Hall effect in thin films

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The theory of analytic functions is used to develop a method of qualitative and quantitative study of the Hall effect in nonuniform thin films that actually amounts to extending the properties of the film to the empty space surrounding the latter. The method is applicable to infinite, semi-infinite and striplike films, for which appropriate sets of stationary solutions have been obtained.

INTRODUCTION

In the last few years, studies of various effects associated with the flow of electric current through thin films have been highly popular. This is due to the practical importance as well as the purely physical attractiveness of the item studied, i.e., a 2-*D* medium in 3-*D* space.[1] However, most of the work in this area has dealt with the local properties of thin films, whereas because of the free interaction of charges and currents via the electromagnetic field through a vacuum, films are a curious example of a nonlocal medium.[2] The present article investigates the influence of this nonlocality on the ordinary Hall effect in nonuniform films of various configurations (in uniform films, as will be shown below, the corresponding term drops out of the equation, and the Hall effect does not affect the distribution of current in the film). It turns out that despite the complex form (nonlinear because of the influence of the self-field and integrodifferential because of nonlocality) of the equation describing this phenomenon, one can nevertheless make substantial progress in its solution with the aid of methods of the theory of analytic functions.

Let a film of small thickness $\delta \to 0$ be located in the xy plane (see Fig. 1a), and let its properties be characterized by surface conductivity Σ, carrier concentration N, and current density J. Then the Hall effect in the film is described by the following system of equations:[2]

$$\mathbf{E} = \frac{1}{Nec}[\mathbf{J}, \mathbf{e}_z(B_0+B_z)] + \frac{\mathbf{J}}{\Sigma}, \qquad \frac{\partial B_z}{\partial t} = -\mathbf{e}_z c \operatorname{curl} \mathbf{E}, \tag{1}$$

where $B_0\mathbf{e}_z$ is the external constant magnetic field, and B_z is the z component of the intrinsic field. The first equation holds in the presence of only one type of carriers (specifically, electrons); for a solid-state plasma, this signifies compensation of their charge by the lattice (not by holes) and for a gaseous plasma–rapidity of the processes, leaving the massive ions unaffected.[3]

If the current flows in the direction of nonuniformity of the film: $\mathbf{J}\|\mathbf{e}_y$, i.e., $B = (B_x, B_z)$, and $N = N(y)$ ($\Sigma =$ const; for $(\mathbf{J}\nabla N) = 0$, the Hall effect is manifested only in the presence of carrier vortex flows[2]), there follows from Eq. (1) a single 1-*D* equation which holds for $z = 0$:

$$\frac{\partial B_z}{\partial t} = \frac{\partial}{\partial y}\left(\frac{1}{Ne}\right)(B_0+B_z)J - \frac{c}{\Sigma}\frac{\partial J}{\partial x}, \tag{2}$$

in which in the case $\partial/\partial x \gg \partial/\partial y$ one can set

$$\frac{\partial}{\partial y}\left(\frac{1}{Ne}\right) = \text{const.}$$

The nonlocality of the medium is manifested in an integral relationship between J and B_z, i.e., Ampere's law (the displacement current being neglected):

$$B_z(x,0) = \frac{2}{c}\int_{-\infty}^{+\infty} \frac{J(x')}{x'-x}\,dx'.$$

For the treatment below, it is convenient, after using the relation

$$\frac{2\pi}{c}J(x) = B_x(x,+0),$$

to replace J in Eq. (2) by the x component of the magnetic field and rewrite this equation in the dimensionless form

$$\frac{\partial B_z}{\partial t} = (1+2B_z)B_x - \frac{\partial B_x}{\partial x} \qquad z=+0. \tag{3}$$

The choice of the signs of B_0, $\mathbf{B}$, and $\partial N/\partial y$ made here is entirely immaterial, since all the cases can be obtained by means of the substitutions $\mathbf{B} \to -\mathbf{B}$, $x \to -x$, $t \to -t$. Equation (3) is the basic equation studied in this work.

