A note on W_3 algebra

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We consider the reduced Wess-Zumino-Witten $sl(3)$ model by making use of two free fields; the result possesses global $sl(3)$ invariance. In this theory the operators W_2 and W_3 introduced previously are Noether currents. Our analysis directly generalizes the results obtained by Alekseev and Shatashvili, which treated the $sl(2)$ cases (two-dimensional gravitation).

Two-dimensional theories employing specific chiral W_n algebras remain somewhat puzzling. It has recently been suggested[4] that these theories are actually invariant under ordinary $sl(n)$ symmetry operations. The W operators then couple to Casimir combinations of the generators for Kac-Moody algebra, $W_n \sim \mathrm{tr}\, J^n + \ldots$.[6] Furthermore, the W operators themselves can be looked upon as Noether currents responsible for some symmetry of the theory, in the same way as the Sugawara energy-momentum tensor $\sim \mathrm{tr} J^2$ turns out to be a Noether current in the Wess-Zumino-Witten (WZW) model, a consequence of general coordinate transformations. These ideas have been discussed by Alekseev and Shatashvili[4] as they relate to $sl(2)$ symmetry. Here we extend their results to the first nontrivial example of $sl(3)$ algebra.

1. We review here the basic assertions relating to two-dimensional gravitation made in Ref. 4 (see Refs. 5, 7–9, and a more detailed description in Ref. 10). This theory can be considered a Hamiltonian reduction of the WZW $sl(2)_k$ model. Representing the currents in terms of free fields (see Ref. 11 and references therein),[1)]

$$\begin{aligned} J^+ &= \exp(2^{1/2} i\phi/q)\psi', \\ J_0 &= \exp(2^{1/2}\phi i/q)\psi'\chi - 2^{-1/2} iq\phi', \\ J^- &= \exp(2^{1/2} i\phi/q)\psi'\chi^2 - 2^{1/2} iq\chi\phi' + (2-q^2)\chi', \end{aligned} \tag{1}$$

and the reduction condition $J^+ = 1$ means that

$$\psi' = \exp(-2^{1/2} i\phi/q). \tag{2}$$

Another consequence of reduction is that it is possible to put $\chi = 0$. In this gauge, the action in the WZW model becomes[11]

$$S_\varphi = \frac{1}{2}\int\left(\dot\phi\phi' + \frac{2^{1/2} i}{q} R\phi\right) = -\frac{q^2}{4}\int \dot\varphi\varphi' - 2^{-1/2}\int R\varphi, \tag{3}$$

where R is the two-dimensional curvature, $R = (\log\gamma)''$, γ is the two-dimensional metric $(\gamma = \gamma_{\bar z\bar z})$, $\varphi = 2^{1/2} i\phi/q$, $q^2 = k + g = k + 2$. The energy-momentum tensor is determined by the variation of coordinates $z \to z - \varepsilon(z)$, with the field φ transforming as $\delta\varphi = \varepsilon\varphi' + \varepsilon'$ (Ref. 5) (the term ε' is related to the fact that it is ψ and not φ that is a scalar field, transforming according to $\delta\psi = \varepsilon\psi'$) and $\delta\gamma = \varepsilon'\gamma + \ldots$, so that $\delta R = \dot\varepsilon'' + \ldots$ and

$$\begin{aligned} \delta S &= -\int \dot\varepsilon T, \\ T &= \frac{q^2}{4}(\partial\varphi)^2 - \frac{1}{2}(q^2-1)\partial^2\varphi \\ &= -\frac{1}{2}(\partial\phi)^2 - \frac{i}{2^{1/2} q}(1-q^2)\partial^2\phi. \end{aligned} \tag{4}$$

The corresponding central charge is $c = 1 - 6(q - 1/q)^2$. This is the same as the charge in the minimal M_{mn} model if $q^2 = m/n$ (Ref. 10).

In place of (2), we may rewrite the action in terms of the field ψ:

$$S_\psi = -\frac{k}{2}\int \frac{\dot\psi'\psi''}{(\psi')^2}. \tag{5}$$

Note that at the quantum level, S_ϕ is not the same as S_ψ: the difference stems from the change in quantum measure in the functional integral. This difference shows up in the replacement of the parameter k by $q^2 = k + g = k + n = k + 2$, and in the appearance of the term R_φ in Eq. (2) (a detailed explanation may be found in Secs. 2.3 and 3.3 of Ref. 11). All quantities expressed in terms of ψ and φ (in particular, the Noether currents) are changed by the replacement of k by q^2 and by a contribution associated with the term $R\varphi$ (the latter contribution to the Noether current appears any time a symmetry transformation fails to leave the metric invariant). The representation (5) has an advantage over (3): antiholomorphic $sl(2)$ symmetry is obtained quite simply—the action $S = S_\psi$ is invariant under the substitution

$$\delta\psi = \varepsilon_+ + \varepsilon_0\psi + \varepsilon_-\psi^2 \tag{6}$$

[which is the infinitesimal form of the piecewise-linear transformation $\psi \to (a\psi + b)/(c\psi + d)$]. We may therefore treat ψ as a complex coordinate in the space CP^1, and S as the invariant action, which depends solely on the homomorphic coordinates in that same space (no recourse to antiholomorphic coordinates). Of course, the Lagrangian in (5) cannot be invariant under $sl(2)$; it constitutes a separate class of Wess-Zumino term, and varies like a total derivative when ψ is replaced by $\psi + \varepsilon_-\psi^2$.

