Analytic solutions to the Bloch equations for amplitude- and frequency-modulated fields

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We have solved the system of Bloch equations with time-dependent coefficients analytically by using a group-theoretic approach based on dynamic SO(3) symmetry. This method allows us to calculate in explicit form the evolution matrix for the Bloch vector when the relaxation rates of the components of this vector are equal, i.e., $\gamma_1 = \gamma_2$. For the case where these rates are different, we present a general formalism in which this matrix is used to calculate the Bloch vector components to any order in the small parameter $(\gamma_1 - \gamma_2)t$. We find conditions which define a class of amplitude- and frequency-modulated functions for which the solutions are expressible in terms of Legendre functions. In particular, these conditions are identified for envelopes in the form of hyperbolic secants, Lorentz functions, and Gaussians. For amplitude modulations $\sim \mathrm{sech}(t/T)$ and frequency modulations $\sim \tanh(t/T)$, the parameters which determine the evolution matrix are given in terms of elementary functions. We discuss the possibility of applying our approach and the results we have obtained to the theory of the interaction of radiation with matter and thereby solving problems with the dynamic symmetry groups SO(3) and SU(1,1).

1. INTRODUCTION

The equations introduced by F. Bloch[1] to describe the interaction of classical radiation with two-level quantum systems are widely used in quantum electronics and optics as well as in the physics of magnetic resonance. This system of equations has a rather simple form in vector notation:

$$\frac{d}{dt}\mathbf{x}=(\hat{A}_0+\hat{A})\mathbf{x}+\mathbf{f}, \quad \mathbf{x}(t=0)=\mathbf{x}_0, \tag{1}$$

where the relaxation matrix and the matrix of coefficients in a rotating coordinate system acquire the following form in the "rotating-wave" approximation:

$$\hat{A}_0=\begin{bmatrix} -\gamma_2 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & -\gamma_1 \end{bmatrix}, \quad \hat{A}=\begin{bmatrix} 0 & -\Delta & \alpha \\ \Delta & 0 & \beta \\ -\alpha & -\beta & 0 \end{bmatrix}, \tag{2}$$

where the vector $\mathbf{f}$ usually has a single nonzero component proportional to the population difference under thermal equilibrium conditions. In the case where the exciting field is monochromatic ($\alpha = 0$) and has a constant frequency (Δ denotes the detuning from resonance) and amplitude β, analytic solutions of the Bloch equations were obtained by Torrey[2] by using Laplace transforms. For constant detuning and a monochromatic field with variable amplitude, analytic solutions are known for such special cases as the Rabi problem[3] ($\Delta = 0$, $\gamma_{1,2} = 0$, and β is an arbitrary function of time), the problem of adiabatic passage ($\Delta t < \gamma_{1,2}^{-1}, \beta \sim \beta/\Delta t$, where Δt is the pulse length), the McCall-Hahn pulse [$\gamma_{1,2} = 0$, $\beta \sim \mathrm{sech}(t/T)$ (Ref. 4) and $\Delta = 0$, $\gamma_{1,2} \neq 0$, $\beta \sim \mathrm{sech}(t/T)$ (Ref. 5)] and certain other cases.

For the case of a monochromatic exciting field whose frequency or amplitude vary periodically, the Bloch equations become a system of equations with periodic coefficients, which can be solved by the Floquet-Lyapunov method. However, this method is also applicable to the case of a polychromatic excitation consisting of a set of monochromatic components with equal amplitudes and identical differences between frequencies of neighboring components, which corresponds under certain assumptions to a monochromatic field with a periodic amplitude. In Ref. 6 a solution was obtained for a bichromatic exciting field in the form of a series in harmonics of the frequency difference of the components, whose amplitudes were expressed in terms of continued fractions. For weak excitation fields we can also use the field as a small parameter to expand in an asymptotic series.[7–9] However, when the amplitude or frequency (or both of these together) of a strong monochromatic field vary aperiodically, or when the polychromatic excitation cannot be reduced to a monochromatic field with periodic amplitude, the methods listed above are inapplicable. Note that the solution of the homogeneous part of the system (1) with variable coefficients can sometimes be written formally as an exponential matrix:

$$\mathbf{x}_{\text{nom}}=\left[\exp\int_0^t(\hat{A}+\hat{A}_0)\,dt'\right]\mathbf{x}_0.$$

However, for this we require that the very strong condition

$$\left[\hat{A}+\hat{A}_0, \int_0^t(\hat{A}+\hat{A}_0)\,dt'\right]=0$$

be fulfilled (the so-called Lappo-Danilevskii criterion; see Ref. 10).