Thus, the constraint (3) is imposed on the magnetic field $\mathbf{B}$ for $z = +0$. On the other hand, outside the film the field satisfies the usual vacuum relations

$$\operatorname{curl} \mathbf{B}=0, \qquad \operatorname{div} \mathbf{B}=0, \tag{4}$$

which suggests the consideration of the complex function $w = B_z + iB_x$, which is analytic outside the film in the complex $\zeta = x + iz$ plane (see Fig. 1b), since, as is well known, the condition of its analyticity $\partial w/\partial\bar{\zeta} = 0$ is identical to Eqs. (4). If one can now write the relationship (3) between the real and imaginary parts of w in the form of an equation for the function itself, then this equation, according to the analytic continuation theorem, can thereby be extended to the entire region of analyticity of w. The resulting local differential equation will now be comparatively simple to solve. It should be emphasized here that from a physical standpoint, there is no local relationship between the components of $\mathbf{B}$ in a vacuum except (4) (the Hall effect is absent in a vacuum), but the magnetic field is so strictly specified by its sources in the film (the analytic function in the region is completely defined by its value at the boundary) that such a relationship actually appears. Below, the indicated program is carried out in three cases.

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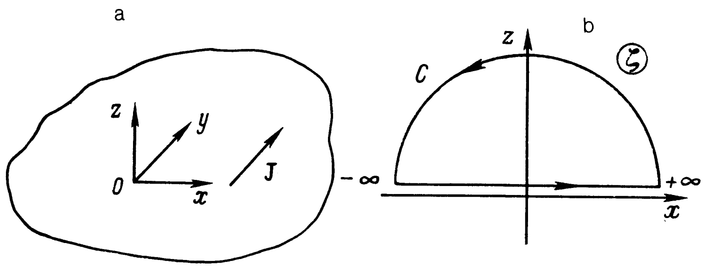

FIG. 1. Geometry of the problem of an infinite film in physical space (a) and on the complex ζ plane (b).

1. INFINITE FILM

The basic results in this geometry were obtained in Ref. 2. Here they are given a somewhat different treatment, convenient for subsequent generalizations.

In an infinite film, the components of the magnetic field in Eqs. (3) are related nonlocally (integrally) to each other by Ampere's law and its converse, which can be found by means of a Fourier transformation:

$$B_z(x,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{B_x(x',+0)}{x'-x}dx',$$
$$B_x(x,+0)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{B_z(x',0)}{x'-x}dx', \quad (5)$$

as in the case of the real and imaginary parts of generalized susceptibility (Kramers-Kronig relations). Thus in this case, the functions w, analytic in the upper halfplane ζ, are the objects of study. For them, the relationship (5) can be obtained from the relation

$$\frac{1}{2\pi i}\oint_C\frac{w(\zeta')}{\zeta'-\zeta}d\zeta'=w(\zeta), \quad (6)$$

where the contour C is indicated in Fig. 1(b).

From a physical standpoint, the change from the vector field $\mathbf{B}$ to the complex function w in this problem signifies the replacement of the sources in the film by the sources in the lower halfplane ζ; such sources are the singular points of w [the simplest examples are $w = i/(\zeta + i)$ and $w = 1/(\zeta + i)$, which correspond to the magnetic field in the region $z > 0$, respectively, produced by the linear magnetic charge and current which are located at the point $(0, -1)$].

The contour integral in Eq. (6) has the remarkable property that, being applied to any function w that has no singular points on the contour C, it converts this function into an analytic one when $\mathrm{Im}\,\zeta > 0$ holds. This makes it the tool best suited for carrying out the indicated program—the extension of (3) to the entire upper halfplane ζ. Actually, this property results in two methods of converting the arbitrary function $u(x)$ specified on the x axis to a function of the class studied, namely, applying to this function the linear operators

$$L_{\mathrm{Re}}u=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(x)}{x-\zeta}dx,$$
$$L_{\mathrm{Im}}u=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)}{x-\zeta}dx, \quad (7)$$