In terms of the original free field φ, $sl(2)$ symmetry can only be realized by nonlocal transformations:

$$\delta\varphi = \varepsilon_0 + 2\varepsilon_-\int e^\varphi dz + \varepsilon_+' e^{-\varphi} + \varepsilon_0' e^{-\varphi}\int e^\varphi dz + \varepsilon_-' e^{-\varphi}\left(\int e^\varphi dz\right)^2. \tag{7}$$

(Notice that the "positive" $sl(2)$ operator does not affect φ if $\varepsilon'_+ = 0$, but it changes $\int e^\varphi dz$: $\delta\int e^\varphi dz = \varepsilon_+ + \ldots$.) We must emphasize that in contrast to the general form of field theory, nonlocal transformations in the conformal theory are well-defined. The problem is usually that after a nonlocal field transformation, the correlators start to depend not just on the points, but on the integration paths as well. In the conformal theory, however, where the conformal blocks depend on the points holomorphically, the integrals no longer depend on the path, and the nonlocal expressions turn out to

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be no worse than the local ones. (In the boson version of the theory, this is ensured by the Feigen-Fuks-Dotsenko-Fateev projection, wherein a delta function is inserted into the correlators, forcing the integrals along unrestricted paths to zero; see Ref. 11 for details.) Be that as it may, it is more convenient to study the symmetry of the theory using the representation (5): most Noether currents are nonlocal functions of φ, but locally they depend on ψ.

Here we are dealing with three Noether currents, corresponding to the three generators of $sl(2)$ algebra, and two currents, associated with the homomorphic and antiholomorphic coordinate transformations:

$$\delta\psi=\varepsilon_{+}+\varepsilon_{0}\psi+\varepsilon_{-}\psi^{2}+\alpha\psi'+\beta\dot{\psi},$$

$$\delta S=-\int(\delta\psi)'\bar{J}_{+}=-\int(\varepsilon_{+}'\bar{J}_{+}+\varepsilon_{0}'\bar{J}_{0}+\varepsilon_{-}'\bar{J}_{-}+\dot{\alpha}T+\beta'\bar{T}). \tag{8}$$

The expressions for the $sl(2)$ currents take the form

$$\bar{J}_{+}=-\frac{k}{2}\frac{1}{\psi'}\left(\frac{\psi''}{\psi'}\right)^{\cdot}=-\frac{k}{2}\left(\frac{\dot{\psi}''}{\psi'^{2}}-\frac{\dot{\psi}'\psi''}{\psi'^{3}}\right),$$
$$\bar{J}_{0}=\psi\bar{J}_{+}+\frac{k}{2}\frac{\dot{\psi}'}{\psi'}=-\frac{k}{2}\left(\frac{\psi\dot{\psi}''}{\psi'^{2}}-\frac{\psi\dot{\psi}'\psi''}{\psi'^{3}}-\frac{\dot{\psi}'}{\psi'}\right), \tag{9}$$
$$\bar{J}_{-}=\psi^{2}\bar{J}_{+}+k\frac{\psi\dot{\psi}'}{\psi'}-k\dot{\psi}$$
$$=-\frac{k}{2}\left(\frac{\psi^{2}\dot{\psi}''}{\psi'^{2}}-\frac{\psi^{2}\dot{\psi}'\psi''}{\psi'^{3}}-2\frac{\psi\dot{\psi}'}{\psi'}+2\dot{\psi}\right),$$

and only the homomorphic energy-momentum tensor is locally a function of the field φ [see (4)]:

$$T=-\frac{k}{2}\left(\frac{\psi'''}{\psi'}-\frac{3}{2}\frac{\psi''^{2}}{\psi'^{2}}\right),$$
$$\bar{T}=-\frac{k}{2}\left(\frac{\dot{\psi}\psi''}{\psi'^{2}}-\frac{\dot{\psi}\dot{\psi}'\psi''}{\psi'^{3}}-\frac{1}{2}\frac{\dot{\psi}'^{2}}{\psi'^{2}}\right). \tag{10}$$

It is easy to show that

$$\bar{T}=-\frac{1}{k}(\bar{J}_{+}\bar{J}_{-}-\bar{J}_{0}^{2}). \tag{11}$$

2. We now repeat these arguments for the case of $sl(3)_k$ algebra. The only Noether currents, which may be expressed locally in terms of φ, will be W_2 and W_3. The nonholomorphic partner of W_3 ought to be identical with $\mathrm{tr}\bar{J}^3$, a condition analogous to (11).

In the general case of the algebra $G = sl(n)$, the action is

$$S_{\varphi}=-\frac{q^{2}}{2}\int\dot{\varphi}\varphi'-\int R\rho\varphi, \tag{12}$$

where φ is a scalar field, as manifested by a vector with rank $G = n - 1$ components, and $q^2 = k + g = k + n$. In general, there are different types of ψ fields. Recall (all details may be found in Ref. 11) and the ψ fields emerge from the Gaussian expansion of the group elements of G; for $n = 3$, we have

$$g=\begin{bmatrix}1&0&0\\ \chi_{1}&1&0\\ \chi_{2}&\chi_{3}&1\end{bmatrix}\begin{bmatrix}e^{-\mu_{1}\varphi}&0&0\\ 0&e^{-\mu_{2}\varphi}&0\\ 0&0&e^{-\mu_{3}\varphi}\end{bmatrix}\begin{bmatrix}1&\psi_{1}&\psi_{2}\\ 0&1&\psi_{3}\\ 0&0&1\end{bmatrix}$$

(the vectors μ—the weights of the fundamental representations—are defined in Fig. 1). These ψ fields are labeled by the positive roots of the algebra $G = sl(n)$. There is an integer characteristic that goes along with each root α—its weight

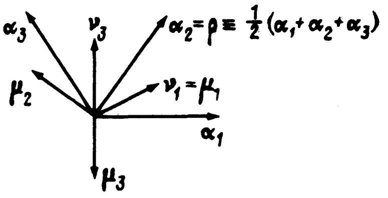

FIG. 1. Roots and weights for $sl(3)$ algebra.