Recently, one of the authors of the present paper used a group-theoretic approach based on identifying the dynamic symmetry group and a parametrization corresponding to it to solve the Bloch equations with time-dependent coefficients (see Ref. 11). Analytic solutions were found for a special class of time dependences of the coefficient matrix $\hat{A}$ when relaxation is neglected. In this paper we will develop a general formalism to solve the Bloch equations (1) with coefficients which depend on time in an arbitrary fashion and including relaxation (Sec. 2). The formalism is based on the

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finding the evolution matrix of the homogeneous part of system (1) in explicit form. In Sec. 3 we use the dynamic symmetry group SO(3) to find conditions under which the Bloch equations have analytic solutions in terms of associated Legendre functions, and identify a whole class of amplitude- and frequency-modulated functions which satisfy these conditions. In Sec. 4 we find amplitude-frequency pairs for envelope fields with the following forms: the hyperbolic secant, the Lorentz function, and the Gaussian, and calculate in explicit form the parameters of the evolution for an amplitude modulation $\sim \operatorname{sech}(t/T)$ and a frequency modulation of the form $\sim \tanh(t/T)$. In Sec. 5 we calculate the contribution of relaxation to the evolution matrix of the Bloch vector to first order in the quantity $(\gamma_1-\gamma_2)t$.

2. GENERAL FORMALISM

The method used to obtain the solution to the inhomogeneous system of Bloch equations (1) is well-known: it involves the evolution matrix of the matrizant $\hat{U}_{\text{hom}}$ of the homogeneous system (see, e.g., Ref. 12):

$$\mathbf{x}=\hat{U}_{\text{hom}}\left[\mathbf{x}_0+\int_0^t \hat{U}^{-1}\mathbf{f}\,dt'\right]$$

where the matrix $\hat{U}_{\text{hom}}$ satisfies the evolution equation

$$\frac{d}{dt}\hat{U}_{\text{hom}}=(\hat{A}_0+\hat{A})\hat{U}_{\text{hom}},\quad \hat{U}_{\text{hom}}(t=0)=\hat{I}, \tag{3}$$

and $\hat{I}$ is the 3×3 unit matrix. In order to solve the matrix equation (3), we transform to the interaction representation

$$\hat{U}_{\text{hom}}=\hat{U}_0\hat{U}_{\text{int}}, \tag{4}$$

where the matrix $\hat{U}_0$ has the form

$$\hat{U}_0=\exp\hat{A}_0 t. \tag{5}$$

The matrix $\hat{U}_{\text{int}}$ satisfies the equation

$$\frac{d}{dt}\hat{U}_{\text{int}}=\hat{\tilde{A}}\hat{U}_{\text{int}},\quad \hat{U}_{\text{int}}(t=0)=\hat{I}, \tag{6}$$

where $\hat{\tilde{A}}=\hat{U}_0^{-1}\hat{A}\hat{U}_0$. We write this matrix in the explicit form

$$\hat{\tilde{A}}=\begin{bmatrix}0 & -\Delta & \alpha e^{\gamma t}\\ \Delta & 0 & \beta e^{\gamma t}\\ -\alpha e^{-\gamma t} & -\beta e^{-\gamma t} & 0\end{bmatrix}, \tag{7}$$

where $\gamma\equiv\gamma_1-\gamma_2$ is the difference in relaxation rates.

From these equations it follows that we can cast the evolution of the "homogeneous" Bloch vector in the following form

$$\mathbf{x}_{\text{hom}}=\exp(\hat{A}_0 t)\mathbf{x}_{\text{int}}, \tag{8}$$

where the vector $\mathbf{x}_{\text{int}}$ in its turn evolves according to the relation

$$\mathbf{x}_{\text{int}}=\hat{U}_{\text{int}}\mathbf{x}_0 \tag{9}$$

and satisfies the equation

$$\frac{d}{dt}\mathbf{x}_{\text{int}}=\hat{\tilde{A}}\mathbf{x}_{\text{int}},\quad \mathbf{x}_{\text{int}}(t=0)=\mathbf{x}_0. \tag{10}$$

