Model of the early universe in $f(R)$ theory

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(Submitted 13 April 1990)
Zh. Eksp. Teor. Fiz. **98**, 1138–1150 (October 1990)

We investigate the cosmology of the early universe with a polarized vacuum of conformal and nonconformal fields. Allowance for the latter yields some novel possibilities for the evolution of the background metric; specifically, it predicts several quasi-de Sitter stages. We use the instanton approach to estimate the probability of the Universe being created in each of these. If the Universe is created in a de Sitter regime with curvature close to the Planck value, then subsequent quasiexponential inflation is feasible if the mass of the nonconformal scalar fields satisfies $m \gtrsim 10^{-5} M_{Pl}$. The quasi-de Sitter evolutionary regime ($R_{ik} \approx g_{ik} R/4$) enables one to describe all known models with vacuum polarization using an effective Lagrangian $L(R) = -R + f(R)$. For this theory, we have quantized scalar perturbations and analyzed their growth during the subsequent evolution of the Universe, taking advantage of the mathematical equivalence between classical cosmology and a scalar field φ with the potential $V(\varphi)$, which is uniquely defined by the form of $f(R)$. Our results can be used both for the R^2 cosmology of the Starobinskii model and for a theory with nonconformal fields.

INTRODUCTION

Inflationary models of the early universe make it possible, in principle, to relate an effective Lagrangian and initial conditions to cosmological data obtained at the present epoch. In addition to the simplest previously investigated models with vacuum polarization of conformally covariant fields and with scalar fields, it would also be of interest to study inflationary models which, apart from the conformally covariant fields, also take into account nonconformal fields in the guise of quantized scalar fields with an effective mass m.

At a quasi-de Sitter stage ($R_{ik} = \frac{1}{4} g_{ik} R$), the effective energy-momentum tensor can be obtained by adding an appropriate function $f(R)$ to the Einstein Lagrangian, such that the overall Lagrangian depends solely on the single scalar curvature R. Foremost among the theories conforming to such a Lagrangian are the R^2 cosmology[1] and the Starobinskii model[2] (R^2 + conformal anomaly). The contribution of nonconformal fields can be taken into account in the same way.[3]

The construction of such a theory is most profitably begun with a general analysis of the Lagrangian $f(R)$, as reflected by the title of the present paper. Another reason that it needs to be examined is the question of the conformal correspondence of cosmological models to a scalar field and to $f(R)$ theory, a property utilized below in quantizing scalar perturbations in $f(R)$ theory. In the final analysis, the growth of such perturbations determines the large-scale structure of the Universe.

In the present paper, we present the results of an investigation of a model of the early universe that incorporates vacuum polarization of both conformally covariant and nonconformal fields. In Section 1 we present a detailed study of background cosmological models and their phase portraits. In Section 2 we obtain the equations of linear scalar perturbation theory for $L(R)$ cosmology. Finally, in Section 3, we construct a theory of the transition from a model with polarized corrections to models with a scalar field.

1. DYNAMICS OF BACKGROUND MODELS

One requirement of all scenarios of the early universe is an inflationary expansion stage in which the Universe, starting out at a size of the order of the Planck length, expands to dimensions of order 1 cm in 10^{-41} sec. The global structures of the observable Universe and the structure of large-scale perturbations are formed in the process.

Among the mechanisms driving the inflationary stage, the two most popular are the filling of the Universe with a quasiclassical high-intensity scalar field,[4] and polarization of the quantum field vacuum in a highly curved Friedmann spacetime model.[2,3] The first inflationary model with a polarized vacuum of conformally covariant fields was the Starobinskii scenario.[2]

The corresponding mean energy-momentum tensor in that model consisted of two parts,

$$\varkappa \langle \underset{1}{T_i^k} \rangle = \frac{1}{M_c^2} \left[2R_{;i}^{;k} - 2\delta_i^k R_{;l}^{;l} - 2RR_i^k + \frac{1}{2} \delta_i^k R^2 \right], \quad (1.1)$$

$$\varkappa \langle \underset{2}{T_i^k} \rangle = -\frac{1}{H_c^2} \left[R_l^k R_i^l - \frac{2}{3} RR_i^k - \frac{1}{2} \delta_i^k R_m^l R_l^m + \frac{1}{4} \delta_i^k R^2 \right], \quad (1.2)$$

where M_c and H_c are parameters that depend on the number and type of conformally covariant fields.

The Einstein equation with $\varkappa \langle \underset{1}{T_i^k} \rangle$ can be obtained via the variational principle from the action

$$S = \frac{1}{16\pi G} \int (-g)^{1/2} L(R)\, d\Omega, \qquad L(R) = -R + \frac{R^2}{6M_c^2}. \quad (1.3)$$

The expression for $\varkappa \langle \underset{2}{T_i^k} \rangle$, the so-called conformal anomaly, does not in principle follow from variation of the Lagrangian. It can be approximated at a quasi-de Sitter stage (see Ref. 5), where

$$R_i^k \approx \delta_i^k R/4, \quad R_{ik} R^{ik} \approx R^2/4. \quad (1.4)$$

 0038-5646/90/100636-07$03.00

In that approximation,

$$\underset{2}{\varkappa\langle T_l{}^l\rangle}=\left(\frac{R^2}{3}-R_{ik}R^{ik}\right)\frac{1}{H_c^2}\approx\frac{R^2}{12H_c^2}. \qquad (1.5)$$