$$h_{\alpha}=\rho\alpha,\quad \rho=\frac{1}{2}\sum_{\alpha\in\Delta_{+}}\alpha$$

(Δ_+ is the set of positive roots). The weight assigned to the simple roots is $h_\alpha = 1$, and for the remaining roots the weight is greater than one. The assumption in the Hamiltonian reduction of Drinfeld and Sokolov is that all currents J^+ responsible for simple roots are equal to unity, and all other J^+ are zero. For $sl(3)$ algebra in the gauge in which $\chi_\alpha = 0$, this means that

$$J_{+\alpha_{1}}=\exp(-\alpha_{1}\varphi)\psi_{1}'=1,$$
$$J_{+\alpha_{3}}=\exp(-\alpha_{3}\varphi)\psi_{3}'=1, \tag{13}$$
$$J_{+\alpha_{2}}=(\psi_{2}'-\psi_{1}\psi_{3}')\exp(-\alpha_{2}\varphi)=0.$$

Therefore,

$$\alpha_{1}\varphi=\log\psi_{1}',\quad \alpha_{3}\varphi=\log\psi_{3}', \tag{14}$$

and

$$\psi_{1}=\psi_{2}'/\psi_{3}'. \tag{15}$$

The action (12) can then be rewritten as

$$S=-\frac{k}{3}\int\left\{\frac{\dot{\psi}_{1}'}{\psi_{1}'}\frac{\psi_{1}''}{\psi_{1}'}+\frac{\dot{\psi}_{1}'}{\psi_{1}'}\frac{\psi_{3}''}{\psi_{3}'}+\frac{\dot{\psi}_{3}'}{\psi_{3}'}\frac{\psi_{3}''}{\psi_{3}'}\right\}, \tag{16}$$

while in terms of the fields φ we have

$$S_{\varphi}=S(\psi\to\psi(\varphi),\ k\to q^{2})-\int R\rho\varphi.$$

The variation of the action takes the form

$$\delta S=\frac{k}{3}\int\left\{\frac{\delta\psi_{1}'}{\psi_{1}'}\left[2\frac{\psi_{1}''}{\psi_{1}'}+\frac{\psi_{3}''}{\psi_{3}'}\right]^{\cdot}+\frac{\delta\psi_{3}'}{\psi_{3}'}\left[\frac{\psi_{1}''}{\psi_{1}'}+2\frac{\psi_{3}''}{\psi_{3}'}\right]^{\cdot}\right\}. \tag{17}$$

Rather than ψ_1 and ψ_3, however, the proper fundamental variables are more likely to be ψ_2 and ψ_3, with ψ_1 being expressed in terms of the latter via (15). These variables are to be considered complex coordinates in the space $CP^{n-1} = CP^2$, and they transform piecewise-linearly under $sl(3)$:

$$\psi_{2}\to\frac{a\psi_{2}+b\psi_{3}+c}{p\psi_{2}+t\psi_{3}+r},\quad \psi_{3}\to\frac{d\psi_{2}+e\psi_{3}+f}{p\psi_{2}+t\psi_{3}+r}, \tag{18}$$

The transformation law for ψ_1 is induced by the transformations in (18), and looks rather less trivial. There is no need to substitute (15) into Eqs. (16) and (17); instead, we derive expressions for the infinitesimal variations of ψ_1, ψ_2, and ψ_3 under (18), and make use of (17) to find the Noether currents. This then yields the infinitesimal version of (18):

$$\delta\psi_3=\varepsilon_1+\varepsilon_3\psi_3+\varepsilon_5\psi_2+\varepsilon_7\psi_3^2+\varepsilon_8\psi_2\psi_3,$$
$$\delta\psi_2=\varepsilon_2+\varepsilon_4\psi_2+\varepsilon_6\psi_3+\varepsilon_7\psi_2\psi_3+\varepsilon_8\psi_2^2,$$
$$\delta\psi_1=\delta\left(\frac{\psi_2'}{\psi_3'}\right)=-\varepsilon_3\psi_1+\varepsilon_4\psi_1-\varepsilon_5\psi_1^2+\varepsilon_6+\varepsilon_7(\psi_2-\psi_1\psi_3)$$
$$+\varepsilon_8(\psi_2-\psi_1\psi_3)\psi_1-\varepsilon_1'\frac{\psi_1}{\psi_3'}+\varepsilon_2'\frac{1}{\psi_3'}-\varepsilon_3'\frac{\psi_1\psi_3}{\psi_3'}+\varepsilon_4'\frac{\psi_2}{\psi_3'}$$
$$-\varepsilon_5'\frac{\psi_1\psi_2}{\psi_3'}+\varepsilon_6'\frac{\psi_3}{\psi_3'}+\varepsilon_7'\frac{\psi_3}{\psi_3'}(\psi_2-\psi_1\psi_3)+\varepsilon_8'\frac{\psi_2}{\psi_3'}(\psi_2-\psi_1\psi_3). \tag{19}$$

The antiholomorphic Noether currents $\bar{J}$, which generate the $sl(3)$ algebra, can be determined from (17):

$$\delta S=-\int\left(\sum_{\mu=1}^{8}\varepsilon_\mu'\bar{J}_\mu\right)=\frac{k}{3}\int\left\{\frac{\delta\psi_1'}{\psi_1'}\dot{Y}_1'+\frac{\delta\psi_3'}{\psi_3'}\dot{Y}_3'\right\}, \tag{20}$$

where (see Fig. 1)

$$Y_1(\psi)=2\log\psi_1'+\log\psi_3'=3\mu_1\varphi=3\nu_1\varphi,$$
$$Y_3(\psi)=\log\psi_1'+2\log\psi_3'=-3\mu_3\varphi=3\nu_3\varphi.$$