Introducing the dimensionless variable $\tau=\gamma t$, we write the solution to this equation in the form of a power series

$$\mathbf{x}_{\text{int}}=\sum_{n=0}^{\infty}\tau^n\mathbf{x}_{\text{int}}^{(n)}. \tag{11}$$

The matrix (7) can be written in the form of the following sum:

$$\hat{\tilde{A}}=\hat{A}+\hat{A}_s\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\tau^{2n+1}+\hat{A}_a\sum_{n=0}^{\infty}\frac{1}{(2n)!}\tau^{2n}, \tag{12}$$

where the symmetric and antisymmetric matrices have the form

$$\hat{A}_s=\begin{bmatrix}0 & 0 & \alpha\\ 0 & 0 & \beta\\ \alpha & \beta & 0\end{bmatrix},\quad \hat{A}_a=\begin{bmatrix}0 & 0 & \alpha\\ 0 & 0 & \beta\\ -\alpha & -\beta & 0\end{bmatrix}. \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (10), we obtain the following system of differential equations for the terms of the series (11)

$$\begin{gathered}\frac{d}{dt}\mathbf{x}_{\text{int}}^{(0)}=\hat{A}\mathbf{x}_{\text{int}}^{(0)},\quad \mathbf{x}_{\text{int}}^{(0)}(t=0)=\mathbf{x}_0,\\ \frac{d}{dt}\mathbf{x}_{\text{int}}^{(1)}=(\hat{A}-t^{-1}\hat{I})\mathbf{x}_{\text{int}}^{(1)}+\hat{A}_s\mathbf{x}_{\text{int}}^{(0)},\quad \mathbf{x}_{\text{int}}^{(1)}(t=0)=0,\\ \cdots\\ \frac{d}{dt}\mathbf{x}_{\text{int}}^{(n)}=(\hat{A}-nt^{-1}\hat{I})\mathbf{x}_{\text{int}}^{(n)}+\mathbf{r}^{(n)},\quad \mathbf{x}_{\text{int}}^{(n)}(t=0)=0,\end{gathered} \tag{14}$$

where the vector $\mathbf{r}^{(n)}$ for even and odd n is expressed in terms of the vector $\mathbf{x}_{\text{int}}^{(i)}$ $(i=0,1,2,\ldots,n-1)$ in the following way:

$$\mathbf{r}^{(2n)}=\hat{A}_a\sum_{m=0}^{n-1}\frac{\mathbf{x}_{\text{int}}^{(2m)}}{(2n-2m)!}+\hat{A}_s\sum_{m=0}^{n-1}\frac{\mathbf{x}_{\text{int}}^{(2m+1)}}{(2n-2m-1)!}, \tag{15}$$

$$\mathbf{r}^{(2n+1)}=\hat{A}_s\sum_{m=0}^{n}\frac{\mathbf{x}_{\text{int}}^{(2m)}}{(2n-2m+1)!}+\hat{A}_a\sum_{m=0}^{n-1}\frac{\mathbf{x}_{\text{int}}^{(2m+1)}}{(2n-2m)!}. \tag{16}$$

It is not hard to show that the solution to the inhomogeneous equation for the n-th term of the series $\mathbf{x}_{\text{int}}^{(i)}$ is found by using the evolution matrix for the first term $\mathbf{x}_{\text{int}}^{(0)}$

$$\mathbf{x}_{\text{int}}^{(n)}=t^{-n}\hat{U}\int_0^t t'^{\,n}\hat{U}^{-1}\mathbf{r}^{(n)}dt', \tag{17}$$

where the matrix $\hat{U}$ satisfies the equation

$$\frac{d}{dt}\hat{U}=\hat{A}\hat{U},\quad \hat{U}(t=0)=\hat{I}. \tag{18}$$

From this we see that the solution to the full system of Bloch equations can be found by using the evolution matrix for the homogeneous part of the system without including relaxation.

