# Gravitational radiation from nongravitational systems

A. I. Nikishov and V. I. Ritus

*P. N. Lebedev Physics Institute, USSR Academy of Sciences*
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We obtain the spectrum of gravitational radiation emitted by a classical charge in a circular orbit, in an unbounded orbit and slightly displaced by the field produced by the magnetic moment, and in a combined Coulomb and magnetic-moment field. In the ultrarelativistic limit and the appropriate frequency range, these spectra are proportional to the respective electromagnetic spectra, with a proportionality factor $4\pi Gm^2\Gamma^2/e^2$ that is independent of the frequency, the direction of the wave vector, the specifics of the motion, and the behavior of the field outside the orbit. We obtain rough estimates of $\Gamma$ for an ultrarelativistic charge in an arbitrary field, demonstrating that to order of magnitude, $\Gamma^2$ is the ratio of the radiative contributions made by nonlocal and local sources. The spectrum of gravitational radiation is derived for a relativistic rotating string with masses at its ends, and we show that the masses and the string contribute approximately equally. In the nonrelativistic limit, the harmonics of the radiation from the rotating systems all display the same behavior.

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## 1. INTRODUCTION

In a previous paper,[1] we examined the spectrum of gravitational radiation (GR) from several simple electromagnetic systems. In those systems, a charge executed bounded motion in an external electromagnetic field that varies smoothly in the vicinity of the orbit over distances comparable to the radius of curvature. It was shown that in the ultrarelativistic limit, the GR spectrum of the system is proportional to its electromagnetic spectrum:

$$8\pi G\left[T_{\alpha\beta}^*(q)T^{\alpha\beta}(q)-\frac{1}{2}\,|T_\alpha{}^\alpha(q)|^2\right]\approx\frac{4\pi Gm^2\Gamma^2}{e^2}\,|j_\alpha(q)|^2. \qquad (1)$$

Here $j_\alpha(q)$ and $T_{\alpha\beta}(q)$ are the Fourier components of the current density and energy-momentum tensor of the system. Equation (1) is interesting not only due to its demonstration of the proportionality of the spectra, but also because the nonlocal properties of the system described by the energy-momentum tensor are lumped into the factor $\Gamma^2$, which is independent of the frequency. This parameter $\Gamma$ was found to be of the order of the Lorentz factor $\gamma$ of the charge, and in general it depended on the direction of the wave vector $\mathbf{q}$ and the behavior of the field outside the orbit, in the region where GR was being produced.

This proportionality between the GR and electromagnetic spectra and the asymptotic equality $\Gamma\sim\gamma$ for $\gamma\gg1$ are closely related to the fact that of the two sources of GR, namely the local energy-momentum tensor $t_{\alpha\beta}$ due to a corporeal body and the nonlocal energy-momentum tensor $\theta_{\alpha\beta}$ inherent to external electromagnetic fields, it is precisely the latter that becomes the dominant one in the ultrarelativistic limit. This means that GR comes about when a charge initially emits either a real or virtual photon along a segment of its trajectory that is $\gamma$ times smaller than its radius of curvature $r$, and the photon is subsequently transformed into a graviton via interaction with a quantum of the external field over a path whose length is of order $l$, the extent of the external field in the direction of photon propagation.

It is straightforward, using Eq. (13) and Eq. (14) below for the energy-momentum tensor of the field, to show that to order of magnitude, the spatial components of the energy-momentum tensor are

$$\theta_{ik}(q)\sim Flj, \qquad (2)$$

where we have $j=j(q-k_{ef})$, and $F$ is the external field strength in the region where the emitted photon is produced. The length $l$ comes from the value of the photon propagator:

$$[(q-k)^2-i\varepsilon]_{ef}^{-1}\sim\frac{1}{|\mathbf{q}|\,|\mathbf{k}|_{ef}}\sim\frac{l}{|\mathbf{q}|}, \qquad (3)$$

i.e., it is the distance traversed by the photon in the field before it turns into a graviton. If for $\gamma\gg1$, Eq. (1) is due to the contribution of the field energy-momentum tensor, then we obtain for the conversion amplitude (which we call $\Gamma$)

$$\Gamma\sim\frac{eFl}{mc^2}. \qquad (4)$$

We consider trajectories with a turning angle at least of order unity (this would include bounded trajectories), for which the radius of curvature, as given by the law of particle motion $eF\sim mv^2\gamma/r$, is less than the extent of the field. If the field varies significantly over distances much greater than $r$, then we have $l\gg r$ and $\Gamma\sim\gamma l/r\gg\gamma$. A similar situation obtains for GR from a charge moving in a circular orbit in a constant, uniform magnetic field that extends far beyond that orbit (see Sec. 3 of Ref. 1). For a field that does vary significantly over distances of the order of the orbital radius [circular motion in a Coulomb field (Sec. 5 of Ref. 1); motion in the field due to a magnetic moment, or in the combined field of a Coulomb center of force and a magnetic moment (treated in Secs. 2 and 3 of the present paper)], we obviously have $\Gamma\sim\gamma\gg1$. Finally, for a field that extends along a trajectory for some distance $l_\parallel\gtrsim r$, but whose transverse dimensions $l_\perp\ll r$—as in modern circular accelerators—we obtain $\Gamma\sim\gamma l/r\ll\gamma$, where $l\sim(l_\perp r)^{1/2}\ll r$; in that regard, see the remark in Sec. 3 below and Eq. (45). Here we are also at odds with the authors of Ref. 2, who maintain that one can estimate the intensity of GR in existing and proposed accelerators by taking the formula for the intensity of

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electromagnetic radiation and replacing $e^2/4\pi$ with $Gm^2\gamma^2$: the small factor $l_\perp/r$ must also be taken into account.

Consider now a trajectory with a small turning angle $\Delta\varphi \lesssim 1$, i.e., an unbounded trajectory in a field of small extent relative to the radius of curvature. The law of motion is then $eF \sim mv\gamma\Delta\varphi/\Delta t$, and making use of the relation $l \sim v\Delta t$, (4) yields $\Gamma \sim \gamma\Delta\varphi \lesssim \gamma$. This estimate holds for $\gamma^{-1} \lesssim \Delta\varphi \lesssim 1$.

For $\Delta\varphi \lesssim \gamma^{-1}$, there are further considerations. Equation (1) relates gravitational radiation to the electromagnetic radiation produced by components of the current that are transverse to $\mathbf{q}$ for $q^2 = 0$. In (2), however, there is also a contribution from components of the current off the mass shell, which are unrelated to particle acceleration. The latter are a factor of $1/\Delta\varphi$ larger than the components associated with acceleration, and they have components orthogonal to $\mathbf{q}$ that are $\sim 1/\gamma\Delta\varphi$ times the transverse components responsible for electromagnetic radiation. In the estimate above, therefore, for $\Delta\varphi \ll \gamma^{-1} \ll 1$, an additional factor of $1/\gamma\Delta\varphi$ appears. For $\Delta\varphi \lesssim 1$, then,

$$\Gamma \sim \gamma\Delta\varphi\left(1+\frac{1}{\gamma\Delta\varphi}\right) = 1+\gamma\Delta\varphi. \qquad (5)$$

In contrast to Ref. 1, where the GR spectrum was calculated in terms of an invariant product of components of the energy-momentum tensor [see left-hand side of Eq. (1)], the spectrum that we derive here is a sum of squares of two independent polarization amplitudes designated $T_+(q)$ and $T_x(q)$. In fact, if we write the invariant expression for the spectrum in a coordinate system in which the wave vector $\mathbf{q}$ points along the 3-axis, then marking tensor components in that system with a prime and taking advantage of the conservation law $q^\alpha T_{\alpha\beta}(q) = q'^\alpha T'_{\alpha\beta}(q') = 0$, as well as $q^2 = q'^2 = 0$, we obtain

$$T_{\alpha\beta}^*(q)T^{\alpha\beta}(q) - \frac{1}{2}|T_\alpha{}^\alpha(q)|^2$$

$$= \frac{1}{2}|T_{11}'(q') - T_{22}'(q')|^2 + 2|T_{12}'(q')|^2. \qquad (6)$$

The expressions

$$T_+(q) = T_{11}'(q') - T_{22}'(q'), \quad T_\times(q) = T_{12}'(q'), \qquad (7)$$

in which the three spatial components of the energy-momentum tensor on the right can be reexpressed in terms of the $T_{ik}(q)$, will also be two transverse components describing the gravitational radiation emitted by the system with independent polarizations.

