Drift of spiral waves on nonuniformly curved surfaces

A. Yu. Abramychev, V. A. Davydov, and V. S. Zykov

Institute of Control Problems of the Academy of Sciences of the USSR

(Submitted 17 November 1989)
Zh. Eksp. Teor. Fiz. **97**, 1188–1197 (April 1990)

A new phenomenon has been discovered: drift of spiral waves on nonuniformly curved surfaces. The phenomenon is studied analytically using an approximate kinematic approach. The results obtained are confirmed by computational experiments. The possibility of observing drift in a medium with a Belousov–Zhabotinskiĭ reaction is discussed.

INTRODUCTION

The considerable and constantly increasing interest in autowave processes in nonlinear excitable media is attributable, first of all, to the prevalence of these phenomena in the most diverse systems (physical, chemical, and biological) and, second, to the possibility of using excitable media as a basis for new devices for information processing.

Excitable media consist of intercoupled nonlinear elements which are capable of forming a pulse in response to an external signal.[1] The coupling between the elements is established, as a rule, by means of diffusion or heat-conduction processes. In an excitable medium there is one uniform rest state that is stable with respect to weak disturbances. Strong disturbances engender a traveling pulse, after the passage of which the medium once again returns to the rest state. Examples of excitable media are a nonequilibrium plasma (including also electron-hole plasma[2]), magnetic superconductors with current,[3] ferroelectric and semiconductor media,[4] solutions with different modifications of the Belousov–Zhabotinskiĭ reaction,[5–7] nerve and muscle tissues, populations of microorganisms,[2] etc.

One of the most important elementary autowave structures in a two-dimensional excitable medium is a spiral wave, arising at locations where the autowavefront cuts off. The cutoff of the autowavefront in the plane rotates around a region, called the nucleus, within which the medium remains in a state of rest. In isotopic media the nucleus is a circle. The shape of the front and the angular velocity of one-armed spiral waves in unbounded media do not depend on the initial conditions and are determined solely by the parameters of the excitable medium.

Thus far the evolution of spiral waves in a plane has been studied in greatest detail. However a two-dimensional excitable medium can have the form of an arbitrary curved surface. In this work we shall study a number of new effects associated with the rotation of spiral waves on nonuniformly curved surfaces, we shall present the results of numerical modeling, and we shall discuss the real possibilities of experimental observation and practical application of these effects.

1. MATHEMATICAL MODEL OF AN EXCITABLE MEDIUM

Excitable media are, as a rule, described by a complicated system of nonlinear partial differential equations of the "reaction-diffusion" type:

$$\dot{\mathbf{u}}=\mathbf{f}(\mathbf{u})+\hat{D}\Delta\mathbf{u}, \tag{1}$$

where $\mathbf{u}=\mathbf{u}(\mathbf{r},t)$ is a vector field characterizing the state of physically small elements of the medium; the vector function $\mathbf{f}(\mathbf{u})$ describes processes ("reactions") within an individual element; and, $\hat{D}$ is a matrix of diffusion coefficients or thermal conductivities. Physically, the components of the vector $\mathbf{u}$ are, as a rule, the concentrations of the reacting substances or the temperature. In most cases of practical interest, by exploiting the differences in the characteristic time intervals over which the components of the vector $\mathbf{u}$ change, the system (1) can be reduced to two or three equations. Thus very informative mathematical models describing many isotropic excitable media contain only two equations:

$$\begin{aligned}\dot{u}&=f_1(u,v)+D_u\Delta u,\\ \dot{v}&=f_2(u,v)+D_v\Delta v,\end{aligned} \tag{2}$$

where the null isocline $f_1(u,v)=0$ has an N-shaped form and the null-isocline $f_2(u,v)=0$ is a monotonic function.