3. ANALYTIC SOLUTIONS TO THE BLOCH EQUATIONS IN THE ABSENCE OF RELAXATION

In this section we will find analytic solutions to Eq. (18) with a matrix $\hat{A}$ of the form (2) with parameters which depend on time. Since this matrix is completely antisymmetric, it can be cast in the form of a sum of three real antisymmetric matrices

$$\hat{A}_1=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad \hat{A}_2=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},$$

$$\hat{A}_3=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}, \tag{19}$$

which form a basis for the Lie algebra of the rotation group SO(3). Passing from the Cartesian basis to the spherical basis:

$$\hat{A}_\pm=i(\hat{A}_1\pm i\hat{A}_2),\quad \hat{A}_0=i\hat{A}_3,$$

we rewrite the matrix $\hat{A}$ in the form

$$\hat{A}=-i\Delta\hat{A}_0+{}^1/_2(\alpha+i\beta)\hat{A}_-+{}^1/_2(-\alpha+i\beta)\hat{A}_+. \tag{20}$$

Because $\hat{A}$ generates the Lie algebra of the group SO(3), the evolution matrix $\hat{U}$, which satisfies Eq. (18), is an element of this group and can be written in terms of its generators. In our case, it is convenient to choose the Wei-Norman parametrization[13] in the form of a product of exponentials:

$$\hat{U}=\exp(g_0\hat{A}_0)\exp(g_-\hat{A}_-)\exp(g_+\hat{A}_+). \tag{21}$$

The explicit form of the evolution matrix in terms of the group parameters of SO(3) is given in the Appendix. Substituting the solution (21) into Eq. (18) with the "Hamiltonian" (20) gives the following system of differential equations for the complex functions g_0, $g_\pm$:

$$\begin{aligned}&-i\Delta=\dot{g}_0-2g_-\dot{g}_+,\quad g_{0,\pm}(0)=0,\\&-\alpha+i\beta=2\dot{g}_+e^{g_0},\quad \dot{g}_0(0)=-i\Delta(0),\\&\alpha+i\beta=2(\dot{g}_--\dot{g}_+g_-^2)e^{-g_0},\\&\dot{g}_\pm(0)={}^1/_2[\mp\alpha(0)+i\beta(0)].\end{aligned} \tag{22}$$

It can be shown[13,14] that the system of equations (22) reduces to a single Riccah equation for the variables $y=\dot{g}_0$, $y(0)=-i\Delta(0)$:

$$\dot{y}+{}^1/_2y^2-\frac{\dot{\alpha}-i\dot{\beta}}{\alpha-i\beta}y+i\left(\dot{\Delta}-\Delta\frac{\dot{\alpha}-i\dot{\beta}}{\alpha-i\beta}\right)+\frac{\Delta^2+\alpha^2+\beta^2}{2}=0. \tag{23}$$

In the case of an exciting field with time-independent detuning Δ_0 and amplitude β_0, the solution to Eq. (23) is found without difficulty and yields the following expressions for the parameters:

$$\begin{aligned}g_0&=2\ln\left(\cos\frac{\Omega_0t}{2}-i\frac{\Delta_0}{\Omega_0}\sin\frac{\Omega_0t}{2}\right),\\ g_-&=\frac{\alpha_0+i\beta_0}{\Omega_0}\sin\frac{\Omega_0t}{2}\left(\cos\frac{\Omega_0t}{2}-i\frac{\Delta_0}{\Omega_0}\sin\frac{\Omega_0t}{2}\right),\\ g_+&=\frac{-\alpha_0+i\beta_0}{\Omega_0}\sin\frac{\Omega_0t}{2}\left[\cos\frac{\Omega_0t}{2}-i\frac{\Delta_0}{\Omega_0}\sin\frac{\Omega_0t}{2}\right]^{-1},\end{aligned} \tag{24}$$

where $\Omega_0\equiv[\Delta_0^2+\beta_0^2+\alpha_0^2]^{1/2}$ is the generalized Rabi frequency. The solution (21) with parameters (24) provides an explicit form of the evolution matrix for stationary fields, which is a generalization of the corresponding matrix in Ref. 15.

Let us now investigate Eq. (23) when the coefficients are time-dependent. We first set $\alpha=0$, which corresponds to excitation of a two-level system by a single field with variable amplitude and frequency:

$$\dot{y}+{}^1/_2y^2-\frac{\dot{\beta}}{\beta}y+i\left(\dot{\Delta}-\frac{\dot{\beta}}{\beta}\Delta\right)+{}^1/_2\Omega^2=0, \tag{25}$$

where $\Omega\equiv[\Delta^2+\beta^2]^{1/2}$ is the variable Rabi frequency. The substitution $y=2\dot{a}/a$ transforms this equation to one of second order:

$$\ddot{a}-\frac{\dot{\beta}}{\beta}\dot{a}+\frac{1}{2}\left(i\dot{\Delta}-i\Delta\frac{\dot{\beta}}{\beta}+\frac{\Omega^2}{2}\right)a=0. \tag{26}$$