Then

$$\bar{J}_1=-\frac{k}{3}\frac{1}{\psi_3'}\left[\dot{Y}_3'+\psi_1\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'\right],\quad \bar{J}_2=\frac{k}{3}\frac{1}{\psi_3'}\left(\frac{\dot{Y}_1'}{\psi_1'}\right)',$$
$$\bar{J}_3=\frac{k}{3}\left[\frac{\psi_1}{\psi_1'}\dot{Y}_1'-\frac{\psi_3}{\psi_3'}\dot{Y}_3'-\frac{\psi_3}{\psi_3'}\psi_1\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'+\frac{\psi_1'}{\psi_3'}\left(\frac{\psi_3'}{\psi_1'}\right)^{\cdot}\right],$$
$$\bar{J}_4=\frac{k}{3}\left[\frac{\psi_2}{\psi_3'}\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'-\frac{\psi_1}{\psi_1'}\dot{Y}_1'+\dot{Y}_1\right],$$
$$\bar{J}_5=\frac{k}{3}\left[\frac{\psi_1^2}{\psi_1'}\dot{Y}_1'-\frac{\psi_2}{\psi_3'}\dot{Y}_3'-\frac{\psi_1\psi_2}{\psi_3'}\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'\right]+k\left[\dot{\psi}_1-\frac{\psi_1\dot{\psi}_1'}{\psi_1'}\right], \tag{21}$$
$$\bar{J}_6=-\frac{k}{3}\left[\frac{1}{\psi_1'}\dot{Y}_1'-\frac{\psi_3}{\psi_3'}\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'\right],$$
$$\bar{J}_7=\frac{k}{3}\left[\frac{\psi_3}{\psi_3'}(\psi_2-\psi_1\psi_3)\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'-\frac{\psi_2-\psi_1\psi_3}{\psi_1'}\dot{Y}_1'-\frac{\psi_3^2}{\psi_3'}\dot{Y}_3'\right]+k\left[\frac{\psi_3\dot{\psi}_3'}{\psi_3'}-\dot{\psi}_3\right],$$
$$\bar{J}_8=\frac{k}{3}\left[\frac{\psi_2}{\psi_3'}(\psi_2-\psi_1\psi_3)\left(\frac{\dot{Y}_1'}{\psi_1'}\right)'-\frac{\psi_1}{\psi_1'}(\psi_2-\psi_1\psi_3)\dot{Y}_1'-\frac{\psi_2\psi_3}{\psi_3'}\dot{Y}_3'\right]$$
$$+k\left[\psi_2\left(\frac{\dot{\psi}_1'}{\psi_1'}+\frac{\dot{\psi}_3'}{\psi_3'}\right)-\psi_1\psi_3\left(\frac{\dot{\psi}_1'}{\psi_1'}\right)+\dot{\psi}_1\psi_3-\dot{\psi}_2\right].$$

The homomorphic energy-momentum tensor $T=W_2$ is the Noether current for transformation of the fields

$$\delta\psi_3=\alpha\psi_3',\quad \delta\psi_2=\alpha\psi_2',\quad \delta\psi_1=\alpha\psi_1', \tag{22}$$

or, in terms of φ,

$$\delta\varphi=\alpha\varphi'+\rho\alpha'. \tag{23}$$

Then

$$\delta S=-\int\dot{\alpha}T,$$
$$W_2=T=k\left[-\frac{\psi_1'''}{\psi_1'}-\frac{\psi_3'''}{\psi_3'}+\frac{4}{3}\left(\frac{\psi_1''}{\psi_1'}\right)^2+\frac{1}{3}\frac{\psi_1''}{\psi_1'}\frac{\psi_3'''}{\psi_3'}+\frac{4}{3}\left(\frac{\psi_3''}{\psi_3'}\right)^2\right], \tag{24}$$

and

$$T_\varphi=T(\psi-\psi(\varphi),\ k\to q^2)+\rho\partial^2\varphi={}^1\!/_2q^2(\partial\varphi)^2-(q^2-1)\rho\partial^2\varphi. \tag{25}$$

The most remarkable fact here is that the holomorphic operator W_3 turns out to be a Noether current associated with an additional symmetry that leaves the action unchanged. In terms of the field ψ, the effects of this symmetry are rather complicated:

$$\delta\psi_3=\varepsilon'\psi_3'+{}^1\!/_3\varepsilon Y_1'\psi_3'=(\varepsilon'+\varepsilon\nu_1\varphi')\psi_3',$$
$$\delta\psi_2=\varepsilon'\psi_1\psi_3'+\varepsilon({}^2\!/_3\psi_1\psi_1''\psi_3'/\psi_1'+{}^1\!/_3\psi_1\psi_3''-\psi_1'\psi_3'), \tag{26}$$
$$\delta\psi_1=-{}^1\!/_3\varepsilon Y_3'\psi_1'=-\varepsilon(\nu_3\varphi')\psi_1'.$$

but in terms of φ, these same equations simplify:

$$\delta(\alpha_3\varphi)=\varepsilon''+2\varepsilon'\nu_3\varphi'+\varepsilon[\nu_1\varphi''+(\nu_1\varphi')(\alpha_3\varphi')];$$
$$\delta(\alpha_1\varphi)=-\varepsilon'\nu_3\varphi'-\varepsilon[\nu_3\varphi''+(\nu_3\varphi')(\alpha_1\varphi')], \tag{27}$$