Variation of (1.3) with $L = L_{ca}(R)$ leads to

$$R\frac{\partial L_{ca}}{\partial R}-2L_{ca}(R)+3\left(\frac{\partial L_{ca}}{\partial R}\right)^{;l}_{;l}=\varkappa T_l{}^l, \qquad (1.6)$$

which at the quasi-de Sitter stage should yield

$$RL_{ca}'(R)-2L_{ca}(R)=\frac{R^2}{12H_c^2}. \qquad (1.7)$$

Hence,

$$L_{ca}=\frac{R^2}{24H_c^2}\ln\frac{R^2}{R_0^2}. \qquad (1.8)$$

In the quasi-de Sitter approximation, the model with conformally covariant fields is described by the Lagrangian

$$L_c(R)=-R+\frac{R^2}{6M_c^2}+\frac{R^2}{24H_c^2}\ln\frac{R^2}{R_0^2}. \qquad (1.9)$$

The corresponding problems for R^2 cosmology $(H_c^2\to\infty)$[6] and for the model in Ref. 2 have been investigated previously.

In the present communication, we investigate the various possibilities that arise in the model described by (1.9) by virtue of the fact that it encompasses nonconformal field vacuum polarization effects. This problem has long been of special interest due to its connection with the bounce problem.[3] The results obtained are also important for the theory developed in Ref. 2.

According to Starobinskii,[3] when the contribution of nonconformal fields is taken into consideration, a term

$$L_n(R)=-\frac{R^2+m^4}{6M_n^2}\ln\frac{R^2+m^4}{m^4} \qquad (1.10)$$

is added to (1.9), where m and M_n are the characteristic mass and a parameter that depends on the number and type of fields.

The contraction of (1.6) for $L = L_c + L_n$ yields a transcendental equation that enables one to search for possible de Sitter solutions:

$$-R+R^2\left(\frac{1}{3M_n^2}-\frac{1}{12H_c^2}\right)-\frac{m^4}{3M_n^2}\ln\frac{R^2+m^4}{m^4}=0. \qquad (1.11)$$

For the $(M_n,H_c\to\infty)R^2$ model, there is a unique de Sitter solution $R=0$, corresponding to a flat Universe. In the Starobinskii model $(M_n\to\infty)$, besides $R=0$ we also have $R=-12H_c^2$.

We now introduce the notation

$$p=-\frac{1}{12H_c^2}+\frac{1}{3M_n^2},\qquad \rho=2pR. \qquad (1.12)$$

For $p<0$, i.e.,

$$4H_c^2\leqslant M_n^2, \qquad (1.13)$$

Eq. (1.11), in the notation of (1.12), takes the form

$$F(\rho,k)=(\rho-1)^2+\ln\left(\frac{\rho^2+k}{1+k}\right)\Big/\ln\left(\frac{1+k}{k}\right)=0, \qquad (1.14)$$

where

$$k=4p^2m^4,\qquad \frac{4pm^4}{3M_n^2}=-\left[\ln\frac{1+k}{k}\right]^{-1}. \qquad (1.15)$$

The behavior of $F(\rho,k)$ is shown in Fig. 1 for various values of k.

As a rule, depending on the value of k, a model with nonconformal fields will have either two or four de Sitter solutions, including $\rho=0$. For a closed Friedmann model, the instanton method[5] can be used to calculate the probability of the Universe being created in each of these solutions. To do so, the de Sitter metric

$$ds^2=dt^2-H^{-2}\,\mathrm{ch}^2(Ht)\,[d\chi^2+\sin^2\chi\, d\Omega^2(\theta,\varphi)]$$

is analytically continued to imaginary time by making the substitution $t\to i(\tau+\pi/2H)$. The instanton action is of the form

$$S_i=\frac{24\pi}{G}\left[\frac{2}{\rho_i}\left(\frac{1}{12H_c^2}-\frac{1}{3M_n^2}\right)+\frac{1}{6M_c^2}+\frac{1}{24H_c^2}\times\ln\frac{\rho_i^2}{k}-\frac{1+k\rho_i^2}{6M_n^2}\ln\frac{\rho_i^2+k}{k}\right]. \qquad (1.16)$$

When the inequality in (1.13) holds, we have

$$S_i\approx\frac{4\pi}{G}\left\{\frac{1}{M_c^2}+\left(\frac{1}{4H_c^2}-\frac{1}{M_n^2}\right)\ln\frac{\rho_i^2}{k}\right\}, \qquad (1.17)$$

where $\rho_i\gg k$. By virtue of (1.12), the first term is the dominant one in S_i and the probability of creation $P\sim\exp(-4\pi/GM_c^2)$ is approximately the same for all ρ_i corresponding to the roots of (1.14). That probability also applies to the Starobinskii scenario.