Almost all of the systems that we shall consider here are axially symmetric, which means that in a spherical coordinate system whose polar axis coincides with the symmetry axis, the angular distribution of the gravitational radiation should not depend on the azimuthal angle $\varphi$ of the vector $\mathbf{q}$. Taking $\mathbf{q}$ to lie in the 1–3 plane and denoting its polar angle by $\theta$, we then have

$$T_{11}' - T_{22}' = T_{11}\cos^2\theta - 2T_{13}\sin\theta\cos\theta + T_{33}\sin^2\theta - T_{22}, \qquad (8)$$

$$T_{12}' = T_{12}\cos\theta - T_{32}\sin\theta. \qquad (9)$$

In the present paper, we find the GR spectrum emitted by a charge moving in a circular orbit in the equatorial plane of the field due to a magnetic moment, and we show that in the ultrarelativistic limit, the spectrum is the same as that emitted by a charge in circular motion in a Coulomb field if we replace the charge's field interaction factor $e\mathfrak{M}/r$ by $-ee'$. Furthermore, the GR spectrum of a charge moving ultrarelativistically in a circular orbit in a combined Coulomb and magnetic moment field is given by the same equation, with the quantity on the left-hand side of (38) (see below) as the interaction parameter. In all cases, then, the GR spectrum may be characterized by the conversion amplitude $\Gamma = \gamma$.

We have also derived the spectrum of gravitational radiation from an ultrarelativistic charge that traverses a Coulomb field, or that passes through the equatorial plane of the field due to a magnetic moment. The deviation angle $\chi$ in either case is assumed small, $\chi \lesssim \gamma^{-1} \ll 1$. We have shown that despite small differences between the trajectories in the region of space where the gravitational radiation is produced, the spectra are essentially the same in the critical range of wave vectors, and are characterized by an amplitude conversion factor that depends on the direction of the wave vector and the deviation angle of the trajectory; see (104) below.

To clarify the way in which gravitational radiation depends on the spatial distribution of the energy-momentum tensor of the field, we have obtained the GR spectrum of a relativistic string with point masses at its ends, and have demonstrated that the contributions of the masses and the string itself are of the same order of magnitude in the ultrarelativistic limit, notwithstanding the fact that to order of magnitude, the string energy is a factor $\gamma$ higher than that of the masses.

Finally, we have shown that in the nonrelativistic limit, where the wavelength of the gravitational radiation is much greater than the size of the system, the gravitational spectrum of a closed system consisting of a point mass moving in a circle about some attracting center, the spectrum is a universal one: the leading term of each harmonic of the radiation is independent of the nature of the field produced by the center, and for all harmonics with $n \geqslant 2$, the contribution from the energy-momentum tensor of the field is a factor $n-1$ less than the contribution from the energy-momentum tensor of the point mass.

## 2. GRAVITATIONAL RADIATION FROM A CHARGE IN THE FIELD OF A MAGNETIC MOMENT

We consider circular motion of a charge in the equatorial plane of the field

$$\mathbf{H} = \frac{3\mathbf{r}(\mathfrak{M}\mathbf{r}) - \mathfrak{M}r^2}{4\pi r^5} = \nabla(\mathfrak{M}\nabla)\frac{1}{4\pi r} \qquad (10)$$

produced by a magnetic moment $\mathfrak{M}$. The equation of motion,

$$mv^2\gamma = \frac{e\omega\mathfrak{M}_\omega}{4\pi cr}, \qquad (11)$$

tells us that $\mathfrak{M}_\omega$, the projection of the magnetic moment in the direction of the charge's orbital angular velocity, has the same sign as the charge: $e\mathfrak{M}_\omega = |e\mathfrak{M}| > 0$.

Making use of the Fourier components

$$\varphi_{\alpha\beta}(k) = -\frac{\mathfrak{M}\mathbf{k}}{k^2}\varepsilon_{\alpha\beta\gamma 0}k^\gamma 2\pi\delta(k^0) \qquad (12)$$

of the field (10) and the Fourier components of the self-field of the charge,

$$f_{\alpha\beta}(q)=\frac{i}{q^2}[q_\alpha j_\beta(q)-q_\beta j_\alpha(q)], \tag{13}$$

we can easily construct the components of the field energy-momentum tensor

$$\theta_{\mu\nu}(q)=-\int\frac{d^4k}{(2\pi)^4}\Big[\varphi_{\mu\alpha}(k)f^\alpha{}_\nu(q-k)+\varphi_{\nu\alpha}(k)f^\alpha{}_\mu(q-k)$$
$$+\frac{1}{2}g_{\mu\nu}\varphi_{\alpha\beta}(k)f^{\alpha\beta}(q-k)\Big], \tag{14}$$

omitting quadratic terms in $\varphi\varphi$ and $ff$ for reasons explained in Ref. 1. The integration over wave vectors **k** of the external field is most conveniently carried out using causal space-time representations for the propagators $k^{-2}$ and $(q-k)^{-2}$, and the representation

$$j_\alpha(q)=e\int_{-\infty}^{\infty}d\tau\,\dot x_\alpha(\tau)e^{-iqx(\tau)} \tag{15}$$

for the current density $j_\alpha(q-k)$. Then $\theta_{\mu\nu}(q)$ is given by

$$\theta_{\mu\nu}(q)=\frac{e\mathfrak{M}_\omega}{8\pi|\mathbf{q}|}\int_0^1du\int_{-\infty}^{\infty}d\tau\,e^{if}a_{\mu\nu}, \tag{16}$$

in which $f=-q_\alpha x^\alpha(\tau)+u[\mathbf{q}\cdot\mathbf{x}(\tau)+|\mathbf{q}|r]$, $u=t(s+t)^{-1}$, where $s$ and $t$ are the proper times of quanta of the external and intrinsic fields, and the $\alpha_{\mu\nu}$ are quadratic polynomials in the coordinates $x_{1,2}(\tau)$ or velocities $\dot x_{1,2}(\tau)$.

Taking advantage of Eqs. (8), (9), and (16), we obtain for the two transverse components of the field energy-momentum tensor

$$\theta_A(q)=\sum_n 2\pi\delta(q^0-n\omega)m\gamma v^2\int_0^1du\,e^{i\zeta u}[\alpha_AJ_n(z_1)+\beta_AJ_n'(z_1)]. \tag{17}$$

The subscript $A$ here can be either $+$ or $\times$, and the $\alpha_A$ and $\beta_A$ are

$$\alpha_+=\frac{3}{2}(1-i\zeta u)\sin^2\theta-\Big(\frac{n^2}{z_1^2}-\frac{1}{2}\Big)$$
$$\times[2\zeta^2u(1-u)\cos^2\theta+(i\zeta u-1)(1+\cos^2\theta)], \tag{18}$$

$$\beta_+=\frac{1}{z_1}[2\zeta^2u(1-u)\cos^2\theta+(i\zeta u-1)(1+\cos^2\theta)]$$
$$+i\sin\theta[i\zeta(1-u)-\zeta u(i+\zeta u)\cos^2\theta], \tag{19}$$

$$\alpha_\times=\frac{n}{2z_1}(-\zeta^2u^2+i\zeta-2i\zeta u)\sin\theta\cos\theta$$
$$-\frac{in}{z_1^2}\Big[\frac{1}{2}\zeta^2u(1-u)(1+\cos^2\theta)+i\zeta u-1\Big]\cos\theta, \tag{20}$$

$$\beta_\times=\frac{in}{z_1}\Big[\frac{1}{2}\zeta^2u(1-u)(1+\cos^2\theta)+i\zeta u-1\Big]\cos\theta, \tag{21}$$

$$z_1=(1-u)z,\quad z=|n|v\sin\theta,\quad \zeta=|\mathbf{q}|r=|n|v. \tag{22}$$

To obtain the transverse components $T_+$ and $T_\times$ of the total energy-momentum tensor, we must add the transverse components of the energy-momentum tensor of the orbiting body to (17):

$$t_+(q)=\sum_n 2\pi\delta(q^0-n\omega)m\gamma v^2\Big\{\Big[\frac{n^2}{z^2}-1+\frac{n^2}{z^2}\cos^2\theta\Big]J_n(z)$$
$$-\frac{1}{z}(1+\cos^2\theta)J_n'(z)\Big\}, \tag{23}$$

$$t_\times(q)=\sum_n 2\pi\delta(q^0-n\omega)m\gamma v^2\frac{in}{z}\Big[\frac{1}{z}J_n(z)-J_n'(z)\Big]\cos\theta. \tag{24}$$

These expressions follow from Eq. (12) of Ref. 1 and Eqs. (7)–(9) of the Introduction.