the first of which makes it the real part, and the second, the imaginary part of a function analytic in the upper halfplane when $\mathrm{Im}\,\zeta = 0$. In this geometry, the simple relationship $L_{\mathrm{Im}} = iL_{\mathrm{Re}}$ exists between these operators. It is evident from the above that

$$L_{\mathrm{Re}}B_z(x,\ 0)=L_{\mathrm{Im}}B_x(x,\ +0)=w. \quad (8)$$

If the operator L_{Im} is now applied to Eq. (3), it follows that since the square and the derivative of an analytic function are analytic in the same region, then, as can readily be verified, it is transformed into the equation

$$\frac{\partial}{\partial t}L_{\mathrm{Im}}B_z(x,0)=w+w^2-\frac{\partial w}{\partial \zeta}, \quad (9)$$

which is integrated completely because of the indicated simple relationship between L_{Im} and L_{Re}. (In principle, the operator L_{Re} could also be applied to Eq. (3), in which case w would appear on the left-hand side of the equation obtained, but the present notation is convenient for subsequent generalizations.) Apparently, this method was first proposed in Ref. 4 for an equation completely analogous to Eq. (2), but in somewhat different terms applicable to a vortex in an ideal fluid (see also Ref. 2).

An interesting property of Eq. (9)—the nonlocal Hall effect on an infinite nonuniform film—is the presence of a specific stationary solution. This property exists only in the absence of an external magnetic field—the first linear term on the right-hand side of Eq. (9) (here and below, by solutions are meant those analytic in the proper region)—and is written in the form

$$w=-\frac{1}{\zeta+a},\qquad \mathrm{Im}\,a>0,$$

where a is a complex constant. This corresponds to a current with a Lorentzian profile of arbitrary width ($\mathrm{Im}\,a$), which transfers the carriers to the region where their concentration increases, and with a fixed integral

$$I=\int_{-\infty}^{\infty}J\,dx=\frac{1}{2}\left[J(x)=\frac{1}{2\pi}\,\mathrm{Im}\,w(x,+0)\right].$$

Such a current is in a state of neutral equilibrium because of the balance of the two terms in the equation. In the case of smaller total current in Eq. (9), dissipation (the term with the derivative) predominates, and the profile J spreads out with time, and in the case of a larger current, the nonlinearity predominates, and J contracts in a finite time.[2]

2. SEMI-INFINITE FILM

For semi-infinite films (see Fig. 2), the components of B on the film are, as before, nonlocally related to each other

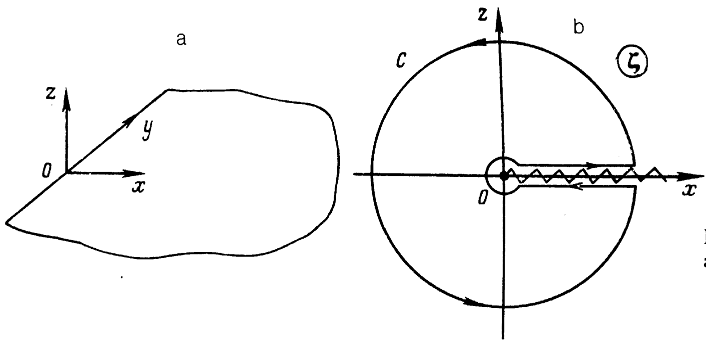

FIG. 2. Geometry of the problem of an infinite film in physical space (a) and on the ζ complex plane (b).

by Ampere's law and its converse, which now is obtained by means of a Mellin transformation [cf. Eqs. 5]:

$$B_z(x,0)=\frac{1}{\pi}\int_0^\infty\frac{B_x(x',+0)}{x'-x}dx',$$
$$B_x(x,+0)=-\frac{1}{\pi}\int_0^\infty\frac{B_z(x',0)}{x'-x}\left(\frac{x}{x'}\right)^{1/2}dx', \tag{10}$$