[here α_1, α_2, α_3 are positive roots of the Lie algebra $sl(3)$, while ν_1 and ν_3 are the fundamental weights: $\alpha_1\nu_3=\alpha_3\nu_1=0$ (see Fig. 1)]. Equations (27) come as a more or less natural generalization of (23). The variation of the action is

$$\delta S=-\int\dot{\varepsilon}W_3^{cl}, \tag{28}$$

$$\frac{3}{k}W_3^{cl}=\left(\frac{\psi_1''}{\psi_1'}+2\frac{\psi_3''}{\psi_3'}\right)''-\left(\frac{\psi_1''}{\psi_1'}+2\frac{\psi_3''}{\psi_3'}\right)\left(\frac{\psi_3''}{\psi_3'}\right)'+\frac{2}{9}\left(\frac{\psi_1''}{\psi_1'}\right)^3$$
$$+\frac{2}{9}\left(\frac{\psi_3''}{\psi_3'}\right)^3-\frac{1}{3}\left(\frac{\psi_1''}{\psi_1'}\right)^2\left(\frac{\psi_3''}{\psi_3'}\right)+\frac{1}{3}\left(\frac{\psi_1''}{\psi_1'}\right)\left(\frac{\psi_3''}{\psi_3'}\right)^2, \tag{29}$$

while in terms of the free fields ϕ (see Fig. 1),

$$W_{3\phi}^{cl}\sim\frac{1}{q}\left[(\partial\phi_\perp)^3-3(\partial\phi_\perp)(\partial\phi_\parallel)^2+3i\left(\frac{3}{2}\right)^{1/2}q\partial\phi_\parallel\partial^2\phi_\parallel+\frac{3i}{2^{1/2}}q\,\partial\phi_\perp\partial^2\phi_\parallel+\frac{9iq}{2^{1/2}}\partial^2\phi_\perp\partial\phi_\parallel+3i\left(\frac{3}{2}\right)^{1/2}q\,\partial\phi_\perp\partial^2\phi_\perp-3q^2\partial^3\phi_\perp-3^{1/2}q^2\partial^3\phi_\parallel\right] \tag{30}$$

(the subscripts $\parallel$ and $\perp$ identify components of the vector $\vec{\phi}$ corresponding to the directions of $\alpha_2=\rho$ and μ_2; see Fig. 1).

3. In Eq. (30), the operator W_3 is really just the classical approximation to the formula given by Fateev and Luk'yanov,[3] which was obtained via a Miura quantum transformation. In order to appreciate this connection with a well-known result, recall how the quantum W_3 algebra of Zamolodchikov[2] was constructed:

$$T(z)T(0)=\frac{c/2}{z^4}+\frac{2T(z/2)}{z^2}+:T^2:(z/2)+o(z^2),$$

$$T(z)W_3^{FZ}(0)=\frac{3}{z^2}W_3^{FZ}(0)+\frac{1}{z}W_3^{FZ\prime}(0)+o(1),$$

$$W_3^{FZ}(z)W_3^{FZ}(0)$$

$$=\frac{c/3}{z^6}+\frac{2}{z^4}T(0)+\frac{1}{z^3}T'(0)+\frac{{}^3/_{10}T''(0)+2\Lambda_1(0)}{z^2}$$

$$+\frac{{}^1/_{15}T'''(0)+\Lambda_1'(0)}{z}+o(1)$$

$$=\frac{c/3}{z^6}+\frac{2T(z/2)}{z^4}+\frac{2\Lambda(z/2)}{z^2}+o(1). \tag{31}$$

Here

$$c=2-12\rho^2\alpha_0^2=2-24(q-1/q)^2=50-24(q^2+1/q^2),$$

$$\alpha_0=q-\frac{1}{q},\quad \Lambda=\frac{2:T^2:}{4-15\alpha_0^2},\quad \Lambda_1\equiv\Lambda+\frac{1}{40}T''. \tag{32}$$

The free-field representation for T and W_3^{FZ} becomes[2]

$$T=W_2=-\frac{1}{2}(\partial\phi_\perp)^2-\frac{1}{2}(\partial\phi_\parallel)^2+2^{1/2}i\alpha_0\partial^2\phi_\parallel$$

$$=q^2[{}^1/_2(\partial\varphi_\perp)^2+{}^1/_2(\partial\varphi_\parallel)^2-2^{1/2}\partial^2\varphi_\parallel]+2^{1/2}q^0\partial^2\varphi_\parallel,$$

$$W_3^{FZ}=\frac{1}{6i(1-{}^{15}/_4\alpha_0^2)^{1/2}}\Big\{(\partial\phi_\perp)^3-3\partial\phi_\perp(\partial\phi_\parallel)^2$$

$$+\frac{3i}{2^{1/2}}\alpha_0\,\partial\phi_\perp\,\partial^2\phi_\parallel$$

$$+\frac{9i}{2^{1/2}}\alpha_0\,\partial^2\phi_\perp\,\partial\phi_\parallel+3\alpha_0^2\,\partial^3\phi_\perp\Big\}$$

$$=\frac{3^{3/2}\cdot 5^{1/2}\,i}{(1-34/15q^2+1/q^4)^{1/2}}\Big\{q^2\Big[(\partial\varphi_\perp)^3$$

$$-3\partial\varphi_\perp(\partial\varphi_\parallel)^2+\frac{3}{2^{1/2}}\partial\varphi_\perp\,\partial^2\varphi_\parallel+\frac{9}{2^{1/2}}\partial^2\varphi_\perp\,\partial\varphi_\parallel-3\partial^3\varphi_\perp\Big]$$

$$-q^0\Big[\frac{3}{2^{1/2}}\partial\varphi_\perp\,\partial^2\varphi_\parallel+\frac{9}{2^{1/2}}\partial^2\varphi_\perp\,\partial\varphi_\parallel-6\partial^3\varphi_\perp\Big]-\frac{1}{q^2}\partial^3\varphi_\perp\Big\}. \tag{33}$$

