Two-dimensional turbulent diffusion

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The problem of convective transport in a 2-*D* incompressible nonstationary flow is discussed. Particular attention is focused on the low-frequency asymptotic limit, when the frequency ω with which velocity field changes is much lower than the frequency of particle motion in the flow. Thus arises the problem of the statistical properties of the isolines of random functions, which is treated with the aid of percolation theory. The asymptotic forms of the turbulent diffusion coefficient $D_t \sim \omega^{3/10}$ and the growth rate $\gamma_s \sim \omega^{1/2}$ of the stochastic instability are obtained analytically. With allowance for small molecular diffusion D, an interpolation formula for D_t is derived that includes both effects, and the mixing length is determined.

1. INTRODUCTION

The turbulent diffusion problem is one of the classical ones and has important applications in hydrodynamics and plasma physics.[1-3] In its most general form, it is formulated as follows. Consider a given flow $\mathbf{v}(\mathbf{r},t)$ having certain properties of randomness ("turbulence"), determine the distribution, asymptotic in time, of an ensemble of points satisfying the equation

$$d\mathbf{r}/dt=\mathbf{v}(\mathbf{r},\ t), \qquad (1)$$

i.e., moving together with the flow. In the limit $\langle \mathbf{r}^2(t)\rangle/t \to \text{const}$ for $t\to\infty$, one speaks of turbulent diffusion with the appropriate coefficient, and the averaging is carried out over an ensemble of flows, or, more transparently, over the initial conditions.

Another approach to this problem consists in carrying out the corresponding averaging in the equation of transport of a passive impurity

$$\partial n/\partial t+\mathbf{v}(\mathbf{r},\ t)\nabla n=D\Delta n, \qquad (2)$$

where n is the density of the impurity, the temperature, or some other scalar field, and D is the coefficient of the bare (molecular or Coulombic) diffusion, to which would correspond a random Langevin force on the right-hand side of Eq. (1). In the majority of cases of practical importance, the diffusion term is small compared to the convective term in Eq. (2), but in some cases it may be fundamental.[3] In the remainder of this section of the article we assume $D = 0$.

In the majority of cases, the flow is assumed to be incompressible, div $\mathbf{v} = 0$, and the present work is confined to this case. In addition, it will be assumed that on average, the fluid is at rest, $\langle \mathbf{v}\rangle = 0$.

By the average (2) is meant a transition to the equation for the average density of the impurity,

$$\partial \langle n\rangle/\partial t = D_T\Delta\langle n\rangle,$$

which holds on spatial scales over which the averaging is carried out, $R \gg a_T$, where a_T is the turbulent mixing length, defined below.

The dependence of the flow velocity $\mathbf{v}$ on the position and time is determined by appropriate hydrodynamic or other equations, depending on the model. In the case of hydrodynamic turbulence of an ideal fluid, we have the equation for the frozen-in condition of a vortex

$$\partial\, \mathrm{rot}\, \mathbf{v}/\partial t=\mathrm{rot}[\mathbf{v}\ \mathrm{rot}\, \mathbf{v}],$$

whence it follows that the characteristic pulsation frequency ω is expressed in order of magnitude in terms of the amplitude of the velocity v and characteristic spatial scale λ by $\omega \approx v/\lambda$. It is evident from dimensional considerations that in this case, the turbulent diffusion coefficient (if diffusion indeed takes place) is estimated from

$$D_T\approx\lambda v. \qquad (3)$$

The majority of studies deal precisely with this case and propose more accurate expressions which define D_T in terms of the turbulence parameters.

The more complicated models, for example, in plasma physics, deal with flows whose frequency is a free parameter. Examples are problems of random electric drifts in a strong magnetic field,[4,5] diffusion of the lines of force of a stochastic magnetic field,[6] or conductivity of three-dimensionally inhomogeneous Hall media.[7] All these problems reduce to that of 2-*D* turbulent diffusion.

An estimate of turbulent diffusion in the high-frequency case $\omega \gg v/\lambda$ is not difficult. During a correlation time ω^{-1}, the particle is displaced over a distance $r\approx v/\omega$, and successive displacements are independent; this yields

$$D_T\approx r^2\omega\approx v^2/\omega,\quad \omega\gg v/\lambda. \qquad (4)$$

Of greatest conceptual and methodological interest is the low-frequency limit $\omega \ll v/\lambda$, for which the asymptotic form of the turbulent diffusion coefficient is by no means trivial, since this form is determined by long streamlines of the flow under consideration and requires the use of methods of percolation theory.[8-10] In contrast to the results (3) and (4), the solution to the transport problem in the low-frequency limit depends on the number d of spatial dimensions. The index ξ in the asymptotic expression

$$D_T\approx\lambda v\,(\lambda\omega/v)^{\xi}$$

is a function of d:

$$\xi=\xi(d).$$

In the 3-*D* case ($d = 3$), the lines of an incompressible field can densely fill whole regions, and the property of stochastic instability of the trajectories is typical (structurally stable) for such fields.[11] It can be assumed that the measure