A specific example of the system (2), which is very convenient for analytic studies and computer calculations, is the model proposed in Refs. 8 and 9:

$$f_1(u,v)=f(u)-v,\quad f(u)=\begin{cases}-k_1u, & u<\sigma,\\ k_f(u-p), & \sigma\leqslant u\leqslant 1-\sigma,\\ k_r(1-u), & 1-\sigma<u,\end{cases}$$
$$f_2(u,v)=\varepsilon_k(k_gu-v),\qquad \varepsilon_k=\begin{cases}\varepsilon, & k_gu-v>0,\\ \varepsilon k_\varepsilon, & k_gu-v<0\end{cases} \tag{3}$$

where coefficients k_g, k_f, ε, k_ε, p, and σ determine the form of the functions f_1 and f_2 and the coefficients k_l and k_r are chosen so as to make the function $f(u)$ continuous.

2. KINEMATICS OF AUTOWAVES ON TWO-DIMENSIONAL SURFACES

To calculate autowave processes in excitable media in each separate case it is necessary to know the specific form of the nonlinear functions f_1 and f_2 of the system (2). However the remarkable similarity of autowave processes observed in media of the most diverse nature and in computational experiments with different models of excitable media[1,9] suggests that the evolution of these structures is based on simple universal mechanisms whose operation does not depend or depends weakly on the specific form of the functions f_1 and f_2. This idea, which has been convincingly confirmed by numerous full-scale and computational experiments, forms the basis of the so-called kinematic approach[9,10] to the study of autowave structures. The kinematic approach is an effective method for studying different autowave regimes.[9–12,14,15] Using this method we shall construct an analytic description of the effects studied in this paper.

 0038-5646/90/040666-06$03.00

On the basis of the kinematic approach an autowave is completely determined by specifying the line of its front. As time passes each section of the front moves in the direction of the normal with the velocity V, determined by the local geodesic curvature K of this section. Assuming the curvatures of the fronts are small we can set approximately

$$V=V_0-DK. \tag{4}$$

The form of the line of the front can be specified by its natural equation $K=K(l,t)$, relating the geodesic curvature K and the arc length l at the time t.

In addition, at the point of cutoff the front can advance or contract in the tangential direction with a velocity C, which depends on the curvature of the front K_0, as the point of cutoff is approached. There exists a critical value of the curvature K_{cr} for which the velocity of advance vanishes. For curvatures K_0 close to K_{cr} the velocity of advance of the front can be given approximately as follows:

$$C=\gamma(K_{cr}-K_0), \tag{5}$$

where the coefficient γ_0 must be positive in order to guarantee a stable solution in the form of a spiral wave. Therefore the front advances if $K_0<K_{cr}$ and contracts if $K_0>K_{cr}$.

The basic equation of the kinematic model was derived in Ref. 11. This equation permits describing the evolution of the form of an autowave as it propagates along a curved two-dimensional surface:

$$\frac{\partial K}{\partial t}+\left[\int_0^l KV(K)\,dl+C\right]\frac{\partial K}{\partial l}+K^2V(K)+\frac{\partial^2V(K)}{\partial l^2}=-kV, \tag{6}$$

where k is the Gaussian curvature of the surface and the arc length l is measured from the point of cutoff of the front.

The natural equation determines the form of the front, but not its position on the surface. To describe the evolution of the front uniquely it is sufficient, aside from determining its form, to indicate how the front moves, i.e., the motion of the point with $l=0$. The specific form of the law of motion of the end point depends on which coordinates are chosen on the curved surface. Thus on a plane (Gaussian curvature $k=0$) the coordinates of the point of cutoff of the front $x_0(t)$ and $y_0(t)$ change according to the following equations:

$$\dot{x}_0=-V(0)\sin\alpha_0(t)-C\cos\alpha_0(t), \tag{7}$$

$$\dot{y}_0=V(0)\cos\alpha_0(t)-C\sin\alpha_0(t),\qquad \dot{\alpha}_0=\left.\frac{\partial V}{\partial l}\right|_{l=0}+CK_0, \tag{8}$$

where α_0 is the angle between the x-axis and the tangent to the front at the point $l=0$.