in which, after the substitution $x\to x^2$, one can again recognize Kramers-Kronig relations for the even real and odd imaginary parts of generalized susceptibility. This means that here the object of study becomes functions w analytic in the ζ plane with a cut along the real semiaxis (see Fig. 2), and on this cut Re w is continuous and Im w changes sign [i.e., on the film $B_x(x,+0)=-B_x(x,-0)$]. The simplest example of such a function is $w=i/(\zeta^{1/2}+i)$—in this geometry, the singular points (sources of the field) are located on the second sheet of a Riemannian surface. For such functions, the relation (10) follows from [see Eq. (6)]

$$\frac{1}{2\pi i}\oint_C\frac{w(\zeta')}{\zeta'-\zeta}\left[1+\left(\frac{\zeta}{\zeta'}\right)^{1/2}\right]d\zeta'=2w(\zeta). \tag{11}$$

Here the contour C is indicated in Fig. 2, and the function $\zeta^{1/2}$ takes real positive values at the upper edge of the cut.

Now the arbitrary function $u(x)$ is changed into a function belonging to the class under investigation by means of the operators

$$L_{\mathrm{Re}}u=\frac{1}{\pi i}\int_0^\infty\frac{u(x)}{x-\zeta}\left(\frac{\zeta}{x}\right)^{1/2}dx,$$
$$L_{\mathrm{Im}}u=\frac{1}{\pi}\int_0^\infty\frac{u(x)}{x-\zeta}dx \tag{12}$$

(the operator L_{Im} makes it the imaginary part of the analytic function at the upper edge of the section), between which there is no longer the simple relationship that existed in the first case. From a mathematical standpoint, this is the "price" for the symmetry of the functions, and from a physical standpoint, for access to the film on both sides. Nevertheless, Eqs. (8) and (9) remain valid, and the term

$$-\frac{B_x(0,+0)}{\pi\zeta}$$

is added only to the right-hand side of Eq. (9). The temporal equation is not integrated here, since it remains nonlocal in the ζ plane as well, but the stationary Hall effect in the semi-infinite nonuniform film is studied just as simply as in the infinite film. The stationary Hall effect is described by the Riccati equation

$$\frac{dw}{d\zeta}-w^2-w-\frac{\alpha}{\zeta}=0,\qquad \alpha=-\frac{B_x(0,+0)}{\pi}. \tag{13}$$

The set of its solutions is substantially richer than for a film without boundaries—any additional limitation promotes a variety of stationary states. In this case, equilibrium is possible even without allowing for the self-field [the nonlinear term in Eq. (13)], although with a different sign of either the external field or $\partial N/\partial y$, or when the film is located in the region $x<0$ [after the substitution $x\to -x$ in Eq. (13)]. In the latter case $J\propto\exp x$. For $J(0)\neq0$, the self-field on the boundary is logarithmically large ($\ln\delta$), and its inclusion may prove necessary. In the absence of the external field and for $J(0)\propto\alpha=0$ we obtain from Eq. (13)

$$w=-\frac{1}{\zeta-a},\qquad \mathrm{Im}\,a=0,\qquad \mathrm{Re}\,a>0$$

is a δ-shaped current with the integral $I=1/2$ (cf. the infinite film). Without the external field and for $J(0)\neq0$, the solution exists only for $\alpha>0$, i.e., at such a current at the boundary that transfers the carriers to where their concentration decreases. For this direction of the current in the infinite film, nonlinearity and dissipation are acting in the same direction, and the profile of the current spreads and can be "supported" against the edge of the film. In this case, on substitution $w=-v'/v$, (13) changes into

$$v''+\frac{\alpha v}{\zeta}=0,$$

whence

$$v=\zeta^{1/2}\{iH_1^{(1)}[2(\alpha\zeta)^{1/2}]-AJ_1[2(\alpha\zeta)^{1/2}]\}.$$

Here $H_1^{(1)}$ corresponds to a Hankel function of the first kind on the upper edge of the cut and of the second kind on the lower edge, J_1 is a Bessel function, and A is a positive (otherwise, on the negative semiaxis Im $\zeta=0$, where v, naturally, is real, a zero of the function v will appear, and hence, a pole of w) real constant. For $A=0$, $\alpha=1$, this corresponds to

$$J=\frac{1}{2\pi}\mathrm{Im}\,w(0,+0)=-\{2\pi^2x[N_1^2(2x^{1/2})+J_1^2(2x^{1/2})]\}^{-1},$$

where N_1 is a Neumann function.