An analysis of the flat background model for the action (1.3) with the Lagrangian $L=L_c(R)+L_n(R)$ has been carried out for the 00 component of the generalized Einstein equation. Assuming $M_c\ll H_c,M_n$, that equation can be reduced to the form

$$-H^2-12pH^4+M_c^{-2}(\dot H^2-6H^2\dot H-2\ddot H H)+\frac{m^4}{36M_n^2}\ln\frac{R^2+m^4}{m^4}=0, \qquad (1.18)$$

where $H=\dot a/a$.

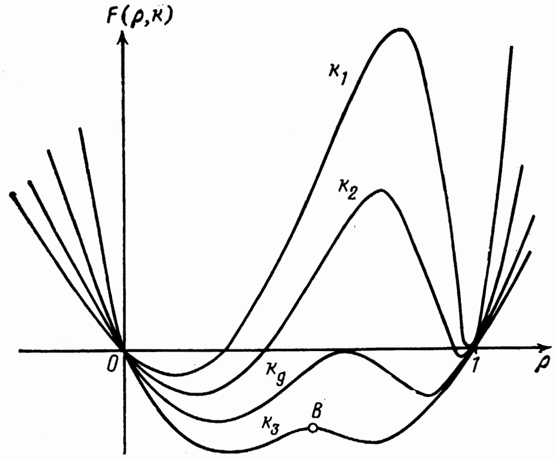

FIG. 1. $F(\rho,k)$ as a function of ρ for different values of k: $k_3>k_g>k_2>1$.

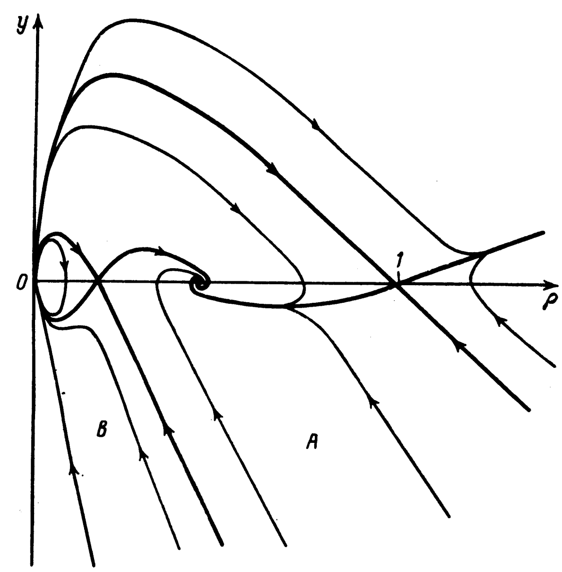

FIG. 2. Phase portrait for the case of four de Sitter solutions.

A qualitative study of this equation in the phase plane can be carried out most advantageously in dimensionless variables corresponding to the notation of (1.12):

$$y=\dot H/M_c^2, \quad \rho=-24pH^2.$$

Equation (1.18) then takes the form

$$\frac{dy}{d\rho}=-\left\{y^2+6By\rho-\frac{B}{2}\left[\rho(\rho-2)+C\ln\frac{\rho^2+k}{k}\right]\right\}\frac{1}{4y\rho},$$
$$B=-(24pM_c^2)^{-1}, \quad C=-\left[\ln\frac{k}{1+k}\right]^{-1}. \quad (1.19)$$

Figures 2 and 3 contain phase portraits for the case of four and two de Sitter solutions, respectively.

In the first instance there is a trapping region A, a stable focus from which the Universe as a whole cannot escape to a scalar stage and form our type of universe. We therefore concentrate on an analysis of trajectories in region B with regular Friedmann asymptotic behavior.

It is clear from Fig. 2 that some subset of the trajectories in that region pass near the exact de Sitter solution; for these, there is a quasi-de Sitter regime in which

$$a(t)\propto\exp\left(\int H(t)dt\right), \quad |\dot H|\ll H^2, \quad |\ddot H|\ll H|\dot H|. \quad (1.20)$$

Equation (1.18) can then be simplified:

$$H^2-H_0^2\left(1-\frac{m^4}{36M_n^2H^2}\ln\frac{R^2+m^4}{m^4}\right)=\frac{6H_0^2}{M_c^2}\dot H,$$
$$H_0^2=-1/12p. \quad (1.21)$$

For the exact de Sitter solution $R_*=-12H_*^2\,(\dot H_*=0)$, the second term on the left-hand side of (1.21) equals H_*^2. For solutions near the de Sitter solution ($H\lesssim H_*$), therefore, Eq. (1.21) can be written in the form

$$H^2-H_*^2=\frac{6H_0^2}{M_c^2}\dot H. \quad (1.22)$$

This equation can easily be integrated:

$$H(t)=H_*\,\mathrm{th}\left[\gamma-\frac{M_c^2H_*}{6H_0^2}t\right], \quad (1.23)$$

where γ can be expressed in terms of the deviation of the Hubble "constant" at the initial time $t=0$ from the exact value H_*:

$$\gamma=-\frac{1}{2}\ln\frac{H_*-H_i}{H_*+H_i}\approx-\frac{1}{2}\ln\frac{\delta R}{2R_*}\sim-\ln\frac{H_*^2}{M_{Pl}^2}. \quad (1.24)$$

This deviation is due to quantum fluctuations in the curvature of the new born universe.[5]