Thus on the basis of the kinematic approach the propagation of autowaves in any two-dimensional excitable medium is described by a small number of phenomenological parameters, such as the velocity of the plane front V_0, the critical curvature K_{cr}, the parameter D determining the dependence of the velocity V on the curvature, and the parameter γ determining the velocity of advance C. These parameters must be determined either experimentally or from the system (2), which for the kinematic model plays in the role of "microscopic" equations. Of course, the values of the kinematic parameters depend strongly on the form of the nonlinear functions f_1 and f_2 in Eqs. (2). Under some conditions, however, the values of the parameters D and γ do not depend on the form of the functions f_1 and f_2. Thus if $D_u=D_v$, then the parameter D is equal to D_u or D_v, while the parameter γ vanishes.[12] It is this case, for which the diffusion coefficient of the activator is practically equal to that of the inhibitor (the variable v), that is realized in an excitable medium with the Belousov–Zhabotinskiĭ chemical reaction.

3. SPIRAL WAVES ON THE SURFACE OF A SPHERE

A sphere is a surface with a constant positive Gaussian curvature $k_0=\text{const}>0$, and $k_0=1/R_0^2$, where R_0 is the radius of the sphere. We shall use the basic equation of kinematics (6) to study the autowave regimes on a spherical surface. We first study the stationary solution of Eq. (6). In the stationary case Eq. (6) has the form

$$\frac{\partial K}{\partial l}\int_0^l KV(K)\,dl+K^2V(K)+\frac{\partial^2V(K)}{\partial l^2}=-k_0V. \tag{9}$$

Equation (9) can be solved analytically if $DK_{cr}/V_0\ll 1$, which corresponds to weakly excitable media. If the radius of the sphere is sufficiently large, so that $R_0K_{cr}\gg 1$, then the stationary equation (9) can be studied by the methods of the theory of singular perturbations, developed for equations with small parameters in the term with the highest order derivative.[13] As shown in Ref. 11, the solution of Eq. (9) is a spiral wave that is symmetric relative to the equator, rotating with constant angular velocity on the surface of a sphere, and having, unlike the two -dimensional case, two nuclei (at the northern and southern poles of the sphere). The curve describing the autowave-front, for example, in the northern hemisphere, is almost everywhere the so-called evolvent of a circle on a sphere, the natural equation of which has the following form:

$$K=(A/V_0-k_0l)\,(2Al/V_0-k_0l^2)^{-1/2}, \tag{10}$$

where $A=(\omega^2-V_0^2k_0)^{1/2}$ and ω is the angular rotational velocity of the spiral wave. The form of the front differs from the evolvent only in narrow boundary layers near the nucleus (where the curvature is a linear function of l) and the equator (where K is a quadratic function of l). Figure 1 illustrates a spiral wave on a sphere.

An important property of a spiral wave on the surface of

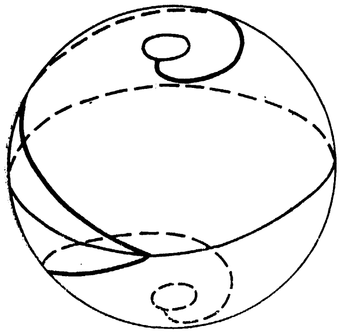

FIG. 1. The form of the front of a stationary rotating wave on the surface of a sphere.

a sphere is that its angular rotational velocity is higher than in a plane, and for small k_0 ($k_0/K_{cr}^2 \ll 1$) it is proportional to the Gaussian curvature:

$$\omega=\omega_0\left(1+\frac{1}{2\xi^2}\frac{V_0}{DK_{cr}}\frac{k_0}{K_{cr}^2}\right), \qquad (11)$$

where ω_0 is the angular velocity of the spiral wave in a plane and is given by the expression

$$\omega_0=\xi V_0 K_{cr}(DK_{cr}/V_0)^{1/2},$$

with $\xi = 0.685$.

It should be noted that the fact that the angular rotational velocity increases as the Gaussian curvature of the sphere increases was established in Ref. 11. In Ref. 11, however, instead of Eq. (11) an incorrect expression was given for ω, differing by the fact that the second term in parentheses did not have a cofactor of the order of V_0/DK_{cr}. Thus in the case of weakly excitable media ($V_0/DK_{cr} \gg 1$) the dependence of the angular velocity on k_0 is stronger than that indicated in Ref. 11.