In the nonrelativistic limit $|n|v\ll1$, both $\zeta$ and the arguments of the Bessel functions are small: $z_1\sim z\sim\zeta\ll1$. Physically, this means that the orbits are small compared with the wavelength of the radiation. Expanding the Bessel functions, we obtain for the integrals defining the field energy-momentum tensor

$$\int_0^1du\,e^{i\zeta u}(\alpha_+J_n+\beta_+J_n')\approx\begin{cases}\dfrac{z}{16}(1-3\cos^2\theta), & n=1,\\[2ex] \dfrac{z^{n-2}(1+\cos^2\theta)}{2^n(n-1)!}, & n\geqslant2,\end{cases} \tag{25}$$

$$\int_0^1du\,e^{i\zeta u}(\alpha_\times J_n+\beta_\times J_n')\approx\begin{cases}\dfrac{iz}{16}\cos\theta, & n=1,\\[2ex] -\dfrac{iz^{n-2}\cos\theta}{2^n(n-1)!}, & n\geqslant2.\end{cases} \tag{26}$$

For the analogous quantities (23) and (24) in the body's energy-momentum tensor we have

$$\Big(\frac{n^2}{z^2}-1+\frac{n^2}{z^2}\cos^2\theta\Big)J_n-\frac{1}{z}(1+\cos^2\theta)J_n'$$
$$\approx\begin{cases}-\dfrac{z}{8}(3-\cos^2\theta), & n=1,\\[2ex] \dfrac{z^{n-2}(1+\cos^2\theta)}{2^n(n-2)!}, & n\geqslant2,\end{cases} \tag{27}$$

$$\frac{in}{z}\cos\theta\Big(\frac{1}{z}J_n-J_n'\Big)\approx\begin{cases}\dfrac{iz}{8}\cos\theta, & n=1,\\[2ex] -\dfrac{iz^{n-2}\cos\theta}{2^n(n-2)!}, & n\geqslant2.\end{cases} \tag{28}$$

Note here that for harmonics with $n\geqslant2$, the contribution of the field energy-momentum tensor is $n-1$ times smaller than that of the matter tensor.

In the nonrelativistic approximation, then, we have the GR spectrum

$$\frac{1}{2}|T_+|^2+2|T_\times|^2|_{n=1,v\ll 1}$$

$$=t2\pi\delta(q^0-\omega)\frac{9}{64}m^2v^6\sin^2\theta\left(1-\frac{2}{3}\sin^2\theta+\frac{1}{72}\sin^4\theta\right), \tag{29}$$

$$\frac{1}{2}|T_+|^2+2|T_\times|^2|_{n\geq 2,nv\ll 1}$$

$$=t2\pi\delta(q^0-n\omega)m^2v^4\left(\frac{nz^{n-2}}{2^{n-1}(n-1)!}\right)^2$$

$$\times\left(1-\sin^2\theta+\frac{1}{8}\sin^4\theta\right). \tag{30}$$

Not surprisingly, the largest contribution comes from the second harmonic, and it agrees with the value predicted by the Einstein equation.

In the ultrarelativistic limit ($\gamma\gg 1$), the most important harmonics and angles in the GR spectrum are those for which $n\approx z_1\approx z\approx\zeta\sim\gamma^3$ and $\alpha=\theta-\pi/2\sim\gamma^{-1}$, and the most important values of $u$ are those for which $u\sim\gamma^{-3}$. We then obtain for the integrals (25) and (26)

$$\int_0^1 du\, e^{i\zeta u}(\alpha_+J_n+\beta_+J_n')\approx -iJ_n'(z), \tag{31}$$

$$\int_0^1 du\, e^{i\zeta u}(\alpha_\times J_n+\beta_\times J_n')\approx -\frac{1}{2}J_n(z)\cos\theta, \tag{32}$$

where for $J_n(z)$ and $J_n'(z)$ we must use their asymptotic representations in terms of the Airy function:

$$J_n(z)\approx\frac{1}{\pi}\left(\frac{2}{n}\right)^{1/3}\Phi(y),\quad J_n'(z)\approx-\frac{1}{\pi}\left(\frac{2}{n}\right)^{2/3}\Phi'(y),$$

$$\Phi(y)=\int_0^\infty dt\cos\left(yt+\frac{t^3}{3}\right),\quad y=\left(\frac{n}{2}\right)^{2/3}\left(1-\frac{z^2}{n^2}\right). \tag{33}$$

In the ultrarelativistic limit, Eqs. (27) and (28) for the energy-momentum tensor of the orbiting body become

$$\left(\frac{n^2}{z^2}-1+\frac{n^2}{z^2}\cos^2\theta\right)J_n-\frac{1}{z}(1+\cos^2\theta)J_n'$$

$$\approx 2\left(\cos^2\theta+\frac{1}{2\gamma^2}\right)J_n(z), \tag{34}$$

$$\frac{in}{z}\cos\theta\left(\frac{1}{z}J_n-J_n'\right)\approx -i\cos\theta J_n'(z). \tag{35}$$

The transverse components of the body's energy-momentum tensor are clearly a factor of $\gamma$ smaller than those of the field, and can be neglected. We then obtain for the GR spectrum in the ultrarelativistic limit

$$\frac{1}{2}|T_+|^2+2|T_\times|^2|_{\gamma\gg 1}$$

$$\approx t\sum_n 2\pi\delta(q^0-n\omega)\frac{1}{2}m^2\gamma^2(\alpha^2J_n^2+J_n'^2)=\frac{m^2\gamma^2}{2e^2}|j_\mu(q)|^2. \tag{36}$$

Thus, for the system in question, the gravitational and electromagnetic spectra are related, in the ultrarelativistic limit, by Eq. (1), in which $\Gamma=\gamma$.

## 3. GRAVITATIONAL RADIATION FROM A CHARGE IN THE FIELD OF A CHARGED, FIXED CENTER WITH A MAGNETIC MOMENT

The equation of motion of a charge $e$ in a circular orbit of radius $r$, speed $v$, and angular frequency $\omega$ in the equatorial plane of a fixed center carrying charge $e'$ and magnetic moment $\mathfrak{M}$ is

$$m\gamma v^2=\frac{1}{4\pi r}\left(-ee'+\frac{e\mathfrak{M}_\omega\omega}{c}\right). \tag{37}$$

For such an orbit to exist, it is necessary that

$$-ee'+\frac{e\mathfrak{M}_\omega\omega}{c}>0. \tag{38}$$

The conserved energy-momentum tensor of the systems is made up of the tensor $t_{\alpha\beta}$ of the body, the field tensor $\theta^M_{\alpha\beta}$ proportional to the field of the magnetic moment, and the field tensor $\theta^C_{\alpha\beta}$ proportional to the Coulomb field. The transverse components of the first two tensors are given by Eqs. (23), (24), and (17). The transverse components of the energy-momentum tensors $\theta^C_{+,\times}$ were obtained for the pure Coulomb problem in Ref. 1, but were not written out explicitly. They are (see also Ref. 3)

$$\theta_A{}^C(q)=\sum_n 2\pi\delta(q^0-n\omega)m\gamma v^2$$

$$\times\int_0^1 du\, e^{i\zeta u}[\alpha_AJ_n(z_1)+\beta_AJ_n'(z_1)], \tag{39}$$

$$\alpha_+=\frac{\zeta^2}{z_1^2}(1+\cos^2\theta)-(i\zeta u-1)\left[\frac{n^2}{z_1^2}(1+\cos^2\theta)-\cos^2\theta\right], \tag{40}$$

$$\beta_+=\frac{1}{z_1}(1+\cos^2\theta)(i\zeta u-\zeta^2-1), \tag{41}$$

$$\alpha_\times=in\left[-\frac{1}{2}v^2+\frac{1}{z_1^2}(\zeta^2+1-i\zeta u)\right]\cos\theta, \tag{42}$$

$$\beta_\times=\frac{in}{z_1}(i\zeta u-1-v^2)\cos\theta. \tag{43}$$

All other quantities are the same as in (22).