The expression in curly brackets in the second of Eqs. (33) is not the same as the result obtained by Fateev and Luk'yanov,[3] which was derived from the Miura transformation:

$$(\partial-F_1)\dots(\partial-F_n)$$

$$=\partial^n-\frac{1}{\alpha_0^2}T\partial^{n-2}+\frac{i}{3^{3/2}\cdot 2^{1/2}\alpha_0^3}W_3^{FL}\partial^{n-3}+\dots \tag{34}$$

[here $F_a=i\mu_a\partial\vec{\phi}/\partial\alpha_0$, and μ_a is the fundamental weight of the $sl(n)$ algebra; see Fig. 1 for the case $n=3$]. The resulting relationship is then

$$W_3^{FL}=(\partial\phi_\perp)^3-3\partial\phi_\perp(\partial\phi_\parallel)^2+\frac{3^{3/2}i}{2^{1/2}}\alpha_0\,\partial\phi_\parallel\,\partial^2\phi_\parallel$$

$$+3\cdot 2^{-1/2}i\alpha_0\,\partial\phi_\perp\,\partial^2\phi_\parallel+3^2\cdot 2^{-1/2}i\alpha_0\,\partial^2\phi_\perp\,\partial\phi_\parallel$$

$$+3^{3/2}\cdot 2^{-1/2}i\alpha_0\,\partial\phi_\perp\,\partial^2\phi_\perp$$

$$+3\alpha_0^2\,\partial^3\phi_\perp+3^{3/2}\alpha_0^2\,\partial^3\phi_\parallel=6i(1-{}^{15}/_4\alpha_0^2)^{1/2}W_3^{FZ}-3^{3/2}\cdot 2^{1/2}i\alpha_0\,\partial T. \tag{35}$$

Clearly, this is a general phenomenon: the W operators of Fateev and Luk'yanov are not conformal fields, and must be (slightly) corrected by terms containing derivatives of the "lowest-order" W operators, with coefficients proportional to the powers of α_0 (these corrections were recently discussed in Ref. 12). In terms of $\varphi=-i\phi/q$,

$$\alpha_0^{-1}W_3^{FL}=q^2W_3^{cl}(\varphi)+o(q^0), \tag{36}$$

where W_3^{cl} has been defined in (30). Our Noether current W_3^{cl} is then nothing but the classical approximation to the W_3 operator of Fateev and Luk'yanov (i.e., it is correct only to leading order in q). Of course, in the same classical approximation, the transformations (6) and (27) may be supplemented by a general coordinate transformation to obtain the linear combination (35), and to represent the classical approximation to W_3^{FZ} as a Noether current. Technically, however, this is quite difficult to do at the quantum level.

One important difference between the quantum expressions (33) for $T=W_3$ and the "highest-order" W_k operators is that negative powers of q appear in the latter case. It can be shown to be completely impossible to derive detailed expressions for the variation of our action (12), since the latter yields only contributions containing q^2 and $q^0=1$. In contrast to the general coordinate transformations (23), however, which relate to the energy-momentum tensor, transformations like (27)—which are responsible for the other W operators—are nonlinear, and they induce nontrivial changes in the metric appearing in the functional integral. The corresponding Jacobian has a nontrivial dependence on the φ fields, and it does not reduce to the usual anomaly (which would only affect terms of order q^0). Instead, a calculation of the Jacobian leads to an infinite perturbation series in q^{-2} [as in Eq. (33)]. Naturally, the existence of compact expressions like (33) means that there ought to be a way to sum the series; further study is required.

4. We summarize by formulating the two important statements whose validity has been demonstrated here for an $sl(n)$ theory with $n=3$.

a) There exists an $sl(n)$-invariant action which may be written in terms of inhomogeneous holomorphic coordinates of the type (ψ_2,ψ_3) in the space CP^{n-1}. Under $sl(n)$ transformations, the corresponding Lagrangian changes by a total derivative. The coordinates (ψ_2,ψ_3) comprise the last column of one of the Gaussian expansion matrices. These are so identified that the Drinfeld-Sokolov Hamiltonian reduction expresses all remaining fields ψ and ϕ locally in terms of them (in the gauge $\chi=0$). For example, $\psi_1=\psi_2'/\psi_3'$, but $\psi_2=\int\psi_1\psi_3'dz$; ψ_2 and ψ_3 are therefore the preferable coordinates, and the group $sl(n)$ acts precisely upon the latter via rational transformations (i.e., simply).

b) The indicated action also admits of $n-1$ other (surprising?) symmetries of the type (see the Appendix for details)

$$\delta^{(k)}\phi=\varepsilon^{(k)}+\varepsilon^{(k-1)}\phi'+\varepsilon^{(k-2)}[\phi''+(\phi')^2]+\dots. \tag{37}$$

The first of these, $\delta\varphi=\alpha\varphi'+\rho\alpha'$, is simply a general coordinate transformation, and the corresponding Noether current is the holomorphic energy-momentum tensor $T=W_2$. The generators of the other transformations are the remaining W_k operators, which are known from the theory of W_n algebras. We have not derived the quantum equations for $W_{k\neq 2}$. Apart from the W algebras themselves, the algebra of the above transformations may also be of interest. One can also determine the antiholomorphic analog of W_k, which is identified with $\mathrm{tr}\bar{J}^k$ + corrections.

APPENDIX

We explain here how we derived the equations for the transformations that generate the W_3 operators for arbitrary $sl(n)$ algebras.