The "quasistationary approximation" is very effective for studying the nonstationary regimes of the evolution of spiral waves.[12,14] Because the characteristic restoration time of the front of a spiral wave is different from the characteristic penetration time of a spiral wave this approximation makes it possible to distinguish these processes in a wide range of values of the parameters of the excitable medium.[14] Within a narrow boundary layer the form of the front rapidly adjusts itself to the instantaneous value $K(0, t)$ of the curvature at the end point, after which this value slowly reaches the critical value K_{cr} owing to contraction or advance with velocity C. If the quasistationary approximation is applicable, then to describe the nonstationary evolution of a spiral wave it is sufficient to determine the character of the motion of only one end point ($l = 0$) of the front. This approximation is applicable if

$$\gamma/D \ll (V_0/DK_{cr})^{1/2}. \qquad (12)$$

The condition (12) is very weak, since $(V_0/DK_{cr}) \gg 1$. When the inequality (12) holds the curvature K_0 at the end point of the front satisfies the following equation:

$$\dot{K}_0=-\xi\gamma(V_0/D)^{1/2}K_0^{3/2}(K_0-K_{cr}). \qquad (13)$$

This equation is supplemented by a system of three differential equations describing the motion of the end point of a spiral wave along a two-dimensional surface. On a plane these equations have the form of Eqs. (7) and (8) for the Cartesian coordinates x_0 and y_0; on a sphere the position of the end point is most naturally described by the values of the polar and azimuthal angles θ and ϕ. As a result we obtain a closed system of four first-order ordinary differential equations, whose solutions determine the trajectory of the end point of a spiral wave.

In concluding this section we note that the evolution of spiral waves on the surface of a sphere was recently studied experimentally.[16] Spiral waves were excited in a solution with a Belousov–Zhabotinskiĭ chemical reaction on the surface of a small bead ($R_0 \approx 0.6$ mm). The stationary state examined above was not observed in the course of the experiment. We think this is because, first, the spherical surfaces employed were too small—one loop of the spiral barely fit onto the sphere—and, second, a medium with high refractibility (the characteristic width of the front in the photographs in Ref. 16 is comparable to the radius of the sphere) was used, and in this case nonstationary (cycloidal) regimes of rotation of spiral waves appear in the plane also.[17]

We shall now describe the basic results of this work concerning the evolution of spiral waves on surfaces with variable curvature.

4. EVOLUTION OF A SPIRAL WAVE WHEN THE RADIUS OF THE SPHERE VARIES PERIODICALLY

We shall first study an auxiliary problem (which is, however, of interest in itself) concerning the dynamics of a spiral wave on a sphere whose radius changes periodically in time according to the law

$$R=R_0+R_1\cos(\omega_1 t+\beta), \qquad R_1 \ll R_0. \qquad (14)$$

In this case the Gaussian curvature of the sphere undergoes oscillations with amplitude $k_1 = 2R_1/R_0^3$. In what follows we assume that the diffusion coefficient of the activator is practically equal to that of the inhibitor $D_u \approx D_v = D$. This is based on two facts. First, the case of equal diffusion coefficients is realized in a chemically excitable medium with a Belousov–Zhabotinskiĭ reaction, which is convenient for experimental study of spiral autowaves. Second, as already noted, if $D_u \to D_v$, then the kinematic parameter γ, which determines the velocity of advance, approaches zero. This means that in studying the drift of a spiral wave in this case we can neglect the velocity of advance C compared with the velocity of normal motion V. This significantly simplifies the final formulas for the drift velocities, which in the general case $C \neq 0$ are extremely unwieldy. In addition, taking into account the fact that the velocity C is different from zero does not lead to any qualitatively new results; it merely gives quantitative corrections to the expressions for the drift velocities obtained under the condition $\gamma \to 0$.