In the present problem, with two external fields acting at the same time, the factors $m\gamma v^2$ in (17) and (39) must be replaced by

$$\frac{e\mathfrak{M}_\omega\omega}{4\pi cr}\equiv k^Mm\gamma v^2,\qquad -\frac{ee'}{4\pi r}\equiv k^Cm\gamma v^2$$

in those two equations, respectively, i.e., the field energy-momentum tensor is given by the sum $k^M\theta^M_A+k^C\theta^C_A$.

It can be shown that in the nonrelativistic limit, the integrals

$$\int_0^1 du\, e^{i\zeta u}[\alpha_AJ_n+\beta_AJ_n']$$

for any $n$ are the same as (25) and (26), i.e., they are given by the right-hand sides of those equations. For the effective values of $n$ and $\theta$ in the ultrarelativistic limit, they are given by (31) and (32).

Thus, the components $\theta^M_{\alpha\beta}$ and $\theta^C_{\alpha\beta}$ turn out to be the same in both the nonrelativistic and ultrarelativistic limits; but then we have $k^M\theta^M_A + k^C\theta^C_A \approx \theta^M_A(q) \approx \theta^C_A(q)$, since $k^M + k^C = 1$ holds by virtue of the equation of motion (37). This means that in the nonrelativistic domain, the GR spectrum emitted by a charge moving in this complicated field is given by (29) and (30); in the ultrarelativistic domain, it is given by (36). Note that in the intermediate case, where the velocity of the charge is neither too low nor too high, the transverse components $\theta^M_A(q)$ and $\theta^C_A(q)$ are quite different, and the gravitational spectrum turns out to be sensitive to the details of the field in which the charge is immersed.

One of the prime reasons for undertaking this study has been a desire to elucidate those properties of the field that are most important in determining the conversion amplitude $\Gamma$. Since it defines the proportionality factor between two invariants [see Eq. (1)], $\Gamma$ itself ought to be an integral invariant of the system. In all four electromagnetic systems considered in Ref. 1 and the present paper, for the same charged-particle orbits but different fields (circular motion in a Coulomb field, in the field of a circularly polarized wave, in the field of a magnetic moment, and in the field of a Coulomb center and magnetic moment), the conversion amplitude $\Gamma$ is the same, and is equal to $\gamma$.

Clearly, $\Gamma$ statisfies this simple equality because in a circular trajectory there is no inherent scale length. If instead we consider the gravitational spectrum of a charge in a circular orbit in a screened Coulomb field with potential $(e'/4\pi r)e^{-\eta r}$, then for $\gamma \gg 1$ the leading terms in the transverse components $\theta_A(q)$ of the field energy-momentum tensor will differ from (17) by a factor

$$C = \frac{\eta r}{1+\eta r} e^{\eta r} K_1(\eta r), \qquad (44)$$

where $K_1(x)$ is the modified Bessel function. We then have $\gamma = C\gamma$. As $\eta r$ increases, $C(\eta r)$ falls monotonically from 1 to 0, and for $\eta r \gg 1$ it goes as $(\pi/2\eta r)^{1/2}$. In deriving (44) it has been assumed that $\eta r \ll \gamma^{3/2}$ and $\gamma \gg 1$, which means that it applies to the case $\eta r \gg 1$.

The square-root dependence of the conversion amplitude on the intrinsic scale length $\eta^{-1}$ of the field for $\eta r \gg 1$ is easy to understand if we note that $\Gamma$ is proportional to the length $l$ of the conversion region [see Eq. (4)]. In fact, for $\gamma \gg 1$, the region in which photons are converted into gravitons extends along a line that connects the point of tangency to the circle of radius $r$ with the point at which that tangent intersects another circle of radius $r + \eta^{-1}$ (where the field and the conversion process are considerably weaker); i.e., its length is

$$l \sim [(r+\eta^{-1})^2 - r^2]^{1/2}|_{\eta r \gg 1} \approx r\left(\frac{2}{\eta r}\right)^{1/2}. \qquad (45)$$

Note that bounded relativistic motion of a charge in a Coulomb field can be treated classically if the classical radius of the orbit,

$$r = -\frac{ee'}{4\pi m v^2\gamma}\bigg|_{\gamma\gg 1} \approx -\frac{ee'}{4\pi m c^2\gamma}$$

is larger than $\hbar/p$, or in other words if the Coulomb center has a charge greater than 137. If on the other hand $|e'/e| > 170$, the Coulomb field will produce pairs and screen itself.[4]

We also point out that since for $\gamma \gg 1$ the charge will be orbiting at close to the speed of light, the conversion region engendered by the field of the charge outside its orbit will be transported away at superluminal velocities.

As shown in Sec. 2 of Ref. 1, the contribution of transverse components of the energy-momentum tensor of a material body to gravitational radiation for $\gamma \gg 1$ are of the same order as the contribution of the current to electromagnetic radiation, if we replace $Gm^2$ by $e^2/4\pi$. To order of magnitude, then, $\Gamma$ determines the ratio of the transverse components, $\theta_+$, $\theta_\times$ of the tensor $\theta_{\alpha\beta}$ to the transverse components $t_+$, $t_\times$ of the tensor $t_{\alpha\beta}$. It would be interesting to know how that ratio (or $\Gamma$) depends on the spatial distribution of the tensor $\theta_{\alpha\beta}(x)$. With that in mind, let us examine a nonelectromagnetic system in which the tensor $\theta_{\alpha\beta}(x)$ is confined to the line joining a test particle to a center of rotation.

## 4. GRAVITATIONAL RADIATION FROM A RELATIVISTIC STRING WITH MASSES AT ITS ENDS

A relativistic string with masses at its ends[5] is not an electromagnetic system. It can be viewed as a realistic model of a system containing two bodies connected by a force field confined to the line joining them. Gravitational radiation will therefore be produced not just by local sources—the point masses—but by a distributed source as well, namely the string. It is quite interesting to compare the contributions to gravitational radiation from these two sources, particularly in the ultrarelativistic limit, where the energy confined to the string is approximately $\gamma$ times the energy of the masses at either end.

The system at hand is described by the action

$$S = -\mu \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [(\dot{x}x')^2 - \dot{x}^2 x'^2]^{1/2}$$
$$\sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} d\tau \left\{\left[\frac{dx_\alpha(\tau,\sigma_i(\tau))}{d\tau}\right]^2\right\}^{1/2}, \qquad (46)$$

where $\mu$ is a constant that characterizes the tension in the string, $m_1$ and $m_2$ are the masses at the ends of the string (we shall henceforth assume both to be equal to $m$), and $x^\alpha(\tau,\sigma)$ is the four-vector that parametrizes the world surface of the string. The dot and prime denote partial derivatives with respect to $\tau$ and $\sigma$, respectively.

Equation (46) yields equations of motion with the partial solutions

$$x^1(\tau,\sigma) = \sigma \sin \omega\tau, \quad x^2(\tau,\sigma) = \sigma \cos \omega\tau,$$
$$x^3(\tau,\sigma) = 0, \quad x^0(\tau,\sigma) = t = \tau, \qquad (47)$$

which describe the string as a straight-line segment rotating at angular velocity $\omega$. In (47), we have chosen the evolution parameter $\tau$ to be the coordinate time, while $\sigma$ is the distance of a point under consideration on the string from the center of rotation (with the appropriate sign). The equations of motion of the masses at the ends of the string are the same as the boundary conditions, given by

$$\sigma_2=-\sigma_1=r,\quad \omega r=[1+(m\omega/2\mu)^2]^{1/2}-m\omega/2\mu. \tag{48}$$

The constant $2r$ is the distance between the masses at which the tension in the string dictates rotation at angular velocity $\omega$.