In the general case, the ψ fields are defined in terms of a Gaussian expansion of the g matrices in the WZW model:[11]

$$g=g_L(\chi)g_D(\varphi)g_U(\psi),$$

$$g_U(\psi)=\begin{bmatrix} 1 & \psi_1^{(1)} & \psi_1^{(2)} & \psi_1^{(3)} & \dots & \psi_1^{(n-1)} \\ 0 & 1 & \psi_2^{(1)} & \psi_2^{(2)} & \dots & \psi_2^{(n-1)} \\ 0 & 0 & 1 & \psi_3^{(1)} & \dots & \psi_3^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \qquad \text{(A1)}$$

where $g_L(\chi)$ and $g_D(\varphi)$ are lower-triangular and diagonal matrices, respectively. For $sl(3)$, we make use of a different notation for the variables,

$$\psi_1=\psi_1^{(1)},\ \psi_3=\psi_2^{(1)},\ \psi_2=\psi_1^{(2)},$$

in accordance with the notation of Ref. 11, wherein the interested reader can find complete information on representations of Kac-Moody algebras in terms of free fields. In our new notation, the raised index identifies the weight $\rho\alpha$ for the field ψ_α.

After the reduction and current conditions $J^{(1)}=1$ and $J=g^{-1}\partial g$ have been imposed and the gauge condition $\chi=0$ has been chosen, the action will depend solely on some set of $n-1$ fields ψ. The remaining fields ψ and φ can be locally expressed in terms of these $n-1$ independent variables only if they are elements of the last column of the matrix $g_U(\psi)$:

$$\alpha_i\varphi=\log\partial\psi_i^{(1)},\quad \psi_i^{(1)}=\partial\psi_i^{(2)}/\partial\psi_{i+1}^{(1)},$$

$$\psi_i^{(2)}=(\partial\psi_i^{(3)}-\psi_i^{(1)}\partial\psi_{i+1}^{(2)}+\psi_i^{(1)}\psi_{i+1}^{(1)}\partial\psi_{i+2}^{(1)})/\partial\psi_{i+2}^{(1)}=\partial\psi_i^{(3)}/\partial\psi_{i+2}^{(1)}, \qquad \text{(A2)}$$

$$\cdots$$

$$\psi_i^{(k)}=\partial\psi_i^{(k+1)}/\partial\psi_{i+k}^{(1)},$$

$$\cdots$$

The recursion relations (A2) express any element $g_U(\psi)$ in terms of an element in the next column. The variables $\psi^{(n-i)}\equiv\Psi_i$ appearing in the last column may be looked upon as inhomogeneous complex coordinates in the space CP^{n-1} [this manifold is just the space required for the adjoint action under $sl(n)$].

The group $sl(n)$ operates on these coordinates via piecewise-linear transformations,

$$\Psi_i\to\frac{a_{ij}\Psi_j+a_{i0}}{a_{0j}\Psi_j+a_{00}}. \qquad \text{(A3)}$$

The first problem is now to find an action that depends only on the holomorphic coordinates Ψ_i in the space CP^{n-1}, and that is invariant under $sl(n)$. It is remarkable that such an action exists (and is given by a Kirillov-Kostant-like reduction), but it is simpler to write it out in terms of the unit-weight fields $\psi_i\equiv\psi_i^{(1)}$ rather than $\Psi_i\equiv\psi_i^{(n-i)}$:

$$S=k\int\sum_{i,j=1}^{\mathrm{rank}\,G}K^{ij}\frac{\dot\psi_i'\psi_j''}{\psi_i'\psi_j'}, \qquad \text{(A4)}$$

where $K^{ij}=(Q^{-1})^{ij}$ is the inverse of the Cartan matrix Q_{ij},

$$K={}^1/_2\ \text{for}\ sl(2),\quad K=\frac{1}{3}\begin{pmatrix}2&1\\1&2\end{pmatrix}\ \text{for}\ sl(3),$$

$$K=\frac{1}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}\ \text{for}\ sl(4),\dots.$$

Note, however, that the effect of the group $sl(n)$ on the fields ψ_i is a nontrivial one [see (19) for a simple example], since the only good coordinates in the space CP^{n-1} are the Ψ_i. For variations in either Ψ_i or ψ_i, the action changes by

$$\delta S=-k\sum_{i,j=1}^{n-1}\int\left(\frac{\delta\psi_i'}{\psi_i'}\right)^{\cdot}\frac{K^{ij}\psi_j''}{\psi_j'}=-k\int\sum_{i=1}^{n-1}\left(\frac{\delta\psi_i'}{\psi_i'}\right)^{\cdot}Y_i(\psi)'. \qquad \text{(A5)}$$

Hereafter, we will find it convenient to employ a more compact notation, similar to that introduced in (A2),

$$Y_i(\psi)'=\sum_{j=1}^{n-1}K^{ij}\frac{\psi_j''}{\psi_j'}=\sum_{j=1}^{n-1}K^{ij}\alpha_j\varphi'=\mathbf{v}_i\varphi'. \qquad \text{(A6)}$$

Then

$$\psi_i''/\psi_i'=Q_{ij}Y_j(\psi)', \qquad \text{(A7)}$$

where Q_{ij} is the inverse Cartan matrix of the $sl(n)$ algebra (i.e., $Q_{ii}=2$, $Q_{i-1,i}=Q_{i,i+1}=-1$, all other elements vanish).