Let a spiral wave, the radius of whose nucleus satisfies $r_0 \ll R_0$, circulate on the surface of a sphere whose radius varies periodically according to the law (14). We denote by θ_0 and ϕ_0 the polar and azimuthal angles of the center of the nucleus of the spiral wave and by θ and ϕ the spherical coordinates of the moving end point of the autowavefront. For computational convenience we shall choose the axis of the spherical coordinate system so that the center of the nucleus of the spiral wave lies near the equator: $\pi/2 - \theta_0 \ll 1$.

For $C = 0$ and $r_0 \ll R_0$ the velocities of the end point in spherical coordinates are described by the following expressions:

$$\dot{\theta} = -\frac{V_0}{R}\sin\omega t, \qquad \dot{\phi} = \frac{V_0}{R}\frac{\cos\omega t}{\sin\theta_0}. \qquad (15)$$

Let the frequency of variation of the radius of the sphere be close to the rotational frequency ω of the wave: $|\omega - \omega_1| \ll \omega$. We substitute into Eq. (15) the expression (11) [where k_0 is replaced by $k_0 - k_1\cos(\omega_1 t + \beta)$] and (14). Averaging over the time (i.e., retaining only the terms that oscillate slowly with frequency $|\omega_1 - \omega|$) we find that when the radius of the sphere is modulated periodically the center of the spiral wave should drift along the surface of the sphere with the following angular velocities:

$$\dot{\theta}_0=\frac{V_0R_0k_1}{4}\left(1-\frac{1}{\xi^2}\frac{V_0}{DK_{cr}}\frac{k_0}{K_{cr}^2}\right)\sin[(\omega_1-\omega)t+\beta],$$
$$\dot{\phi}=\frac{V_0R_0k_1}{4\sin\theta_0}\left(1-\frac{1}{\xi^2}\frac{V_0}{DK_{cr}}\frac{k_0}{K_{cr}^2}\right)\cos[(\omega_1-\omega)t+\beta]. \quad (16)$$

As in the case of periodic variation of the parameters of a two-dimensional excitable medium,[14,18] this drift has a resonant character. The trajectory of the center of the nucleus is a closed curve, whose characteristic size increases as ω_1 approaches ω. At $\omega=\omega_1$ the direction of drift is determined by the initial modulation phase β, and its velocity is proportional to the amplitude of the changes in the Gaussian curvature k_1.

5. DRIFT OF A SPIRAL WAVE ALONG A NONUNIFORMLY CURVED SURFACE

We now study the basic problem addressed in this work—to describe the dynamics of a spiral wave on a nonuniformly curved surface.

A surface in a three-dimensional space is characterized by two curvatures: the average curvature H and the Gaussian curvature k.[19] There exist several types of surfaces with constant Gaussian curvature k (but, generally speaking, with variable average curvature H): a plane, a sphere, a pseudosphere, a cone, and a cylinder with arbitrary cross section. Since only the Gaussian curvature k appears in the basic equation of kinematics of the spiral wave (6) the kinematic approach allows for the existence of stationary (time-independent geodesic curvature K) solutions, describing spiral waves, on these surfaces. For this reason, it follows from the equations of kinematics that on a nonuniformly curved surface with a constant Gaussian curvature there will be no drift of spiral waves, in spite of the nonuniformity with respect to the average curvature H. This conclusion can also be drawn by studying the structure of Laplacians in the "microscopic" equations in an excitable medium (2).

As an example we consider a spiral wave rotating on the surface of a cone with half-angle α. Let the axis of the cone be oriented along the z-axis. The position of any point on a conical surface can be specified by two coordinates: $x_1=r$, where r is the distance from the vertex of the cone to the given point, and $x_2=\phi$, where ϕ is the azimuthal angle. The components of the metric tensor g_{ik} on the surface of the cone have the form

$$g_{11}=1, \qquad g_{22}=r^2\sin^2\alpha. \quad (17)$$

The Laplacian operator on the cone is calculated in the standard manner:

$$\Delta u=\frac{1}{g^{1/2}}\left[\frac{\partial}{\partial x_1}\left(\frac{g^{1/2}}{g_{11}}\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{g^{1/2}}{g_{22}}\frac{\partial u}{\partial x_2}\right)\right], \quad (18)$$

where g is the determinant of the metric tensor.