Equation (1.23) describes the initial stage in the evolution of the universe following its "birth," which consumes a time of order $t_*=6\gamma H_0^2/M^2H_*$, after which the approximation (1.20) is no longer valid. The scale factor may be found by integrating (1.20):

$$a(t)\propto a_0\left[\mathrm{ch}\left(\gamma-\frac{M_c^2H_*}{6H_0^2}t\right)\right]^{-6H_0^2/M^2}. \quad (1.25)$$

It is thus clear that for an initial time $\Delta t\lesssim t_*$, the universe passes through a de Sitter stage with

$$a(t)\propto\exp[H_*t], \quad (1.26)$$

following which, for a time $\Delta t_2\approx 6H_0^2/M^2H_*$, it passes through a quasi-de Sitter stage with

$$a(t)\propto\exp\left[-\frac{M_c^2H_*^2t^2}{12}\right]. \quad (1.27)$$

When there are two de Sitter solutions (Fig. 3), the analysis near point 1 is exactly the same as before.

It is also possible to obtain a second inflationary stage. As can be seen from Fig. 1, a situation involving three de Sitter solutions can come about, but in actuality it would be highly unlikely, requiring as it does that the parameters of the theory be tuned extremely carefully. At the same time, the proximity of the peak B in Fig. 1 to the horizontal axis means that an additional inflationary stage is feasible over a rather wide range in k. The phase portrait for this case is

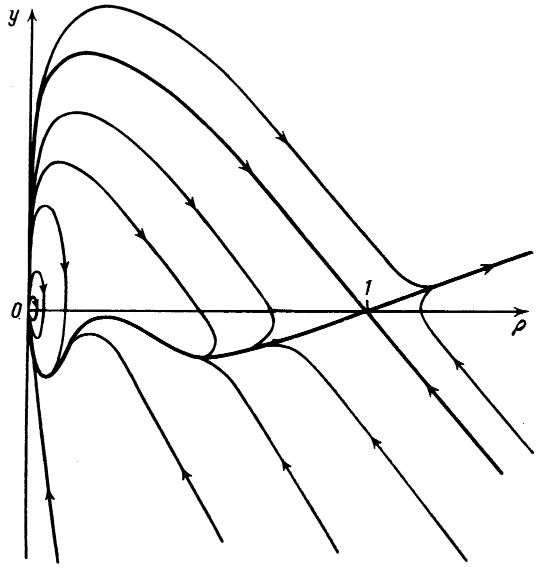

FIG. 3. Phase portrait for the case of two de Sitter solutions.

shown in Fig. 3, where ti is clear that the universe, having been "born" near the exact de Sitter solution, passes in succession through a stage of exponential inflation according to (1.26), quasiexponential inflation according to (1.27), and finally one more inflationary stage.

Let us analyze the latter period of evolution, which may be described as before by Eq. (1.18). Assuming the validity of (1.20), that equation becomes

$$H^2+12pH^4-\frac{m^4}{36M_n^2}\ln\frac{R^2+m^4}{m^4}=-\frac{6H^2\dot{H}}{M_c^2}. \quad (1.28)$$

Substituting $\rho=-24pH^2$ directly into the expression for $F(\rho,k)$ [see (1.14)], we can show that at the peak, the left-hand side of (1.28) equals $F(\rho,k)=F(H^2,k)$ to within some constant factor. That makes it possible to model the left-hand side of (1.28) near the peak as

$$H^2+12pH^4-\frac{m^4}{36M_n^2}\ln\frac{R^2+m^4}{m^4}=-\varepsilon-\beta\Delta H^2. \quad (1.29)$$

Setting $H=H_*-\Delta H$ in (1.29) and equating coefficients of corresponding terms, we have

$$\beta=2, \quad H_*^2=\frac{H_0^2}{2}=-\frac{1}{24p}, \quad (1.30)$$

i.e., the parameter p here determines the value of H_* at the peak B. If we then insert (1.29) into (1.28) and make use of (1.30), we obtain

$$-6H_*^2\Delta\dot{H}/M_c^2=\varepsilon+2\Delta H^2, \quad (1.31)$$

which for small ε has the solution

$$H=-\frac{\varepsilon M_c^2}{6H_*^2}t+H_*. \quad (1.32)$$

Clearly, over a time of order $t\lesssim 6H_*^2/\varepsilon M_c^2$, the universe undergoes inflation for a second time. We may now estimate the parameter ε. As we already noted, at some k_g, one more de Sitter solution H_g makes its appearance (see Fig. 1), which means that at a certain H_g, the left-hand side of Eq. (1.29) goes to zero:

$$H_g^2+12p_gH_g^4-\frac{m^4}{36M_n^2}\ln\frac{R_g+m^4}{m^4}=0. \quad (1.33)$$

Subtracting (1.31) from (1.27) and making use of (1.28), we have for $H=H_*$

$$\varepsilon\sim(H_g^2-H_*^2)/2. \quad (1.34)$$

Furthermore, since $|\dot{H}|\ll H^2$ in the inflationary stage, we obtain with the aid of (1.32) another constraint,

$$\frac{\varepsilon M^2}{6H_*^2}\ll H_*^2. \quad (1.35)$$

Further numerical modeling indicates that this sort of double inflation can occur if $k\gtrsim 10^{-2}$. Assuming that the creation of the universe takes place at curvatures (see Refs. 1,5)