In general, by substituting $w=-v'/v-1/2$, Eq. (13) can be converted into the Laplace equation

$$\zeta v''+\left(\alpha-\frac{\zeta}{4}\right)v=0,$$

for the solution of which the following integral representation is obtained by means of the Laplace method:[5,6]

$$v=\int_K (t-1/2)^{\alpha-1}(t+1/2)^{-\alpha-1}\exp(\zeta t)\,dt,$$

where the following condition is imposed on the contour K in the complex plane t:

$$\int_K \frac{d}{dt}\left[\exp(\zeta t)\left(\frac{t-1/2}{t+1/2}\right)^{\alpha}\right]dt=0.$$

For integer values of α, as the contours for two independent solutions, one can select the contour going around the pole of the integrand, and the contour going from its zero to infinity, where $\exp(\zeta t)\to 0$, and the solution is written in quadratures; for example, for $\alpha=1$ we have

$$v=\left(\int d\zeta\,\zeta^{-2}\exp\zeta+A\right)\zeta\exp(-\zeta/2),$$

where A is a positive real constant, and the constant in

$$\int d\zeta\,\zeta^{-2}\exp\zeta=-i\pi-\frac{1}{\zeta}+\ln\zeta+\frac{\zeta}{2}+\ldots$$

is chosen so that the function v is real on the semiaxis $x<0$.

In concluding this section, we note that Eq. (13) can be generalized to the case of a nonuniform external field B_0 produced by a source located on the semiaxis $x<0$. For example, for a conductor with a current at $x=-1$, the term $-w$ changes into

$$\frac{-w(\zeta)+w(-1)}{\zeta+1}$$

[$w(x,0)$ for $x<0$ is a purely real function !]

3. BAND

For the band located at $-1\leqslant x\leqslant 1$ (see Fig. 3), Ampere's law is written in the form

$$B_z(x,0)=\frac{1}{\pi}\int_{-1}^{1}\frac{B_x(x',+0)}{x'-x}dx'.$$

By analogy with the above, it is easy to see that the object of study here should be the functions w, analytic in the ζ plane with a cut from -1 to $+1$, on which $\operatorname{Re} w$ is continuous, and $\operatorname{Im} w$ changes sign. For these functions, instead of Eqs. (6) and (11), one can write

$$\frac{1}{2\pi i}\oint_C \frac{w(\zeta')}{\zeta'-\zeta}\left[1+\left(\frac{1-\zeta^2}{1-\zeta'^2}\right)^{1/2}\right]d\zeta'=2w(\zeta). \qquad (14)$$

The contour C is indicated in Fig. 3, and the function $(1-\zeta^2)^{1/2}$ takes positive real values on the upper bank of the cut. The simplest example of such a function is

$$w=i\left[\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2}+i\right]^{-1},$$

although for this function, since the integrand decreases at an insufficient rate as $|\zeta'|\to\infty$, it is necessary to add to the right-hand side of Eq. (14) the residue at infinity, equal to $w(\infty)=1/2$.

The conversion operators assume the form

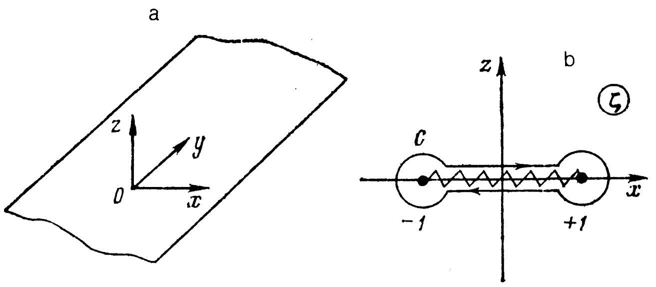

FIG. 3. Geometry of the problem of a band in physical space (a) and on the complex ζ plane (b).