Passing now to a new variable $z(t)$, Eq. (26) can be written in the following form:

$$a''+\frac{a'}{\dot{z}}\frac{d}{dt}\ln\frac{\dot{z}}{\beta}+a\frac{1}{2\dot{z}^2}\left(i\beta\frac{d}{dt}\frac{\Delta}{\beta}+\frac{\Omega^2}{2}\right)=0, \tag{27}$$

where the prime denotes differentiation with respect to the new variable z. The dynamic symmetry of the Bloch equation with respect to the group SO(3) allows us to assume that the solutions can be expressed in terms of spherical functions or Legendre functions of the first and second kind $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$. Comparing Eq. (27) with the Legendre equation

$$(1-z^2)a''-2za'+[\nu(\nu+1)-\mu^2/(1-z^2)]a=0, \tag{28}$$

we obtain the conditions for identity of these two equations:

$$\begin{aligned}&\frac{d}{dt}\ln\frac{\dot{z}}{\beta}=-\frac{2z\dot{z}}{1-z^2},\\&(1-z^2)\left(i\beta\frac{d}{dt}\frac{\Delta}{\beta}+\Omega^2/2\right)=2\dot{z}^2[\nu(\nu+1)-\mu^2/(1-z^2)].\end{aligned} \tag{29}$$

The following solutions for the field amplitude and detuning satisfy these equations:

$$\beta=\frac{\theta\dot{z}}{1-z^2},\quad \Delta=\eta z\beta, \tag{30}$$

where θ and η are complex numbers such that

$$\nu=-{}^1/_2(1\pm(1+i\theta\eta)),\quad \mu=\pm{}^1/_2i\theta(1+\eta^2)^{1/2}.$$

Reverting to the substitution $g_0=2\ln a+\text{const}$, and to Eqs. (30), (26), (25), and (22) with the initial conditions, we can explicitly calculate all the parameters $g_{0,\pm}$ of the evolution matrix (21). From this we see that the homogeneous system of Bloch equations without relaxation has analytic solutions in terms of associated Legendre functions of the first kind $P_\nu^\mu(z)$ and second kind $Q_\nu^\mu(z)$ for an infinite set of temporal dependences of the amplitude β and frequency Δ of the exciting field, connected to each other by the relation (30). Solving this system yields the evolution matrix (18), from which we have a solution to the complete system of Bloch equations (1) in the sense of Eq. (17). Setting $\eta \neq \pm i$, we can obtain solutions in terms of Legendre functions $P_\nu(z)$ *and* $Q_\nu(z)$ with zero upper indices.

We have obtained solutions to Eq. (25) with $\alpha = 0$. Returning to the original equation (23), we will show that it has exact solutions in terms of Legendre functions under the condition (30) in the following cases:

1. The case obtained above, i.e., $\alpha = 0, \beta \neq 0$;
2. $\beta = 0, \alpha \neq 0$;
3. α and β are real quantities;
4. α and β are pure imaginary quantities;

In the latter two cases, Eq. (23) takes the form

$$\dot{y}+{}^1/_2 y^2-\frac{\dot{\varepsilon}}{\varepsilon}y+i\left(\dot{\Delta}-\Delta\frac{\dot{\varepsilon}}{\varepsilon}\right)+\frac{\Delta^2+|\varepsilon|^2}{2}=0,$$

where

$$\varepsilon\equiv\alpha-i\beta.$$

4. VARIOUS PULSE SHAPES AND FREQUENCY MODULATIONS OF THE FIELD

From Eq. (30) we can find physically interesting pulse shapes for the field and the corresponding frequency modulation functions for which the analytic solutions obtained in the previous section are valid. We recall that these solutions belong to one of the four cases listed at the end of the section. For definiteness we will assume that $\alpha = 0$. In principle all the quantities entering into (30) can be complex. For a case where β and Δ are real, which has a transparent physical meaning, the Hamiltonian (20) is an anti-Hermitian matrix, and consequently the evolution matrix is unitary. For real β and Δ, condition (30) is fulfilled if

1. z, θ, η are real quantities, μ and ν are pure imaginary numbers.