The system energy-momentum tensor is comprised of the energy-momentum tensor $t_{\mu\nu}$ of the masses at the ends of the string, and the energy-momentum tensor of the string itself:

$$\theta_{\mu\nu}(x)=\mu\int d\tau\, d\sigma[(\dot{x}x')^2-\dot{x}^2x'^2]^{-1/2}\delta(x-x(\tau,\sigma))\{x'^2\dot{x}_\mu\dot{x}_\nu +\dot{x}^2x_\mu'x_\nu'-(x'\dot{x})(\dot{x}_\mu x_\nu'+\dot{x}_\nu x_\mu')\}. \tag{49}$$

The latter expression can be simplified by imposing the gauge condition $x'_\alpha\dot{x}^\alpha=0$. Then

$$\theta_{\mu\nu}(x)=\mu\int d\tau\, d\sigma\left(-\frac{x'^2}{\dot{x}^2}\right)^{1/2}\delta(x-x(\tau,\sigma))\left(\dot{x}_\mu\dot{x}_\nu+\frac{\dot{x}^2}{x'^2}x_\mu'x_\nu'\right). \tag{50}$$

Making use of (47) and (48), we obtain the energy density and the energy of the string:

$$\theta_{00}(x)=\mu\int_{-r}^{r}\frac{d\sigma}{(1-\omega^2\sigma^2)^{1/2}}\delta(\mathbf{x}-\mathbf{x}(\tau,\sigma)), \tag{51}$$

$$E^{str}=\int d^3x\,\theta_{00}(x)=\frac{\mu}{\omega}\int_{-v}^{v}\frac{dx}{(1-x^2)^{1/2}}=\frac{2\mu}{\omega}\arcsin\omega r$$
$$=2mv\gamma^2\arcsin v. \tag{52}$$

In the latter expression, we have used the relationship

$$\frac{\mu}{\omega}=mv\gamma^2,\qquad \gamma=(1-v^2)^{-1/2} \tag{53}$$

between the tension and the velocity $v=\omega r$ at the loaded end of the string, which follows from (48). Interestingly, in ultrarelativistic motion ($\gamma\gg1$), the string energy $E^{\rm str}$ is $\pi\gamma/2$ times the energy $E^{\rm mass}=2m\gamma$ of the masses at its ends.

Let us now proceed from (49) to the Fourier components, and using them and Eqs. (7)–(9), construct the transverse components of $\theta_A(q)$, which describe the gravitational radiation from the string:

$$\theta_+(q)=\sum_n 2\pi\delta(q^0-n\omega)mv\gamma^2 \times\int_{-v}^{v}\frac{dx}{(1-x^2)^{1/2}}\left\{\left[\left(\frac{n^2}{z^2}-\frac{1}{2}\right)(1+\cos^2\theta) -\left(x^2-\frac{1}{2}\right)\sin^2\theta\right]J_n(z)-(1+\cos^2\theta)\frac{1}{z}J_n'(z)\right\}, \tag{54}$$

$$\theta_\times(q)=\sum_n 2\pi\delta(q^0-n\omega)mv\gamma^2 i \times\cos\theta\int_{-v}^{v}\frac{dx}{(1-x^2)^{1/2}}\frac{n}{z}\left[\frac{1}{z}J_n(z)-J_n'(z)\right]. \tag{55}$$

Here $z=|n|x\sin\theta$, and $x=\omega\sigma$ is the velocity at the point on the string with coordinate $\sigma$.

We point out that the energy-momentum tensor of a string can be represented by a sum of two terms corresponding to the two halves of the string, i.e., to $-r\leqslant\sigma<0$ and $0<\sigma\leqslant r$. In particular,

$$\theta_A(q)=\theta_A(q,-v)+\theta_A(q,v),$$
$$\theta_A(q,\pm v)=\sum_n 2\pi\delta(q^0-n\omega)\theta_{An}(q,\pm v). \tag{56}$$

The relation

$$\theta_{An}(q,v)=(-1)^n\theta_{An}(q,-v) \tag{57}$$

leads to interference between the gravitational radiation fluxes from opposite halves of the string, as a result of which the amplitudes $\theta_{An}(q)$ for odd harmonics vanish, while the even harmonics have double the amplitude emitted by each half. Similar interference takes place in the gravitational radiation emitted by the two masses at the ends of the string. The transverse components of the energy-momentum tensor of this system consist of a sum of components,

$$t_A(q)=t_A(q,-v)+t_A(q,v), \tag{58}$$

which come from Eqs. (23) and (24) with opposite signs for $v$.

We note further that the sum $t_A(q,v)+\theta_A(q,v)$ is the amplitude for gravitational radiation with polarization $A$ coming from a different object—a string of length $r$, fixed at the end $\sigma=0$ and loaded by a mass $m$ at the end $\sigma=r$, and rotating about the fixed end at angular velocity $\omega$.

Let us now analyze the behavior of $\theta_A(q,v)$ in the nonrelativistic and ultrarelativistic limits. For $nv\ll1$, expanding the Bessel functions in (54) and (55), we obtain

$$\theta_{+n}(q,v)\approx\begin{cases}\dfrac{mv^3}{16}(1-3\cos^2\theta)\sin\theta, & n=1,\\[2mm] \dfrac{mv^n(n\sin\theta)^{n-2}}{2^n(n-1)!}(1+\cos^2\theta), & n\geqslant2,\end{cases} \tag{59}$$

$$\theta_{\times n}(q,v)\approx\begin{cases} i\dfrac{mv^3}{16}\sin\theta\cos\theta, & n=1,\\[2mm] -i\dfrac{mv^n(n\sin\theta)^{n-2}}{2^n(n-1)!}\cos\theta, & n\geqslant2.\end{cases} \tag{60}$$

These expressions are identical with the nonrelativistic harmonics of the transverse components of the field energy-momentum tensor in the systems considered above; see Secs. 2 and 3. The coincidence, however, is not mere chance. It can be shown that the conserved tensor $T_{\alpha\beta}(q)$ satisfies the equation

$$q_iq_j\frac{\partial^2}{\partial q_k\partial q_l}T_{ij}(q)+2q_i\frac{\partial}{\partial q_l}T_{ik}(q)+2q_i\frac{\partial}{\partial q_k}T_{il}(q)+2T_{kl}(q) =-q^{02}\int d^3x\,e^{-i\mathbf{qx}}x_kx_lT^{00}(\mathbf{x},q^0). \tag{61}$$

For a closed nonrelativistic system, we can insert $T^{00}(\mathbf{x},q^0)\approx t^{00}(\mathbf{x},q^0)$ on the right-hand side of Eq. (61). Assuming that the system contains a single point mass moving about a center of force, we seek a solution of the resulting equation in the form

$$T_{ij}(q)=t_{ij}(q)+m\int dt\, e^{iq^0t}f(\mathbf{qx})(\ddot{x}_ix_j+\ddot{x}_jx_i)+\ldots, \qquad (62)$$

where the ellipsis denotes terms like those in (98) below that can be dispensed with for our purposes. We then obtain for the function $f(z)$ an equation and solution:

$$zf'(z)+f(z)=\frac{1}{2}e^{-iz}, \quad f(z)=\frac{1-e^{-iz}}{2iz}. \qquad (63)$$

Expanding $f(z)$ as a power series in $z$ and assuming circular motion, we obtain the nonrelativistic expressions for all harmonics $T_{ij}(q)$. Their transverse components have been given in Sec. 2 [see also (59) and (60)].

For $\gamma\gg 1$ in Eqs. (54) and (55), the most important ranges of values are $n\approx z\sim\gamma^3$, $v-|x|\sim\gamma^{-2}$, and $\alpha\equiv\pi/2-\theta\sim\gamma^{-1}$. Carrying out the appropriate expansions and making use of the representation (33), we obtain

$$\theta_{+n}(q,v)\approx\frac{2m\gamma}{\pi n}\int_1^\infty\frac{d\xi}{\xi^{1/2}}\Phi''(y), \qquad (64)$$

$$\theta_{\times n}(q,v)\approx i\frac{m\gamma\alpha}{2\pi}\left(\frac{2}{n}\right)^{2/3}\int_1^\infty\frac{d\xi}{\xi^{1/2}}\Phi'(y),$$

$$y=\left(\frac{n}{2\gamma^3}\right)^{2/3}(\xi+\gamma^2\alpha^2). \qquad (65)$$

These ultrarelativistic transverse components of the string's energy-momentum tensor turn out to be of the same order of magnitude as the transverse components of the energy-momentum tensor for the material body at the end of the string [compare Eqs. (23), (24) and (33), (35) with Eqs. (64) and (65)].