Substituting Eqs. (17) into Eq. (18) we obtain the Laplacian on a cone:

$$\Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin^2\alpha}\frac{\partial^2 u}{\partial\phi^2}. \quad (19)$$

Making the substitution of variables $d\phi'=\sin\alpha d\phi$ we transform to a Laplacian in the plane. But there is no drift of spiral waves in a plane. Therefore there will be no drift on the surface of the cone. Analogously, by making the corresponding substitution of variables, it can be shown that there is no drift on other surfaces with constant Gaussian curvature. For this reason nonuniformity of the average curvature is not a sufficient condition for drift of a spiral wave; a necessary condition for drift of a spiral wave is that the Gaussian curvature of the surface must be nonuniform.

Consider a spiral wave on a sphere whose surface is slightly deformed, so that the Gaussian curvature k is a function of the polar angle θ. By slight deformation of the sphere we mean that the Gaussian curvature k does not change much over a distance of the order of the size of the nucleus of the spiral wave, so that $(dk/d\theta)(r_0/R_0)\ll k_0$ holds. For motion along such a nonuniformly curved surface the end point of the spiral wave passes successively through a region with different values of the Gaussian curvature. For this reason it moves as if the Gaussian curvature of the surface varied as a function of time according to the law

$$k=k_0+(dk/d\theta)|_{\theta=\theta_0}(r_0/R_0)\cos\omega t. \quad (20)$$

We have thus reduced the problem of the evolution of a spiral wave on a nonuniformly curved surface to the problem studied above concerning the motion of a spiral wave along a spherical surface with a periodically varying radius. In addition, the change in the curvature governed by Eq. (14) is characterized in this case by the parameters

$$k_1=(dk/d\theta)|_{\theta=\theta_0}(r_0/R_0), \quad \beta=0, \quad \omega_1=\omega.$$

For these values of the parameters we obtain from Eq. (16) the following expressions for the drift velocities:

$$\dot{\theta}_0=0, \quad (21)$$
$$\dot{\phi}_0=\frac{V_0r_0}{4\sin\theta_0}\left(1-\frac{1}{\xi^2}\frac{V_0}{DK_{cr}}\frac{k_0}{K_{cr}^2}\right)\frac{dk}{d\theta}\bigg|_{\theta=\theta_0}.$$

Analysis of the relations (21) leads to the formulation of the following laws of drift of spiral waves along nonuniformly curved surfaces. First, the drift velocity of a spiral wave is proportional to the gradient of the Gaussian curvature. Second, the motion occurs in a direction perpendicular to the gradient. We also note that the term $Vk_0/\xi^2DK_{cr}^3$ in the parentheses in Eq. (21) is small compared with unity, since it is of the order of $(r_0/R_0)^2$. For this reason the sign of the angular velocity $\dot{\phi}_0$ is determined only by the sign of $dk/d\theta$. Thus on the surface of a prolate ellipsoid of revolution a spiral wave rotating counterclockwise in its northern "hemisphere" should drift with angular velocity $\dot{\phi}_0<0$.

We now estimate the drift velocity for a medium with a Belousov–Zhabotinskiĭ reaction. We shall take the typical values $V_0=3$ mm/min and $r_0=0.5$ mm. Let the spiral wave rotate on the surface of a prolate ellipsoid of revolution, take for the minor semiaxis $a=2$ mm, and for the major semiaxis $b=3$ mm, and let the position of the center of the nucleus on the sphere be determined by the angle $\theta_0\approx\pi/6$. The Gaussian curvature of the ellipsoid of revolution has the form

$$k(\theta)=\frac{b^2}{(a^2\cos^2\theta+b^2\sin^2\theta)^2}. \quad (22)$$

Differentiating Eq. (22) with respect to θ and substituting the derivative into Eq. (21) gives the angular velocity $\dot{\phi}_0=0.035$ min^{-1}, which corresponds to the drift velocity $V_d\approx0.04$ mm/min. Thus under these conditions the center of the spiral wave moves over a distance of the order of the radius of the nucleus approximately over ten revolutions of the spiral wave.