2. z, θ, η are pure imaginary numbers, ν is a pure imaginary number, and μ is a real number if $\eta^2 < 1$ and a pure imaginary number if $\eta^1 > 1$. Setting $\theta = -ib_0$ and $\eta = ic_0$, we obtain the following expressions for the indices ν and μ of the Legendre functions:

$$\nu=\frac{i}{2}b_0c_0,\quad \mu=\pm\frac{b_0}{2}(1-c_0^2)^{1/2}. \tag{31}$$

Taking into account the initial conditions $a[z(t=0)]=1$ and $\dot{a}[z(t=0)]=0$, the solutions to the Legendre equation with index 1/2 have the form

$$a(z)={}^1/_2(1-z^2)^{-1/4}\{[(1-z^2)^{1/2}\pm iz]^{\nu+1/2}+[(1-z^2)^{1/2}\pm iz]^{-\nu-1/2}\}. \tag{32}$$

We take the variable z in the form $z = i\,\mathrm{tg}\varphi$, where φ is an arbitrary real differentiable function of time. Then the amplitude and frequency modulations take the simple forms:

$$\beta=b_0\dot{\varphi},\quad \Delta=-b_0c_0\dot{\varphi}\,\mathrm{tg}\,\varphi. \tag{33}$$

From this it is not difficult to obtain the frequency modulation functions for field envelope shapes which are interesting from a physics point of view, e.g.,

The hyperbolic secant $\beta = \mathrm{sech}\, b_1 t$:

$$\Delta=-c_0\,\mathrm{sech}\,b_1t\,\mathrm{tg}\,\varphi_1,\quad \varphi_1=2(b_0b_1)^{-1}(\mathrm{arctg}\,e^{b_1t}-\pi/4); \tag{34}$$

The Lorentz function $\beta = 1/(1+t^2)$:

$$\Delta=-[c_0/(1+t^2)]\,\mathrm{tg}\,\varphi_2,\quad \varphi_2=b_0^{-1}\,\mathrm{arctg}\,t, \tag{35}$$

The Gaussian function $\beta = \exp(-b_2^2t^2)$:

$$\Delta=-c_0\exp(-b_2^2t^2)\,\mathrm{tg}\,\varphi_3,\quad \varphi_3=(2b_0b_2)^{-1}\pi^{1/2}\mathrm{erf}\,(b_2t), \tag{36}$$

where $b_{1,2}$ are arbitrary real quantities with the dimension of frequency.

Once we specify the form of the substitution $z(t)$, we can find an infinite set of such β and Δ pairs. Let us calculate in explicit form the parameters $g_{0,\pm}$, for the substitution $z = i\,\mathrm{sh}(t/T)$, $\theta = -ibT$, $\eta = ic/b$, which leads to a solution well-known in the theory of phase modulation[15,16] in the form of a hyperbolic secant for the envelope field and a hyperbolic tangent for its frequency:

$$\beta=b\,\mathrm{sech}\,(t/T),\quad \Delta=-c\,\mathrm{th}\,(t/T)$$

(here the parameter T determines the length of the pulse).

The parameters of the evolution in this case are expressed in terms of elementary functions if we have $\mu = \pm 1/2$. From (31) it follows that this condition is fulfilled if the modulation depth of the amplitude and frequency are connected with each other in the following way:

$$b^2=c^2+1/T^2.$$

Using the solution (32), we obtain the following expressions for the evolution parameters starting at the time $t = -\infty$:

$$g_0=\ln\mathrm{sech}\,(t/T)-(T^{-1}+ic)t,$$
$$g_+=\frac{ib}{2(T^{-1}+ic)}\exp(T^{-1}+ic)t, \tag{37}$$
$$g_-=-(\mathrm{th}\,(t/T)+1)(T^{-1}+ic)(ib)^{-1}\exp[-(T^{-1}+ic)t].$$

Substituting these expressions into the evolution matrix $\hat{U}$ (see Appendix), we can obtain an explicit form for the components of the Bloch vector for an atom which is initially found, let us say, in the ground state $\mathbf{x}(t=-\infty) = (0,0,-1)$:

$$x_1=\frac{c}{b}\,\mathrm{sech}\frac{t}{T},\quad x_2=-\frac{1}{bT}\,\mathrm{sech}\frac{t}{T},\quad x_3=\mathrm{th}\frac{t}{T}. \tag{38}$$