$$L_{\mathrm{Re}}u=\frac{1}{\pi i}\int_{-1}^{1}\frac{u(x)}{x-\zeta}\left(\frac{1-\zeta^2}{1-x^2}\right)^{1/2}dx,$$

$$L_{\mathrm{Im}}u=\frac{1}{\pi}\int_{-1}^{1}\frac{u(x)}{x-\zeta}dx \qquad (15)$$

and for them, Eqs. (8) and (9) remain in force again, and the terms

$$-\frac{1}{\pi}\left[\frac{B_x(1,+0)}{1-\zeta}+\frac{B_x(-1,+0)}{1+\zeta}\right],$$

are added to the right-hand side of Eq. (9), i.e., for a steady state in this geometry, instead of Eq. (13), one can write a somewhat different Riccati equation

$$\frac{dw}{d\zeta}-w^2-w-\frac{\alpha_+}{1-\zeta}-\frac{\alpha_-}{1+\zeta}=0 \qquad (16)$$

with a natural definition of α_+ and α_-. This equation is also convenient to use by converting it to a second-order linear equation. Thus, in the absence of an external field, for a symmetric distribution of current in the band, $(\alpha_+=\alpha_-=\alpha)$ the following equation is obtained for $v(w=-v'/v)$:

$$(1-\zeta^2)v''+2\alpha v=0;$$

its solutions are expressed in terms of associated Legendre functions. If $\alpha=1$, then

$$v=2\zeta+(1-\zeta^2)\ln\frac{\zeta+1}{\zeta-1}+A(1-\zeta^2), \qquad (17)$$

where the logarithm is defined so that v is purely real on the real axis outside the cut, and to prevent v from becoming zero in this region, the constant A should be set equal to zero; such a requirement (cf. preceding case) is a consequence of the formation of a second edge. This expression for v corresponds to a current

$$J=-2\Big/\left\{\left[2x+(1-x^2)\ln\frac{1+x}{1-x}\right]^2+\pi^2(1-x^2)^2\right\}.$$

The transformation from w to v is convenient to use not only for finding the profile of a stationary current in a nonuniform film, but also for calculating its integral. Actually, a consequence of the physical behavior and properties of analyticity and symmetry of the function w is the following chain of inequalities:

$$I=\int_{-1}^{1} J(x)\,dx=\frac{1}{4\pi i}\oint_C w\,d\zeta=\frac{1}{4\pi i}\oint_R \frac{v'}{v}\,d\zeta,$$

where R is a circle of infinite radius, traveled counterclockwise. By virtue of the argument principle, the latter integral is equal to $\Delta_{\arg} v/4\pi$, where $\Delta_{\arg} v$ is the increment of the argument v for a single revolution with respect to R. For Eq. (17), as $|\zeta|\to\infty$, we have $v\approx 4/3\zeta$ and $\Delta_{\arg} v=-2\pi$, i.e., $I=-1/2$. Incidentally, the residue of w at infinity can also be used for these purposes.

CONCLUSION

It should be noted in conclusion that the results obtained in this work are generalized without any particular difficulty to the case of infinite films of any shape with one or two edges: bent once, crimped, spiral, etc. However, the corresponding formulas are represented in a visible form only in the case of a simply described conformal transformation of the regions discussed in this paper to new ones. Concerning the transition from the case of an infinite planar film to a cylindrical one, see Ref. 7.

Thus, use of methods of the theory of analytic functions permits a complete study of stationary Hall flow of current in nonuniform thin films, despite the nonlinear integrodifferential form of the initial equation. A very similar equation (with a single dissipative term) also describes the process of Maxwellian relaxation of charge in a film[8] where a nonlocal interaction takes place via an electric, not a magnetic, field in a vacuum, but the authors of this work,[8] having used the linearity of their problem, very quickly performed a Fourier transformation, depriving themselves of the opportunity of working with an integral equation.

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