6. COMPUTATIONAL EXPERIMENT ON A "REACTION-DIFFUSION" MODEL

We also studied the drift of spiral waves along nonuniformly curved surfaces in numerical experiments on a model of the type (2) with the functions f_1 and f_2 given in the form (3) with the following coefficients:

$$k_f=1.7,\ k_g=2,\ p=0.1,$$
$$\varepsilon=0.15,\ k_\varepsilon=6,\ \sigma=0.01,\ D_u=D_v=1.$$

For this medium the velocity of propagation is $V_0 \approx 0.4$ and the radius of the nucleus is $r_0 \approx 2$.

The calculations were performed for the case of a prolate ellipsoid with semiaxis $a=2$ and $b=2.9$. The Laplacian operator on an ellipsoid, calculated in accordance with Eq. (18), had the form

$$\Delta u = \frac{1}{a^2 \sin^2\theta} \times \left\{ \frac{\sin\theta}{(1+\mu \sin^2\theta)^{1/2}} \frac{\partial}{\partial\theta} \left[\frac{\sin\theta}{(1+\mu\sin^2\theta)^{1/2}} \frac{\partial n}{\partial\theta} \right] + \frac{\partial^2 u}{\partial\phi^2} \right\}$$

with $\mu = (b/a)^2 - 1$.

A square difference grid with step $\Delta\theta = \Delta\phi = 0.005$ was chosen. An explicit difference scheme with a time step $\Delta t = 0.04$ was employed. The motion of the spiral wave in the northern "hemisphere" of the ellipsoid was modeled. The initial position of the center of the nucleus of the spiral wave was determined by the angles $\theta_0 \approx 0.56$ and $\varphi_0 \approx 0.2$.

The trajectory of the end point of the spiral wave, obtained as a result of the calculations, is shown in Fig. 2. One can see from the figure that an individual loop of the trajectory has the form of a strongly prolate and almost closed ellipse. The entire trajectory can be represented as the motion of the end point along an ellipse whose center moves slowly. The loops of the trajectory are elliptical because angular coordinates were employed. In metric coordinates the boundary of the nucleus of the spiral wave is a circle. The observed displacement of the center of the nucleus confirms the prediction of the foregoing analysis. Indeed, as one can see from Fig. 2, the nucleus of a spiral wave drifts only along the parallel of the ellipsoid ($\theta_0 = 0$) in a direction determined by the formula (21). In addition, the estimate of the drift velocity using the formula (21) ($\dot{\phi}_0 = 2.8 \times 10^{-5}$) agrees well with the results of numerical calculations ($\dot{\phi}_0 = 3.6 \times 10^{-5}$). In the computational experiments performed with ellipsoids of a different shape the drift velocity was also observed to increase as the gradient of the Gaussian curvature of the surface increased.

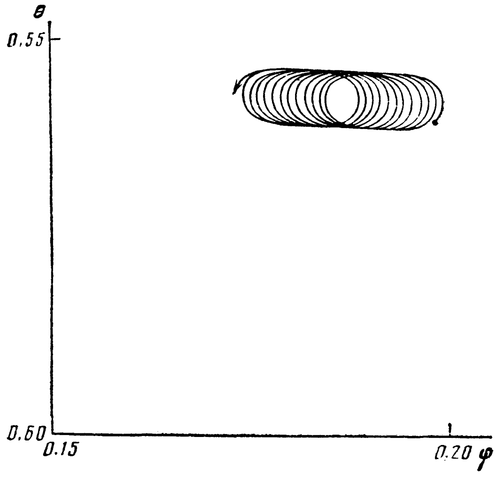

FIG. 2. The trajectory of ...e end point of a spiral wave on the surface of a prolate ellipsoid.