5. THE CONTRIBUTION OF RELAXATION

Within the framework of our algebraic approach, we can take into account variable relaxation when the relaxation rates for the longitudinal and transverse components of the Bloch vector are equal, i.e., $\gamma_{1,2}(t) = \gamma_0(t)$. For this case we must add the unit matrix $\hat{I}$ to the basis (19) of the algebra SO(3) and include an additional exponential factor

$$\exp\left\{-\hat{I}\int_0^t\gamma_0(t)\,dt\right\}$$

in the evolution matrix (21). Unfortunately, using this approach it is difficult to obtain exact analytic solutions to the Bloch equations when these relaxation rates are different, because in this case the Bloch matrix $\hat{A} + \hat{A}_0$ generates the Lie albegra $S1(3,C)$ or the Algebra SU(3), whose higher dimensionality (eight) considerably complicates the procedure for finding the group parameters g. In this section, within the framework of the general formalism of Sec. 2 we will compute the contribution of relaxation of the longitudinal and transverse components of the Bloch vector to the evolution of the latter. The calculations are carried out to first order in the small parameter γt, i.e., they are valid for small times or for small differences in the relaxation rates. From the expressions of the general formalism (17), (16), and (8), it follows that the evolution of the homogeneous part of the Bloch vector in this approximation appears in the following form:

$$\mathbf{x}_{\text{hom}} \simeq \exp(\hat{A}_0 t)\, \hat{U}\Big(\hat{I}+\gamma \int_0^t t' \tilde{\hat{A}}_s\, dt'\Big)\mathbf{x}_0. \tag{39}$$

Because the evolution matrix $\hat{U}$ is unitary for real functions β, Δ, and α [see (20)], the matrix $\tilde{\hat{A}}_s = \hat{U}^{-1}\hat{A}_s \hat{U}$ is Hermitian.[17] The matrix $\hat{A}_s$ has the form (13). It is not difficult to calculate the components of the Bloch vector $\mathbf{x}_{\text{hom}} = (x_1, x_2, x_3)$ in the case where the atom is initially found in the ground state $(0, 0, -1)$. They are expressed in terms of the matrix elements g_{ij} of the evolution matrix in the following way:

$$x_j = -e^{-\gamma_\sigma t}\Big(g_{j3} + \gamma \sum_{k=1}^{3} g_{jk}\tilde{a}_{k3}\Big),$$
$$\gamma_\sigma = \begin{cases} \gamma_2, & j=1,2, \\ \gamma_1, & j=3, \end{cases} \tag{40}$$

where $\tilde{a}_{k3}$ is the third column of a Hermitian matrix whose elements for $\alpha = 0$ equal

$$\tilde{a}_{k3} = \int_0^t t' (g_{2k}^* g_{33} + g_{3k}^* g_{23})\,\beta\, dt'.$$

The matrix elements g_{ij} in their turn are expressed in terms of the group parameters $g_{0,\pm}$ whose explicit form was given at the end of the previous section for a field envelope and its frequency modulation in the form of a hyperbolic secant and hyperbolic tangent, respectively.

6. CONCLUSIONS AND DISCUSSION OF RESULTS

We have used a group-theoretic approach to calculate the evolution matrix which determines the dynamics of the Bloch vector, and to find analytic solutions to the Bloch equations with arbitrarily varying coefficients. In the case of equal relaxation times these solutions are expressed in terms of associated Legendre functions of the first and second kind if the variable amplitude of the field and its variable frequency are related to each other in a definite way.

There is an infinite set of shapes for such amplitude and frequency modulations, specific examples of which, having physical meaning, were presented in the Section 4. In particular, the dynamic problem of interaction of a two-level system with an external field whose envelope varies as a hyperbolic secant and whose frequency varies as a hyperbolic tangent was solved in terms of elementary functions. For the case $T_1 \neq T_2$ a general formalism was proposed to solve the Bloch equation which in principle allows us to calculate the components of the Bloch vector to any order in the quantity γt, if the evolution matrix is known which corresponds to the problem in the absence of attenuation. In the fifth section we presented an example of calculating the contribution of relaxation to the evolution matrix to first order in the small quantity γt.

The matrix form of the solution to the Bloch equation is especially convenient for calculating the time evolution of a two-component system for a many-pulse excitation. In this case the calculation of the signal from a multipulse echo (light or spin) for a field with amplitude and frequency modulation reduces essentially to multiplication of the evolution matrices. Due to the general character of the solution, the results we have obtained are valid not only for the Bloch equations in classical form but also for any system of linear differential equations with variable coefficients, whose dynamic symmetry group (algebra) is SO(3) or SU(1,1).