The reason why the gravitational radiation from a string turns out, for $\gamma\gg 1$, to be of the same order as gravitational radiation from a mass at its end is as follows. The condition $v-|x|\sim\gamma^{-2}$ means that radiation is launched from small segments near the ends of the string moving at velocities $x$ such that the corresponding Lorentz factor $\gamma(x)\equiv(1-x^2)^{-1/2}$ is of order $\gamma$. Even though the string energy is more than $\gamma$ times the energy of the mass, it is distributed along the string in such a way that the energy moving through space with a Lorentz factor $\gamma$ constitutes only a fraction $\gamma^{-1}$ of the total energy of the string:

$$\frac{\mu}{\omega}(\arcsin v-\arcsin v')\,|_{\gamma'\sim\gamma\gg1}\approx\frac{\mu}{\omega}\left(\frac{1}{\gamma'}-\frac{1}{\gamma}\right)\sim\frac{\mu}{\omega}\gamma^{-1}, \qquad (66)$$

i.e., it is precisely of the same order of magnitude as the energy of the mass at the end of the string [see (53)].

## 5. GRAVITATIONAL RADIATION FROM A STRING WITH UNLOADED ENDS

Before proceeding to the limit $m=0$ (or $v=1$), we rewrite Eqs. (54) and (55) in the form

$$\theta_+=\sum_n 2\pi\delta(q^0-n\omega)\frac{\mu}{\omega}\int_{-v}^{v}\frac{dx}{(1-x^2)^{1/2}}\left\{\frac{1}{4}(1+\cos^2\theta)[J_{n+2}(z)+J_{n-2}(z)]-\sin^2\theta\left(x^2-\frac{1}{2}\right)J_n(z)\right\}, \qquad (67)$$

$$\theta_\times=\sum_n 2\pi\delta(q^0-n\omega)\times\frac{\mu}{\omega}\frac{i}{4}\cos\theta\int_{-v}^{v}\frac{dx}{(1-x^2)^{1/2}}[J_{n+2}(z)-J_{n-2}(z)]. \qquad (68)$$

We now take the limit as $v\to 1$, integrating the Bessel functions with the aid of 7.7.2(11) from Ref. 6. We obtain

$$\theta_+=\sum_k 2\pi\delta(q^0-2k\omega)\frac{\pi\mu}{2\omega}\left\{J_{k+1}^2(x)+J_{k-1}^2(x)-\frac{1}{2}\sin^2\theta[J_{k+1}(x)+J_{k-1}(x)]^2\right\}$$
$$=\sum_k 2\pi\delta(q^0-2k\omega)\frac{\pi\mu}{\omega}\{\mathrm{ctg}^2\theta J_k^2(x)+J_k'^2(x)\}, \qquad (69)$$

$$\theta_\times=\sum_k 2\pi\delta(q^0-2k\omega)\frac{i\pi\mu\cos\theta}{4\omega}[J_{k+1}^2(x)-J_{k-1}^2(x)]$$
$$=\sum_k 2\pi\delta(q^0-2k\omega)\left(-\frac{i\pi\mu}{\omega}\right)\mathrm{ctg}\,\theta J_k(x)J_k'(x), \quad x=k\sin\theta. \qquad (70)$$

The GR spectrum is given by the combination

$$\frac{1}{2}|\theta_+|^2+2|\theta_\times|^2=t\sum_k 2\pi\delta(q^0-2k\omega)\left(\frac{\pi\mu}{\omega}\right)^2\left\{\frac{1}{2}[\mathrm{ctg}^2\theta J_k^2(x)+J_k'^2(x)]^2+2\,\mathrm{ctg}^2\theta J_k^2(x)J_k'^2(x)\right\}. \qquad (71)$$

The energy emitted in a time $t\gg\omega^{-1}$ is

$$\mathscr{E}=t\cdot 4\pi G\mu^2\sum_{k=1}^{\infty}k^2\int d\Omega\{\ldots\}, \qquad (72)$$

where {...} denotes the expression in curly brackets in Eq. (71).

Consider now the behavior of terms in this series for $k\gg 1$. Instead of the Bessel functions, we can then use the Airy functions [see (33)]. Bearing in mind that the dominant contribution comes from $\cos\theta\sim(2/k)^{1/2}$, we have

$$k^2\int d\Omega\{\ldots\}\approx\frac{8}{\pi^3k}\int_0^\infty\frac{dy}{y^{1/2}}[y^2\Phi^4(y)+\Phi'^4(y)+6y\Phi^2(y)\Phi'^2(y)]. \qquad (73)$$

The series (72) thus diverges logarithmically. The divergence clearly goes away when quantum effects are taken into account; the latter are important precisely for the emission of the higher harmonics.

## 6. GRAVITATIONAL RADIATION FROM A CHARGE TRAVERSING A COULOMB + MAGNETIC MOMENT FIELD

The trajectory followed by a charge $e$ as it passes a Coulomb center with charge $e'$ is a plane curve, which can be described by the parametric equations

$$x_1(\xi)=r(\xi)\cos\varphi(\xi),\quad x_2(\xi)=r(\xi)\sin\varphi(\xi),$$
$$t(\xi)=\frac{b}{(1-\gamma^{-2})^{1/2}}\left(\operatorname{sh}\xi+\frac{\varkappa}{\gamma^2}\xi\right), \tag{74}$$

where $r$ is the distance between the charge and the center, $\varphi$ is the angle with respect to the symmetry axis (the 1-axis), $t$ is the time, and

$$r(\xi)=a+b\operatorname{ch}\xi,\quad \varphi(\xi)=\frac{1}{(1-\nu^2)^{1/2}}\arcsin\frac{(1-\varkappa^2)^{1/2}\operatorname{sh}\xi}{\operatorname{ch}\xi+\varkappa}. \tag{75}$$

These equations were derived using the approach presented in §39 of Ref. 7.

The motion of the charge may be characterized by three independent parameters: the dimensionless impact parameter $\beta$, the dimensionless Lorentz factor $\gamma$ of the charge at infinity, and the dimensionless ratio $\nu=\alpha/M$ of the product of the charges $\alpha=ee'/4\pi$ to the particle angular momentum $M$. All other parameters are functions of these three:

$$a=\frac{\nu}{(1-\gamma^{-2})^{1/2}}\beta,\quad b=\beta\left(1+\frac{\nu^2}{\gamma^2-1}\right)^{1/2},\quad \varkappa=\frac{a}{b}. \tag{76}$$

In addition, the expression for the scattering angle is

$$\chi=\pi-[\varphi(\infty)-\varphi(-\infty)]=\pi-\frac{2}{(1-\nu^2)^{1/2}}\arccos\varkappa. \tag{77}$$

Let us first look at the electromagnetic spectrum emitted by the charge. We characterize the direction of the wave vector $\mathbf{q}$ by the angle $\delta$ it makes with the (1,2) plane, and the angle $\psi$ between the $(\mathbf{q},3)$ plane and the 2-axis, so that

$$\mathbf{q}=|\mathbf{q}|(\cos\delta\sin\psi,\ \cos\delta\cos\psi,\ \sin\delta). \tag{78}$$

Then taking advantage of current conservation, $q^\alpha j_\alpha(q)=0$, and $q^2=0$, we obtain for the electromagnetic spectrum

$$|j_\alpha(q)|^2=(1-\cos^2\delta\sin^2\psi)|j_1|^2+(1-\cos^2\delta\cos^2\psi)|j_2|^2 -2\cos^2\delta\sin\psi\cos\psi\operatorname{Re}j_1j_2^*. \tag{79}$$

The components of the current density are defined by the integrals

$$j_1(q)=eb\int_{-\infty}^{\infty}d\xi\, e^{-if(\xi)}\left[\operatorname{sh}\xi\cos\varphi(\xi)-\left(\frac{1-\varkappa^2}{1-\nu^2}\right)^{1/2}\sin\varphi(\xi)\right], \tag{80}$$

$$j_2(q)=eb\int_{-\infty}^{\infty}d\xi\, e^{-if(\xi)}\left[\operatorname{sh}\xi\sin\varphi(\xi)+\left(\frac{1-\varkappa^2}{1-\nu^2}\right)^{1/2}\cos\varphi(\xi)\right], \tag{81}$$

$$f(\xi)=q_1r(\xi)\cos\varphi(\xi)+q_2r(\xi)\sin\varphi(\xi) -q^0b(1-\gamma^{-2})^{-1/2}\left(\operatorname{sh}\xi+\frac{\varkappa}{\gamma^2}\xi\right). \tag{82}$$

From here on, we limit our attention to the ultrarelativistic case $\gamma\gg1$, and furthermore, we require that the parameter $\nu$ be at most of order $\gamma^{-1}$; that is,

$$\nu\lesssim1/\gamma\ll1. \tag{83}$$

Then $\varkappa\simeq\nu\ll1$, and to fourth-order accuracy in the small parameters that appear in (83),

$$\left(\frac{1-\varkappa^2}{1-\nu^2}\right)^{1/2}\approx1,\quad b\approx\beta.$$