$$|R|\geqslant 10^{-4}M_{Pl}^2, \quad (1.36)$$

(1.12) and (1.15) yield the following constraint on the effective mass of nonconformal fields:

$$m\geqslant 10^{-5}M_{Pl}. \quad (1.37)$$

2. LINEAR PERTURBATIONS IN MODELS WITH VACUUM POLARIZATIONS

The quasi-de Sitter approximation required to provide a sufficiently protracted inflationary stage will also ensure that the conditions of (1.4) are met. Using those conditions, most of the known corrections to the Einstein Lagrangian can be reduced to the form $L=R+f(R)$. The problem that arises at that point is to derive the fundamental equations of relativistic perturbation theory for $L(R)$ models. That investigation was first undertaken by Nariai[7] for regular cosmological models with a bounce;[8] the theory was constructed by analogy with the well-known paper by Lifshitz.[9] For gravitational and rotational perturbations, we have, in the notation of that paper,

$$h_{0i}=0, \quad \underset{1}{h_\beta{}^\alpha}=\nu(\eta)G_\beta{}^\alpha, \quad \underset{2}{h_\beta{}^\alpha}=w(\eta)[S_\beta^{;\alpha}+S^\alpha_{;\beta}], \quad (2.1)$$

where G^α_β and $S^{;\alpha}_\beta$ are a harmonic tensor and vector, which satisfy

$$\nabla^2S_\alpha=-k^2S_\alpha, \quad S^{;\alpha}_{;\alpha}=0, \quad \nabla^2G_\beta{}^\alpha=-k^2G_\beta{}^\alpha, \quad G_\alpha{}^\alpha=G^\alpha_{\beta,\alpha}=0.$$

Nariai's results reduce to two equations for the amplitudes:

$$\nu''+\left[2\frac{a'}{a}+\frac{L_R'}{L_R}\right]\nu'+k^2\nu=0,$$

$$w''+\left[2\frac{a'}{a}+\frac{L_R'}{L_R}\right]w'=0.$$

From here on, perturbations are considered superposed on a flat Friedmann metric, and a prime denotes differentiation with respect to conformal time.

No equations for second-order scalar perturbations were obtained in the papers cited. This program was implemented for the special case of the R^2 model in Refs. 1 and 6. We present that result below for an arbitrary $L(R)$ model, preserving the notation of the original paper.

Perturbations are considered in the conformal Newtonian gauge

$$ds^2=a^2(\eta)\left[(1+2\Phi)d\eta^2-(1-2\Psi)(dx^2+dy^2+dz^2)\right]. \quad (2.2)$$

With the notation (see Ref. 1)

$$\alpha^2-\alpha'=\frac{F''}{2F}-\alpha\frac{F'}{F}, \quad \alpha=\frac{a'}{a}, \quad F=\frac{dL}{dR}=L_R,$$

$$y=\Phi+\Psi, \quad \delta R=-\frac{L_R}{L_{RR}}(\Phi-\Psi), \quad (2.3)$$

$$u=\frac{F^{1/2}}{F'}ay, \quad z=\frac{(aF^{1/2})'}{a^2F'}$$

we obtain an equation for u from the linearized equations for $L(R)$:

$$u''-\Delta u-\frac{z''}{z}u=0. \quad (2.4)$$

In the short-wavelength approximation $k^2\gg z''/z$,

$$u\propto\exp\left(i\int k\,dt/a\right)+\text{c.c.}$$

In the most important long-wavelength case $(k^2\ll z''/z)$,

$$u\propto A\left[\frac{F^{1/2}}{\dot{F}}-\frac{(aF^{1/2})^{\cdot}}{a^2\dot{F}}\int aF\,dt\right]. \quad (2.5)$$

In this same approximation,

$$\Phi=A\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot},\quad \Psi=\Phi+A\frac{\dot F}{aF^2}\int aF\,dt, \quad (2.6)$$

where a dot denotes differentiation with respect to t.

The equations derived in Ref. 1 for a special case remain valid for arbitrary $F(R)$ as well.

3. CONFORMAL CORRESPONDENCE BETWEEN MODELS WITH VACUUM POLARIZATION AND MODELS WITH A SCALAR FIELD; QUANTIZATION OF SCALAR PERTURBATIONS

The physical analogy between cosmological scenarios with vacuum polarization and cosmology with a quasiclassical scalar field poses the problem of identifying a mathematical analogy between the two approaches. The problem is also an important one because a number of problems are easier to solve in cosmology with a scalar field than in models with vacuum polarization. In particular, scalar perturbation theory, which relies on a great deal of mathematical machinery, falls into that category (see Section 2).

