\tau} g_2^{1/2}\cos\Phi\right) \\ &+\left(\frac{\partial}{\partial\sigma}\sin\Phi - \frac{\partial}{\partial\tau}\cos\Phi\right)\frac{1}{(g_2 g_3)^{1/2}} \\ &\times\left[\frac{\partial}{\partial\sigma}(g_2 g_3)^{1/2}\cos\Phi + \frac{\partial}{\partial\tau}(g_2 g_3)^{1/2}\sin\Phi\right]\Big\}.\end{aligned} \tag{40}$$

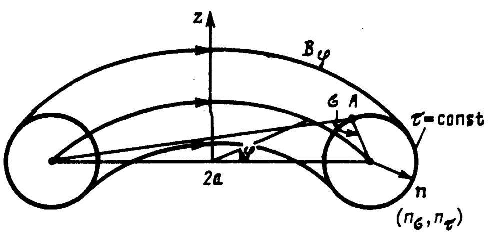

FIG. 2.

For the solutions near the axis ($\tau \gg 1$), Eq. (40) is rewritten in the form

$$\Phi_{tt} = \frac{e^{2\tau}}{8a^2}[(\sin 2\Phi)_{\sigma\sigma} - (\sin 2\Phi)_{\tau\tau} - 2(\cos 2\Phi)_{\sigma\tau} + 2(\cos 2\Phi)_\sigma + 2(\sin 2\Phi)_\tau]. \tag{41}$$

Examining singular solutions which are independent of the angular variable σ, we see that Eqs. (41) reduces to (39) if we restrict the discussion to the higher derivatives with respect to x. Again in the case of a toroidal θ-pinch, we thus find a square-root singularity for the toroidal magnetic field, $\propto (\xi - \xi_0)^{-1/2}$.

The toroidal vortex structures (Figs. 1 and 2) which we have been discussing here are topologically different from the structures which we studied previously.[2-5] The nature of the peak in the magnetic field turns out to be the same as in straight systems.[2-5] The magnetic-field singularity arises first either at the axis, $\tau \to \infty$ ($V/\xi_0 > 0$), or at the boundary of the region under consideration, $\tau_1 < \tau$ ($V/\xi_0 < 0$).

6. In summarizing these results, we wish to emphasize that the discussion above has proved that there may exist an explosive class of solutions describing nonlinear self-consistent toroidal vortex structures in the form of an magnetic helix under conditions corresponding to the development of the Weibel instability. This result has been established on the basis of the dynamic system of equations for the quasistatic magnetic field and the anisotropy vector. This description is valid for large-scale vortices ($l > c/\omega_{Le}$). The evolution of these vortices results in a change in the direction of the anisotropy axis without a change in the degree of anisotropy.

According to (10), in the linear hydrodynamic stage of the Weibel instability the condition for a substantial pressure anisotropy should hold:[6]

$$\frac{P_\parallel - P_\perp}{P_\perp} \gg 1.$$

In the stage of the self-similar singularity, condition (10) requires the satisfaction of the inequality

$$mn_0 V^2/\xi^2 \gg P_\perp, \tag{42}$$

since we have $v_p \sim V/\xi$. If condition (42) holds in the initial stage of the peaking, then it will not be violated when the singularity forms (since we have $|\xi| \geqslant |\xi_0|$). Explosive growth of the magnetic field corresponds to a decrease in the length scale l of the vortices; this decrease puts the evolution of the instability in a short-wave regime, in which a further peaking is possible.[4] In this case, however, the absence of conservation laws (16) indicates a change in the magnitude of the anisotropy (this suggestion is supported by the one-dimensional theory[2]). Correspondingly, as time elapses we would expect to find a transition to isotropy and an increase in the characteristic thermal velocity, with the result that the condition $v_p \sim v_T$ would hold. That case requires a kinetic analysis.

We note in conclusion that we have not taken up the transition of the solutions to a *self-similar* regime. It would be interesting in this connection to see a systematic study of this question, similar to that which has been carried out for the case of Langmuir collapse.[10]

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