Moreover, the scattering angle $\chi$ and the effective values of the angles $\delta$ and $\psi$ are small:

$$\chi\approx2\nu;\quad |\delta|,|\psi|\sim\frac{1}{\gamma}\ll1. \tag{84}$$

Carrying out the appropriate expansions in Eqs. (74) and (80)–(82), we obtain

$$x_1(\xi)=\beta\left[1+\nu\operatorname{ch}\xi-\frac{1}{2}\nu^2\operatorname{sh}\xi\arcsin(\operatorname{th}\xi)+\ldots\right], \tag{85}$$

$$x_2(\xi)=\beta\left[\operatorname{sh}\xi-\frac{1}{2}\nu^2\operatorname{sh}\xi+\frac{1}{2}\nu^2\arcsin(\operatorname{th}\xi)+\ldots\right],$$

$$j_1\approx e\beta\nu S,\quad j_2\approx e\beta C, \tag{86}$$

$$|j_\alpha(q)|^2\approx e^2\beta^2[\nu^2|S|^2+(\delta^2+\psi^2)|C|^2-2\nu\psi\operatorname{Re}(SC^*)], \tag{87}$$

where

$$(S,C)=\int_{-\infty}^{\infty}d\xi(\operatorname{sh}\xi,\operatorname{ch}\xi)e^{-if(\xi)}, \tag{88}$$

$$f(\xi)=\eta-z\operatorname{sh}\xi+w\operatorname{ch}\xi+s\arcsin(\operatorname{th}\xi), \tag{89}$$

$$\eta=\beta q^0\psi,\quad z=\frac{1}{2}\beta q^0\left(\delta^2+\psi^2+\nu^2+\frac{1}{\gamma^2}\right),$$
$$w=\beta q^0\nu\psi,\quad s=\frac{1}{2}\beta q^0\nu^2. \tag{90}$$

Since the scattering angle $\chi$ is so small that (83) and (84) are satisfied, the Coulomb field essentially affects the motion of the charge only over a distance of the order of the impact parameter $\beta\approx b$. Consequently, the effective values of $\xi$ lie in the range $\xi\sim1$ [see (74), (75), (85)]. It is then clear from (89) that the dominant values in the integrals (88) are $z,w,s\sim1$, and accordingly, $q^0\sim\beta^{-1}\gamma^2$; see §77 of Ref. 7.

The integrals in (88) cannot be expressed in terms of elementary functions. When the deviation angle satisfies $\chi\ll\gamma^{-1}$, however, we have

$$S\approx2ie^{-i\eta}K_1(z),\quad C\approx2i\frac{w}{z}e^{-i\eta}K_1(z),$$
$$\chi\ll\gamma^{-1}\ll1. \tag{91}$$

In that event, the electromagnetic spectrum will be given by

$$|j_\alpha(q)|^2=4e^2\nu^2\beta^2K_1^2(z)\left[1-\frac{4\psi^2}{\gamma^2(\delta^2+\psi^2+1/\gamma^2)^2}\right], \tag{92}$$

and the total emitted energy will be

$$\mathscr{E}_{EM}=\int\frac{d^3q}{16\pi^3}|j_\alpha(q)|^2=\frac{\pi e^4e'^2\gamma^2}{4(4\pi)^3m^2\beta^3}, \quad (93)$$

which is consistent with the equation in Problem 1 of §73 in Landau and Lifshitz.[7]

We now go on to calculate the GR spectrum, which is determined by the two transverse component of the full energy-momentum tensor. These can very simply be expressed in terms of the components of the energy-momentum tensor in a coordinate system $K'$ whose 3'-axis lies in the same direction as **q**; see (6). The transformation to the $K'$ system from the $K$ system that we have been dealing with, in which trajectories lie in the (1,2) plane symmetrically about the 1-axis and the wave vector of the radiation is specified by the angles $\delta$ and $\psi$ [see (78)], can be accomplished through two spatial rotations. Specifically, these are $K\to K''$, a rotation by $\psi$ about the 3-axis that carries **q** into the (3",2") plane of the intermediate $K''$ coordinate system (3" = 3), and $K''\to K'$, a rotation by $\theta=\pi/2-\delta$ about the 1" axis such that **q** winds up lying along the 3'-axis in $K'$. Writing the components $T'_{ij}(q')$ of the energy-momentum tensor in $K'$, which appears in (6), in terms of the components $T_{ij}(q)$ in $K$, we obtain

$$\begin{aligned}T_+(q)=&(\cos^2\psi-\sin^2\delta\sin^2\psi)T_{11}-2\sin\psi\cos\psi(1+\sin^2\delta)T_{12}\\&+(\sin^2\psi-\sin^2\delta\cos^2\psi)T_{22}\\&+2\sin\delta\cos\delta(\sin\psi T_{13}+\cos\psi T_{23})-\cos^2\delta T_{33}, \quad (94)\end{aligned}$$

$$\begin{aligned}T_\times(q)=&\sin\delta\sin\psi\cos\psi(T_{11}-T_{22})+\sin\delta(\cos^2\psi-\sin^2\psi)T_{12}\\&-\cos\delta(\cos\psi T_{13}-\sin\psi T_{23}). \quad (95)\end{aligned}$$

To construct the spatial components $T_{ij}=t_{ij}+\theta_{ij}$, we make use of Eqs. (3), (18), (43), and (44) from Ref. 1, yielding

$$t_{ij}(q)=m\int_{-\infty}^{\infty}d\xi\,x_i'x_j'(t'^2-\mathbf{x}'^2)^{-1/2}e^{-if(\xi)},\quad i,j=1,2, \quad (96)$$

$$\theta_{ij}(q)=-i\alpha|\mathbf{q}|\int_{-\infty}^{\infty}d\xi\,e^{-if(\xi)}\int_0^1du\,e^{iu(\mathbf{qx}+|\mathbf{q}|r)}\times\left\{\frac{x_ix_j'+x_jx_i'}{2r}+t'\frac{x_ix_j}{r^2}\left(u+\frac{i}{|\mathbf{q}|r}\right)\right\}+\ldots, \quad (97)$$

where the prime denotes a derivative with respect to $\xi$. The ellipsis in Eq. (97) stands for terms of the form

$$(q_ib_j+q_jb_i)A,\quad \delta_{ij}B, \quad (98)$$

where $A$ and $B$ are rotation-invariant functions that depend on **q** and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, which characterize the actual trajectory of the charge, the direction of the trajectory's symmetry axis, and the tangent to the trajectory at its apex; **b** is one of the three vectors **q**, $\mathbf{e}_1$, or $\mathbf{e}_2$.

In the coordinate rotation $K\to K'$, the expressions in (98) transform into

$$(q_i'b_j'+q_j'b_i')A,\quad \delta_{ij}B, \quad (99)$$

thereby making no contribution to the transverse components of the energy-momentum tensor (7), since $q_1'=q_2'=0$. We can therefore refrain from calculating both the terms in (98) and the components $T_{13}$, $T_{23}$, $T_{33}$, which consist solely of such terms.[8]

Going on to examine the ultrarelativistic case—more specifically, (83)—we have the corresponding expressions for the components $t_{ij}$, $\theta_{ij}$, $i,j=1,2$:

$$t_{11}\approx\alpha v\int_{-\infty}^{\infty}d\xi\,\mathrm{th}\,\xi\,\mathrm{sh}\,\xi\exp[-if(\xi)],\qquad t_{12}\approx\alpha S,\quad t_{22}\approx\frac{\alpha}{v}C, \quad (100)$$

$$\begin{aligned}\theta_{11}&\approx\alpha v\int_{-\infty}^{\infty}d\xi\,\mathrm{th}\,\xi\exp[-\xi-if(\xi)],\\ \theta_{12}&\approx\frac{1}{2}\alpha\int_{-\infty}^{\infty}d\xi\exp[-\xi-if(\xi)], \quad (101)\\ \theta_{22}&\approx\alpha\int_{-\infty}^{\infty}d\xi\,\mathrm{sh}\,\xi\exp[-\xi-if(\xi)].\end{aligned}$$

Here $f(\xi)$ is given by Eq. (89), and we have used $m\gamma\beta v\approx\alpha$. The components $\theta_{ij}$ are clearly of order $t_{ij}$, except for $\theta_{22}\sim vt_{22}\ll t_{22}$. For the $T_{ij}$ we obtain

$$T_{11}\approx\alpha vS,\quad T_{12}\approx\frac{1}{2}\alpha(S+C),\quad T_{22}\approx\frac{\alpha}{v}C. \quad (102)$$

Substituting these components into (94) and (5), omitting all terms containing $T_{13}$, $T_{23}$, $T_{33}$, and bearing in mind that $\delta$ and $\psi$ are small angles [see (84)], we finally have the GR spectrum:

$$8\pi G\left(\frac{1}{2}|T_+|^2+2|T_\times|^2\right)=\frac{4\pi Gm^2\gamma^2}{e^2}[(v-\psi)^2+\delta^2]|j_\alpha(q)|^2. \quad (103)$$

Here the spectrum $|j_\alpha(q)|^2$ is given by Eq. (87).