The first example of a conformal correspondence was constructed for R^2 cosmology, which was rigorously reduced to the Einstein model with a scalar field.[10,11]

In general form, the correspondence between models with vacuum polarization and those with a scalar field is incomplete. In the quasi-de Sitter regime, however, as pointed out in Section 1, all vacuum polarization effects can be incorporated into an additional term $f(R)$ in the Lagrangian. This makes possible a general method for transforming between the two models. Let us consider a conformal transformation of the metric

$$\tilde g_{\mu\nu}(x)=\Omega^2(x)g_{\mu\nu}(x). \quad (3.1)$$

The scalar curvature then transforms as

$$R=\Omega^2\tilde R-6\Omega^3\tilde\Box\frac{1}{\Omega}, \quad (3.2)$$

where

$$\tilde\Box=\frac{1}{(-\tilde g)^{1/2}}\frac{\partial}{\partial x^\mu}\left[(-\tilde g)^{1/2}\tilde g^{\mu\nu}\frac{\partial}{\partial x^\mu}\right],\quad \tilde g=\Omega^8 g. \quad (3.3)$$

The transition from the original $L=R+f(R)$ theory with polarization corrections (f theory) to a scalar field theory (Ψ theory) is accomplished through a succession of transformations:

$$S=-\frac{1}{16\pi G}\int d^4x(-g)^{1/2}[R+f(R)]$$
$$=-\frac{1}{16\pi G}\int\frac{d^4x(-\tilde g)^{1/2}}{\Omega^4}[R+\alpha(R)R-\alpha(R)R+f(R)]$$
$$=-\frac{1}{16\pi G}\int d^4x(-\tilde g)^{1/2}\left[\frac{R(1+\alpha(R))}{\Omega^4}-\frac{\alpha(R)R-f(R)}{\Omega^4}\right]. \quad (3.4)$$

Setting $1+\alpha(R)=\Omega^2$ and making use of (3.2), after some straightforward manipulation of (3.4) we find

$$S=-\frac{1}{16\pi G}\int d^4x(-\tilde g)^{1/2}\left(\tilde R-\frac{6}{\Omega^2}\Omega_{,\mu}\Omega^{,\mu}-\frac{\alpha(R)R-f(R)}{\Omega^4}\right)$$
$$-\frac{1}{16\pi G}\int d^4xD, \quad (3.5)$$

where

$$D=\frac{\partial}{\partial x^\mu}\left[\Omega(-\tilde g)^{1/2}\tilde g^{\mu\nu}\frac{\partial}{\partial x^\nu}\frac{1}{\Omega}\right]$$

is a divergence. Introducing the scalar function $\Psi=\ln[1+\alpha(R)]=\ln\Omega^2$, we have

$$\Omega^2=e^\Psi,\quad \Omega_{,\mu}={}^1/_2\Omega\Psi_{,\mu}. \quad (3.6)$$

Substitution of (3.6) into (3.5) then yields

$$S=-\frac{1}{16\pi G}\int d^4x(-\tilde g)^{1/2}\left[\tilde R-\frac{3}{2}\Psi_{,\mu}\Psi^{,\mu}-V(\Psi)\right], \quad (3.7)$$

where we have omitted the divergence term D and carried out the following series of transformations:

$$\frac{\alpha(R)R-f(R)}{\Omega^4}=\frac{\alpha(R(\Psi))R(\Psi)-f(R(\Psi))}{\Omega^4},$$
$$\frac{F(\Psi)}{\Omega^4}=F(\Psi)e^{-2\Psi}=V(\Psi). \quad (3.8)$$

Finally, with the notation

$$\Psi=\left(\frac{16\pi G}{3}\right)^{1/2}\varphi=\ln\Omega^2=\ln' F,\quad \tilde V(\varphi)=-\frac{V(\Psi)}{16\pi G}, \quad (3.9)$$

we obtain the Lagrangian for the scalar-field theory,

$$L=-\frac{\tilde R}{16\pi G}+\frac{1}{2}\varphi_{;\mu}\varphi^{;\mu}-V(\varphi). \quad (3.10)$$

In order to determine the exact form of $\alpha(R)$, we proceed as follows. Varying the action containing the Lagrangian (3.10), we obtain

$$\tilde R_\nu{}^\mu-{}^1/_2\delta_\nu{}^\mu\tilde R=8\pi G[\varphi^{;\mu}\varphi_{;\nu}-\delta_\nu{}^\mu({}^1/_2\varphi^{;l}\varphi_{;l}-V(\varphi))]. \quad (3.11)$$

Using the inverse of the metric $g_{\mu\nu}=\Omega^{-2}g_{\mu\nu}$, the latter relation can be reduced to

$$R_\nu{}^\mu-\frac{1}{2}\delta_\nu{}^\mu R+\alpha R_\nu{}^\mu-\delta_\nu{}^\mu\frac{f}{2}+\alpha\frac{\delta_\nu{}^\mu}{2}-\alpha_{;\nu}^{;\mu}+\alpha_{;l}^{;l}\delta_\nu{}^\mu=0. \quad (3.12)$$

Comparing (3.12) with the equations of the Einstein $f(R)$ theory,

$$R_\nu{}^\mu-\frac{1}{2}\delta_\nu{}^\mu R+\frac{\partial f}{\partial R}R_\nu{}^\mu-\delta_\nu{}^\mu\frac{f}{2}+\frac{\partial f}{\partial R}\frac{\delta_\nu{}^\mu}{2}$$
$$-\left(\frac{\partial f}{\partial R}\right)_{;\nu}^{;\mu}+\left(\frac{\partial f}{\partial R}\right)_{;l}^{;l}\delta_\nu{}^\mu=0, \quad (3.13)$$

we find

$$\alpha(R)=\partial f/\partial R. \quad (3.14)$$

Likewise, it can be shown that the equation for the scalar field,

$$\tilde\Box_\varphi+V_{,\varphi}=0 \quad (3.15)$$

is transformed into the contraction of the equations (3.13).