The realization of the algebras SO(3)$\sim$SU(2) and SU(1,1) by using angular momentum operators and bilinear components of Bose operators allows us to use these analytic solutions in various problems of quantum optics and electronics, where it is required to calculate the corresponding evolution operator. In Ref. 21, as an example of bilinear parametric processes it is shown how to calculate the evolution operator for SU(1,1), once it is known for SU(2).

In particular, this method can find application in the theory of two-level lasers with phase modulation,[18] where the matrix of coefficients of Eq. (1) has the form of the matrix $\hat{A}$ from Eq. (2) with $\beta = 0$.

As we noted in the third section, the equation for the quantity g_0 in this case is equivalent to Eq. (25). Serving as another example is the dynamic theory of the interaction of two-level atoms with light in a compressed state, in which a central role is played by the system of quantum Langevin equations for the psuedospin operators of an atom.[19] For this case, the matrix of coefficients has the form of the matrix A with $\Delta = 0$ and α and β are random operator functions of time which commute with each other. The technique is applicable also for calculating evolution not only in time but in space.

In the theory of resonant reflection of light from the boundary of an ionized gas, which is described by a two-component model,[20] a system of equations was obtained for the elements of the density matrix of the moving atoms which formally coincides with the system of Bloch equations (1), (2), but with a derivative of the Bloch vector with respect to the coordinate instead of time. In Ref. 20 the detuning from resonance, which includes the Doppler shift, and the field amplitude both have constant values. Inclusion of the spatial modulation of the amplitude and phase of the field leads formally to the results we have obtained for the temporal behavior of the Bloch vector.

The possibility of wide application of our group-theoretic approach based on representations of evolution matrices (operators) in the form of products of exponentials in order to solve dynamic problems in various areas of physics (we note here the theory of echo processes,[14] propagation of waves in nonuniform media,[22] the theory of free-electron lasers,[23] bilinear parametric processes,[21] and compressed states[24]) is based on the fact that the evolution of a physical process is determined by the structure of the corresponding dynamic Lie group (or algebra), and not the conditions imposed on the coefficients, i.e., for a Hamiltonian which is, let us say, independent of time within the method of Laplace or for a periodic time dependence within the Floquet-Lyapunov method.

On the other hand, the system of differential equations for the parameters of the evolution is, generally speaking, nonlinear. Such a system is integrable for the class of soluble Lie algebras of any dimensionality,[13] but the difficulties of solving this system increase with the dimensionality. For a simple algebra of low dimensionality it can be reduced to a single nonlinear equation, as in the case of SU(2) and SU(1,1). If the algebra is not soluble and is not simple, then its decomposition into a semisimple subalgebra and a radical, and subsequent decomposition into simple subalgebras (if possible) leads to a decomposition of the system of equations for the parameters (for details on this see Refs. 13 and 14).

APPENDIX: EXPLICIT FORM OF THE EVOLUTION MATRIX U IN TERMS OF THE GROUP PARAMETERS OF SO(3)

$$U_{11}={}^1/_2e^{-g_0}(1-g_-{}^2)+{}^1/_2e^{g_0}(1-g_+{}^2)+g_-g_+e^{-g_0}(1+{}^1/_2g_-g_+),$$
$$U_{12}={}^1/_2ie^{-g_0}(1+g_-{}^2)-{}^1/_2ie^{g_0}(1+g_+{}^2)+ig_-g_+e^{-g_0}(1+{}^1/_2g_-g_+),$$
$$U_{13}=-g_+e^{g_0}+g_-e^{-g_0}(1+g_+g_-),$$
$$U_{21}=-{}^1/_2ie^{-g_0}(1-g_-{}^2)+{}^1/_2ie^{g_0}(1-g_+{}^2)-ig_-g_+e^{-g_0}(1+{}^1/_2g_-g_+),$$
$$U_{22}={}^1/_2e^{-g_0}(1+g_-{}^2)+{}^1/_2e^{g_0}(1+g_+{}^2)-g_-g_+e^{-g_0}(1+{}^1/_2g_-g_+),$$
$$U_{23}=-ig_+e^{g_0}-ig_-e^{-g_0}(1+g_+g_-),$$
$$U_{31}=g_+-g_-(1-g_-g_+),$$
$$U_{32}=ig_-+ig_+(1+g_-g_+),$$
$$U_{33}=1+2g_+g_-.$$

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