CONCLUSIONS

In this paper we predicted and calculated the drift of spiral waves along nonuniformly curved surfaces. The significance of this study is increased by the fact that real possibilities for observing the effect examined above have now appeared. This is connected with the appearance of modifications of the Belousov–Zhabotinskiĭ reaction with an immobilized catalyst. In this modification the reaction occurs not in the entire volume of the solution, but rather only in a thin layer at the bottom. We recall that the resonance effects predicted for spiral waves on a plane on the basis of the kinematic approach on Ref. 12 have now been directly confirmed experimentally.[17] This also confirms the existence of the drift effect discovered in this work, and the estimates made show that the displacement of the spiral wave can easily be observed visually. For this reason, if the catalyst is immobilized on a nonuniformly curved surface with a given relief, it is possible to obtain fundamentally new possibilities for controlling spiral waves. In particular, it is possible to achieve motion of spiral waves in required directions, stationary rotation in a definite region, or annihilation of pairs of spiral waves with different topological charges. It is very important that in so doing there is no need to make the properties of the excitable medium itself nonuniform, and this makes it much easier to implement such methods for controlling spiral waves.

1. V. A. Vasil'ev, Yu. M. Romanovskiĭ, and V. G. Yakhno, *Autowave Processes in Kinetic Systems*, Reidel, Amsterdam (1987).
2. B. S. Kerner and V. V. Osipov, Zh. Eksp. Teor. Fiz. **83**, 2201 (1982) [Sov. Phys. JETP **56**, 1275 (1982)].
3. A. I. Buzdin and A. S. Mikhaĭlov, *ibid.* **90**, 294 (1986) [Sov. Phys. JETP **63**, 169 (1986)].
4. Yu. I. Balkareĭ, M. I. Elinson, and M. G. Nikulin, in *Problems in Modern Radio Engineering and Electronics* [in Russian], edited by V. A. Kotel'nikov, Nauka, Moscow (1980), p. 431.
5. A. M. Zhabotinskiĭ, *Concentration Autooscillations* [in Russian], Nauka, Moscow (1974).
6. A. T. Winfree, Science **175**, 634 (1972).
7. S. C. Muller, J. Plesser, and B. Hess, Physica D **29**, 71 (1987).
8. I. Krinskiĭ, A. M. Pertsov, and A. N. Reshetilov, Biofizika **17**, 271 (1972).
9. V. S. Zykov, *Simulation of Wave Processes in Excitable Media*, Manchester Univ. Press (1988).
10. V. A. Davydov and A. S. Mikhaĭlov, *Nonlinear Waves: Structures and Bifurcations* [in Russian], Nauka, Moscow (1987), p. 261.
11. P. K. Brazhnik, V. A. Davydov, and A. S. Mikhaĭlov, Teor. Matem. Fiz. **74**, 440 (1988).
12. P. K. Brazhnik, V. A. Davydov, V. S. Zykov, and A. S. Mikhaĭlov, Zh. Eksp. Teor. Fiz. **93**, 1725 (1987) [Sov. Phys. JETP **66**, 984 (1987)].
13. A. B. Vasil'eva and V. F. Butuzov, *Asymptotic Expansions of Solutions of Singularly Perturbed Equations* [in Russian], Nauka, Moscow (1973).
14. V. A. Davydov, V. S. Zykov, A. S. Mikhaĭlov, and P. K. Brazhnik, Izv.

Vyssh. Uchebn. Zaved., Radiofiz. **31**, 288 (1988).
[15] J. J. Tyson and J. P. Keener, Physica D **32**, 327 (1988).
[16] J. Maselko and K. Showalter, Nature **339**, 609 (1989).
[17] V. S. Zykov, Biofizika **31**, 862 (1986).
[18] V. A. Davydov and V. S. Zykov, Zh. Eksp. Teor. Fiz. **95**, 139 (1989) [Sov. Phys. JETP **68**, 80 (1989)].
[19] A. P. Norden, *Theory of Surfaces* [in Russian], Gas. Izd. Nauch-Telch Lit., GINTL, Moscow (1956).
[20] K. I. Agladze, V. A. Davydov, and A. S. Mikhaĭlov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 601 (1987) [JETP Lett. **45**, 767 (1987)].

Translated by M. E. Alferieff