We now show how, with the help of the correspondence thus established, one can proceed to analyze the evolution of scalar perturbations in the f theory. First, we find from (3.9) that

$$g_{\mu\nu}=\Omega^{-2}\tilde g_{\mu\nu}=F^{-1}\tilde g_{\mu\nu}=e^{-\beta\varphi}\tilde g_{\mu\nu},\quad \beta=(3/16\pi G)^{-1/2}. \quad (3.16)$$

Then, by varying the latter, we obtain the relation between perturbations of the metric in f theory and perturbations of the metric and scalar field in Ψ theory:

$$\delta g_{\mu\nu}=e^{-\beta\varphi}\delta\tilde{g}_{\mu\nu}-\beta e^{-\beta\varphi}\tilde{g}_{\mu\nu}\delta\varphi. \qquad (3.17)$$

In a conformal Newtonian reference frame, this last relation takes on an especially simple form:

$$\Phi=\tilde{\Phi}-\frac{\beta}{2}\delta\varphi,\quad \Psi=\tilde{\Psi}+\frac{\beta}{2}\delta\tilde{\varphi}, \qquad (3.18)$$

where Φ and Ψ are perturbations of the metric in f theory, and $\tilde{\Phi}=\tilde{\Psi}$ and $\delta\tilde{\varphi}$ are perturbations of the metric and scalar field in Ψ theory, respectively. For the latter we have the analytic expressions[12]

$$\tilde{\Phi}=\frac{C\varphi'}{\tilde{a}}e^{-ik\eta},$$
$$\delta\tilde{\varphi}=\frac{C}{4\pi G\tilde{a}}\left(\frac{\varphi''}{\varphi'}+\frac{ik}{\tilde{a}}\right)e^{-ik\eta}, \qquad (3.19a)$$

at short wavelengths, and

$$\tilde{\Phi}=\frac{A}{\tilde{a}}\left(\frac{1}{\tilde{a}}\int d\eta\,\tilde{a}^2\right)',$$
$$\delta\tilde{\varphi}=\frac{A\varphi'}{\tilde{a}}\left(\frac{1}{\tilde{a}}\int d\eta\,\tilde{a}^2\right). \qquad (3.19b)$$

at long wavelengths.

Substituting $\varphi'=F'/\beta F$, which follows from (3.9), and the obvious relation $\tilde{a}^2=Fa^2$ into (3.19), we find with (3.18) that

$$\Phi=-\frac{2C}{3\beta F^{1/2}}\left(\frac{\dot{F}}{F}-\frac{5}{2}\frac{\dot{F}}{F}+H+\frac{ik}{a}\right)e^{ik\eta},$$
$$\Psi=\frac{2C}{3\beta F^{1/2}}\left(\frac{\dot{F}}{F}+\frac{1}{2}\frac{\dot{F}}{F}+H+\frac{ik}{a}\right)e^{-ik\eta} \qquad (3.20a)$$

or

$$\Phi=A\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot},$$
$$\Psi=\Phi+\frac{A\dot{F}}{aF^2}\int aF\,dt. \qquad (3.20b)$$

For the R^2 model, these latter relations reduce to familiar expressions.[1]

Using this conformal correspondence, one can then determine the amplitude of quantum (seed) fluctuations as well. We assume that we have crossed over from f theory to Ψ theory, for which an appropriate quantum gauge-invariant perturbation theory has been developed.[13] We can therefore take advantage of that technique, and then revert to the original f theory, as has been accomplished for classical scalar perturbations.

In accordance with (3.18), we may calculate the correlation function for fluctuations of the metric $\Phi=\tilde{\Psi}-1/2\beta\delta\tilde{\varphi}$, $\Psi=\tilde{\Phi}+1/2\beta\delta\tilde{\varphi}$, where $\tilde{\Phi}=\tilde{\Psi}$, and the $\delta\tilde{\varphi}$ are gauge-invariant quantities which in the conformal Newtonian reference frame may be identified with fluctuations of the metric and scalar field. The latter are related by

$$\tilde{\Phi}'+\alpha\tilde{\Phi}=4\pi G\varphi'\delta\tilde{\varphi},\quad \alpha=\tilde{a}'/\tilde{a}, \qquad (3.21)$$

where a prime denotes differentiation with respect to the conformal time η.

In the quantum theory, the quantities $\tilde{\Phi}$, $\delta\tilde{\varphi}$, as well as Φ, Ψ in (3.18), acquire the status of operators.