Thus, when an ultrarelativistic charge passes through a Coulomb field and the deviation angle satisfies $\chi\lesssim\gamma^{-1}$, the spectrum of gravitational radiation is proportional to the electromagnetic spectrum, with a proportionality factor (conversion amplitude)

$$\Gamma=\gamma[(v-\psi)^2+\delta^2]^{1/2} \quad (104)$$

that depends on the direction of the wave vector and the orbital parameters.

Since $v$ and the dominant values of the angles $\delta$ and $\psi$ are constrained by (83) and (84), we have $\Gamma\sim1$, consistent with the estimate in Eq. (5) of the Introduction.

While the proportionality of the GR and electromagnetic spectra results from the fact that we are dealing with an ultrarelativistic system, the increase in $\Gamma$ to $\Gamma\sim1$ stems from the confinement of the region where the GR is produced essentially to the same region as where the electromagnetic radiation is produced. The net result is that the two sources of GR—the local energy-momentum tensor $t_{\alpha\beta}$ of the material body and the nonlocal energy-momentum tensor $\theta_{\alpha\beta}$ of the intrinsic and external electromagnetic fields—make contributions of the same order of magnitude.

For $\nu \ll \gamma^{-1} \ll 1$, if we substitute (92) into (103), we obtain the GR spectrum obtained by Gal'tsov *et al.*[9] to the same approximation. His integration over the wave vector **q** yields a total GR energy $\mathscr{E}_G$ that differs from $\mathscr{E}_{EM}$ in (93) by a factor $4\pi Gm^2/e^2$, a result previously obtained by Peters.[10]

We now consider the gravitational radiation from a charge traversing the field due to a magnetic moment in its equatorial plane. The charge's motion may be characterized by the same parameters $\beta$, $\gamma$ as before, and for the dimensionless interaction parameter $\nu = \alpha/M$ we have

$$\nu = -\frac{e\mathfrak{M}}{4\pi M\beta}, \tag{105}$$

where $\mathfrak{M}$ is the magnitude of the magnetic moment, which points along the 3-axis. The equations of motion can be solved by successive approximations in that parameter, which is assumed to be small. In the zeroth approximation, the particle moves parallel to the 2-axis. Two iterations yield

$$\begin{aligned} x_1(\xi) &= \beta\left[1+\nu \operatorname{ch}\xi - \frac{3}{2}\nu^2 \operatorname{sh}\xi \arcsin(\operatorname{th}\xi)+\ldots\right], \\ x_2(\xi) &= \beta\left[\operatorname{sh}\xi + \frac{1}{2}\nu^2 \arcsin(\operatorname{th}\xi)+\ldots\right], \\ t(\xi) &= \frac{\beta}{(1-\gamma^{-2})^{1/2}}\left(1+\frac{1}{2}\nu^2\right)\operatorname{sh}\xi+\ldots, \\ r(\xi) &= \beta(\operatorname{ch}\xi+\nu+\ldots). \end{aligned} \tag{106}$$

For small $\nu$, the deviation angle satisfies $\chi \approx 2\nu$ [cf. (84)]. Notwithstanding the fact that the solutions (106) differ from (85) and (74) by terms of order $\nu^2$, when (83) holds over the effective range of **q**, the function $f(\xi) = q_\alpha x^\alpha(\xi)$ is in accord with (89). Second-order terms in $\nu$ and $\gamma^{-1}$ in the expressions for $j_1, j_2$ of the electromagnetic current and $t_{11}$, $t_{12}$, $t_{22}$ of the energy-momentum tensor of the material body can be neglected in the factors preceding the exponentials, whereupon $|j_\alpha(q)|^2$ and $t_{ij}$ turn out to be the same as in (87) and (100). The calculations demonstrate that the transverse components $\theta_+$, $\theta_\times$ of the field energy-momentum tensor are identical to the transverse components constructed out of the components in (100) in the Coulomb case. For $\nu \lesssim \gamma^{-1} \ll 1$, then, the GR spectrum of a charge traversing a magnetic-moment field is given by the very same formula (103) as the Gr spectrum of a charge traversing a Coulomb field.

## 7. DISCUSSION AND CONCLUSION

For the electromagnetic systems considered here and in Ref. 1, the GR spectrum is proportional to the electromagnetic spectrum when $\gamma \gg 1$. One argument favoring the universality of this relationship is the virtually plane-wave nature of the external field in the rest frame of the ultrarelativistic charge. For a plane-wave field, Eq. (1) is exact no matter what the velocity of the charge, a result obtained in Ref. 1 for a linearly or circularly polarized monochromatic field that clearly holds for a more general plane-wave field as well. The proportionality of the spectra is quite natural when the energy-momentum tensor of the field dominates that of the material body, in which case the current $j(q-k_{ef})$ entering into the estimate (2) is almost on the mass shell, and (4) yields the estimated conversion amplitude $\Gamma$. If the two energy-momentum tensors are of the same order, then the latter estimate of $\Gamma$ will hold if the components of the currents $j(q-k_{ef})$ and $j(q)$ transverse to **q** are comparable in magnitude. If on the other hand the transverse components of $j(q-k_{ef})$ are much greater than those of $j(q)$, as occurs when the trajectory of the charge is almost a straight line through the field ($\chi \ll \gamma^{-1} \ll 1$), then $\Gamma \sim 1$ [see Eq. (5)].

The way in which the amplitude $\Gamma$ depends upon the characteristics of the force field, the direction of the wave vector, and the constants of the motion of the charged particle—especially its Lorentz factor—governs the behavior of both the external field and the self-field of the charge over a relatively large region, making it difficult to account quantitatively for the conversion of photons into gravitons. In assessing the GR spectrum, it is therefore important to understand the qualitative behavior of $\Gamma$, since the properties of the electromagnetic spectrum $|j_\alpha(q)|^2$ may be assumed to be known.

One point of interest of non-electromagnetic systems is a qualitative comparison of the transverse components $\theta_+$, $\theta_\times$ and $t_+$, $t_\times$. For a rotating relativistic string with mass loading, these components contribute approximately equal amounts of GR, despite the fact that the string energy is $\pi\gamma/2$ the energy of the masses at its ends.

Note that the contribution of the energy-momentum tensor of a material body in circular motion to its gravitational radiation is given by the exact expression (13) in Ref. 1. For arbitrary ultrarelativistic motion of the body, we may estimate the GR contribution made by its energy-momentum tensor at effective values of **q**:

$$8\pi G\left(\frac{1}{2}|t_+|^2+2|t_\times|^2\right) \sim \frac{4\pi Gm^2}{e^2}|j_\alpha(q)|^2. \tag{107}$$

For circular motion, the low-order harmonics in this estimate have an additional factor of $\gamma^2$ on the right-hand side. In that event, the region in which the radiation is produced consists of the whole orbit, and radiation angles of order unity are important. In going from the $j_i$ and $t_{ij}$ to the transverse components, therefore, the latter are not reduced by factors of $\gamma$ and $\gamma^2$, respectively, in contrast to the components in the dominant range of **q**.

Our present results are also applicable to bunches of charged particles that are small compared with the wavelength of the emitted radiation, since the latter is then coherent. Since the wavelength of the fundamental from a single particle is a factor of $\gamma^3$ shorter than the wavelength of the first harmonic, it is possible to imagine a situation in which a bunch of charged particles emit coherently in the lower-order harmonics, and incoherently in harmonics with $n \sim \gamma^3$.


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