A direct test of the $sl(n)$ invariance of the action is an arduous exercise which we shall not pursue here. In order to understand the symmetry transformations associated with the operator W_3, we begin by searching for transformations of the fields ψ_i that leave (A4) unchanged. Next, it must be shown that these transformations turn into local transformations of the "good" coordinates Ψ_i (neither of these steps is trivial, *a priori*!). The desired transformations take the form

$$\delta\psi_i=a_i\varepsilon'\psi_i'+\varepsilon\psi_i'\sum_{j=1}^{n-1}B_{ij}Y_j(\psi)' \qquad \text{(A8)}$$

with some constants a_i and B_{ij}. Then

$$\frac{\delta\psi_i'}{\psi_i'}=a_i\varepsilon''+\varepsilon'\sum_j(a_iQ_{ij}+B_{ij})Y_j'+\varepsilon\sum_jB_{ij}Y_j''$$
$$+\varepsilon\sum_{j,k}B_{ij}Q_{jk}Y_j'Y_k', \qquad \text{(A9)}$$

and after substitution into (A5) and some manipulation, we obtain

$$-\frac{\delta S}{k}=\int\sum_{i=1}^{n-1}\left(\frac{\delta\psi_i'}{\psi_i'}\right)^{\cdot}Y_i'$$
$$=-\sum_{i,j}[B_{ij}+(a_i-a_j)Q_{ij}]\int\varepsilon Y_i''\dot Y_j'$$
$$+\sum_{i,j,k}\int\varepsilon\left[B_{ik}\dot Y_k'(Q_{ij}Y_i'Y_j')+\frac{1}{2}B_{ij}(Q_{ik}-Q_{jk})Y_i'Y_j'\dot Y_k'\right]$$
$$+\ \text{terms proportional to}\ \dot\varepsilon. \qquad \text{(A10)}$$

For (A8) to be a symmetry of the action, it is necessary that the right-hand side of (A10) be equal to $\dot\varepsilon W_3^{\rm cl}$ for all ε. Thus,

$$B_{ij}=-(a_i-a_j)Q_{ij}, \qquad (A11)$$

and the expression in brackets in the second integral in (A10) must be a total time derivative:

$$\sum_{ijk}\Big[B_{ik}(Q_{ij}Y_i'Y_j')+\frac{1}{2}B_{ij}(Q_{ik}-Q_{jk})Y_i'Y_j'\Big]\dot Y_k'=\{\ldots\}^{\cdot}. \qquad (A12)$$

Equation (A11) says that B_{ij} is an antisymmetric matrix built up from the vectors a_i. Naturally, a displacement common to all the components a_i, δa_i = const, leaves the action unchanged, and induces the trivial symmetry $\delta\psi_i$ = const$\cdot\varepsilon'\psi_i'$—a general coordinate transformation with parameter ε'. In terms of the modulus of these displacements, Eqs. (A11) and (A12) admit of a unique solution, with $a_i = i-1$, and

$$B_{ij}=\begin{pmatrix}0 & -1 & 0 & \ldots & 0\\ 1 & 0 & -1 & \ldots & 0\\ 0 & 1 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & 0\end{pmatrix}. \qquad (A13)$$

If we assume that $Y_0 = Y_n \equiv 0$, we may rewrite (8) as

$$\delta\psi_j=(j-1)\varepsilon'\psi_j'+\varepsilon\psi_j'(Y_{j-1}'-Y_{j+1}'). \qquad (A14)$$

We shall not go on to calculate W_3 here; the result agrees with the classical part of the expression obtained by Fateev and Luk'yanov,[3] one α_0 has been replaced by q.

We are still not finished, however. It remains to show that the transformations (A14) turn into local transformations of the fundamental variables Ψ_i, and this is in fact the case. Omitting some straightforward calculations, we give only the final result for the transformation of all ψ fields into $g_U(\psi)$. Equation (A14) takes care of the case $\psi_j = \psi_j^{(1)}$. In general,

$$\begin{aligned}\delta\psi_j&=(j-1)\varepsilon'\psi_j'+\varepsilon\psi_j'(Y_{j-1}'-Y_{j+1}'),\\ \delta\psi_j^{(2)}&=j\varepsilon'\psi_j^{(2)}+\varepsilon[-\psi_j'\psi_{j+1}'+\psi_j\psi_{j+1}'(Y_j'-Y_{j+2}')],\\ \delta\psi_j^{(3)}&=(j+1)\varepsilon'\psi_j^{(3)\prime}+\varepsilon[-\psi_j^{(2)\prime}\psi_{j+2}'+\psi_j^{(2)}\psi_{j+2}'(Y_{j+1}'-Y_{j+3}')],\\ \delta\psi_j^{(k+1)}&=(j+k-1)\varepsilon'\psi_j^{(k+1)\prime}\\ &\quad+\varepsilon[-\psi_j^{(k)\prime}\psi_{j+k}'+\psi_j^{(k)}\psi_{j+k}'(Y_{j+k-1}'-Y_{j+k+1}')].\end{aligned} \qquad (A15)$$

We obtain (26) from (A15) by an appropriate change of notation:

$$\psi_1\equiv\psi_1^{(1)}\to\psi_1,\quad \psi_2\equiv\psi_2^{(1)}\to\psi_3,$$
$$\psi_1^{(2)}\to\psi_2,\quad Y_1\to Y_1,\quad Y_2\to Y_3$$

To check the consistency of (A15) with (A2), it is necessary to use an identity for the inverse of the Cartan matrix for $sl(n)$,

$$-K_{j-2,i}+K_{j-1,i}+K_{j,i}-K_{j+1,i}=\delta_{j-1,i}+\delta_{j,i}, \qquad (A16)$$

whereupon we find that

$$-Y_{j+k-2}'+Y_{j+k-1}'+Y_{j+k}'-Y_{j+k+1}'=\frac{\psi_{j+k-1}''}{\psi_{j+k-1}'}+\frac{\psi_{j+k}''}{\psi_{j+k}'}. \qquad (A17)$$

In similar fashion, one can derive explicit expressions for the more complicated field transformations that leave the action (A4) unchanged, and for which the corresponding Noether currents are consistent with the other operators W_k. These transformations contain $k-1$ derivatives and $k-2$ powers of Y_i' [compare with (A8) for W_3].

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[1)] For consistency with Ref. 4, we employ the notation $\psi'=\partial\psi\equiv\partial\psi/\partial z$ and $\dot\psi=\bar\partial\psi\equiv\partial\psi/\partial\bar z$; the action is everywhere given by an integral over $d^2z=dzd\bar z$.

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