The amplitude $u_k(\eta)$ in the expansion

$$\hat{\tilde{\Phi}}(\eta,x^\alpha)=2^{-1/2}\frac{\varphi'}{\tilde{a}}\int\frac{d^3k}{(2\pi)^{3/2}}\left[u_k^*(\eta)e^{i\mathbf{kx}}\hat{a}_k^-+u_k(\eta)e^{-i\mathbf{kx}}\hat{a}_k^+\right] \qquad (3.22)$$

satisfies the equation (see Eq. (43) of Ref. 13)

$$u_k''(\eta)+\left[k^2-\left(\frac{1}{z}\right)''\Big/\frac{1}{z}\right]u_k(\eta)=0, \qquad (3.23)$$

where $z=\tilde{a}\varphi'/a$, and $\hat{a}_k^+$ and $\hat{a}_k^-$ are creation and destruction operators for quanta of the field Φ.

We first calculate the correlation function for fluctuations of $\tilde{\Phi}$. Using (3.21) to evaluate $\delta\varphi$, and making use of (3.22) as well, we obtain

$$\Phi=\tilde{\Phi}-\frac{\beta}{2}\delta\tilde{\varphi}=2^{-1/2}\int\frac{d^3k}{(2\pi)^{3/2}}\left[\Phi_k^*e^{i\mathbf{kx}}\hat{a}_k^-+\Phi_ke^{-i\mathbf{kx}}\hat{a}_k^+\right],$$
$$\Phi_k=\frac{\varphi'}{\tilde{a}}u_k-\frac{\beta}{8\pi G\tilde{a}}\frac{\varphi''}{\varphi'}u_k-\frac{\beta}{8\pi G}u_k'. \qquad (3.24)$$

Thereafter, the calculation of the correlation function follows along standard lines:

$$\langle 0|\Phi,\Phi|0\rangle=\langle 0|\hat{\Phi}(\eta,x)\hat{\Phi}(\eta,x^\alpha+r)|0\rangle$$
$$=\int|\delta_{k,\Phi}|^2\frac{\sin kr}{kr}\frac{dk}{k},\qquad |\delta_{k,\Phi}|^2=\frac{1}{4\pi^2}|\Phi_k|^2k^3, \qquad (3.25)$$

To find the amplitude $|\delta_{k,\Phi}|$, we must solve Eq. (3.23). In the short-wavelength limit, particle creation is suppressed, so we can assume solutions in the form

$$u_k=Ce^{ik\eta},\quad C=-4\pi G\frac{iN(k\eta)}{k^{3/2}},$$
$$u_k^*=C^*e^{-ik\eta},\quad C^*=4\pi G\frac{iN^*(k\eta)}{k^{3/2}} \qquad (3.26)$$

where $|N|\to 1$ when $k\eta\to\infty$. Expressing φ in (3.24) in terms of F using (3.9) and substituting (3.26) into (3.24), we have

$$\Phi_k=-\frac{2C}{3\beta F^{1/2}}\left(\frac{\dot{F}}{F}-\frac{5}{2}\frac{\dot{F}}{F}+H+\frac{ik}{a}\right)e^{ik\eta}. \qquad (3.27)$$

In the long-wavelength limit we can write

$$u_k(\eta)=A_k\frac{1}{\varphi'}\left(\frac{1}{\tilde{a}}\int\tilde{a}^2\,d\eta\right)',$$
$$u_k^*(\eta)=A_k^*\frac{1}{\varphi'}\left(\frac{1}{\tilde{a}}\int\tilde{a}^2\,d\eta\right)'. \qquad (3.28)$$

In that limit, then, as in the previous case, we have

$$\Phi_k=A_k\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot}. \qquad (3.29)$$

In the frequency range $H(\eta)a(\eta)>k>H_0a_0$, both solutions are valid, and we can match the asymptotic behavior of (3.27) and (3.29):

$$\Phi_k(\eta)=\left\{\frac{2Ce^{ik\eta_0}}{3\beta F^{1/2}}\left(-\frac{\dot{F}}{F}+\frac{5}{2}\frac{\dot{F}}{F}-H-\frac{ik}{a}\right)\right.$$
$$\times\left.\left[\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot}\right]^{-1}\right\}_{k\sim Ha}\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot}, \qquad (3.30)$$

where the subscript $k\sim Ha$ means that its value is to be estimated at such time as the perturbation clears the horizon. Substituting (3.30) and (3.27) into (3.25), we have

$$|\delta_{k,\Phi}|=\begin{cases}\left(\frac{G}{3\pi}\right)^{1/2}\left|H(t)+\frac{ik}{a(t)}\right|\frac{1}{F^{1/2}}, & k_{ph}>H(t),\\ \left(\frac{G}{3\pi}\right)^{1/2}\left\{\frac{H(t)}{F^{1/2}}\left[\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot}\right]^{-1}\right\}_{k\sim Ha} & \\ \times\left(\frac{1}{aF}\int aF\,dt\right)^{\cdot}, & H(t)>k_{ph}>k_0\frac{a_0}{a(t)}.\end{cases} \tag{3.31}$$

Calculating the correlation function for Ψ in similar fashion, we obtain $|\delta_{k,\Psi}|=|\delta_{k,\Phi}|$. In particular, the resulting spectrum for the R^2 model is, up to numerical factors, exactly the same as before.[1] In that case, therefore, all of the estimates for the numerical parameters remain essentially unchanged.

The authors thank A. A. Starobinskii, V. F. Mukhanov, and L. A. Kofman for valuable remarks.

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Translated by Marc Damashek