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of trajectories receding diffusively to infinity is positive. Then we have $D_t \sim \lambda v$; effective diffusion is possible in the stationary case; the index ξ is expressed $\xi(3) = 0$. However, it is not known whether the fundamental assumption that the measure of trajectories receding to infinity is finite is valid. An indication of its validity was obtained in numerical calculations.[12]

In the 2-D case

$$\mathbf{v} = [\nabla \psi(x, y, t), \mathbf{e}_z], \tag{5}$$

which is the main topic of the present study, the streamlines $\psi = \text{const}$ do not posses stochasticity themselves, since a 2-D region cannot be filled with a smooth, nonself-intersecting curve. It follows that for 2-D turbulent diffusion the time-dependence of the flow (or the bare diffusion) is fundamental, i.e., D_T vanishes for $\omega = 0$, so that $\xi(2) > 0$. A lower estimate of D_T can be obtained by virtue of the fact that in a time ω^{-1}, the particle is displaced by at least λ, and hence we have $D_T \gg \lambda^2 \omega$.

Thus for $d = 2$ we have $0 < \xi \leqslant 1$ and, as will be shown below, the second inequality is also strict [$\xi(2) = 3/10$]. At the same time, the results of several studies contradict this conclusion. For example, in a discussion of the low-frequency asymptotic limit in a 2-D problem, Refs. 4 and 6 proposed results which in our notation correspond to the index $\xi = 0$. Outside the dependence on the number of spatial dimensions, Ref. 13 obtained the result $\xi = 0$ (for non-Gaussian turbulence) and $\xi = 1$ (in the opposite case).

The inaccuracy of these results is due to the inexactness of the closure of the moment equations,[4,6] or to the superficial analysis of the integral equation in Ref. 13. In a specified sense, a general reason for this type of discrepancy is the use of standard perturbation series, which in low-frequency limits and also in the case of small molecular diffusion coefficient are outside their radius of convergence. This situation is typical of percolation problems having long-range correlations, in particular, the mixing length $a_T \gg \lambda$. Note that the relationship between the problem of diffusion of lines of magnetic force and percolation theory was pointed out in Ref. 6.

The first section of the present article investigates the statistical properties of the contours of constant random stream function $\psi(x,y,t)$. In particular, we discuss the reduction of the problem to the standard percolation formulation using a periodic grid, for which several exact results exist,[14] and also estimate the lifetime of long streamlines, allowing for the time-dependence of the flow. The second section estimates the turbulent diffusion coefficient. The method of investigation is similar to the approach developed in Ref. 3, which solved the problem of effective diffusion in a stationary random 2-D incompressible flow in the presence of a small diffusion coefficient $D \ll \lambda v$. The third section deals with the relationship between turbulent diffusion and the stochastic instability of particle motion. Finally, we formulate our conclusions. All calculations and reasoning of any degree of complexity are consigned to Appendices.

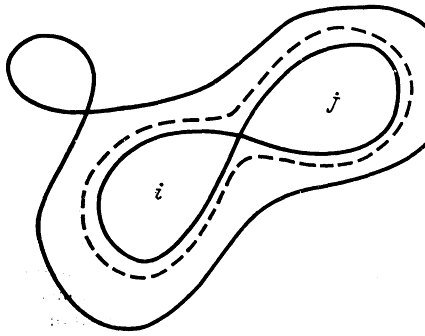

FIG. 1. Separatrices (solid line) and nonsingular isoline (dashed line) of a random function.

1. GEOMETRY OF RANDOM 2-*D* FLOW

This section discusses the statistical properties of random flow streamlines which are the level lines of a random stream function ψ. Concerning the latter, the following is postulated: $\psi(x,y,t)$ is bounded, has a single characteristic spatial scale λ, characteristic amplitude $\psi \approx \lambda v$, and characteristic variation frequency $\omega \ll v/\lambda$; $\psi(x,y,t)$ is not periodic in either space or time; the spatial average is $\langle \psi \rangle = 0$ for any t. The randomness (at a "general location") also assumes that at almost all t no two of the countable set of singular points (saddle points or extrema) have the same height, the mean density of singular points of each type per unit area being of the order of λ^{-2}.

For simplicity, dimensionless variables in which $\psi \approx \lambda \approx v = 1$, $\omega \ll 1$, $D \ll 1$ are used below.

Among the contours $\psi = \text{const}$ there are lines covering an arbitrarily large area; this is particularly clear from the "contour-dressing" procedure shown in Fig. 1. The size distribution function of the contours is conveniently introduced in the form of the two-index probability $P(i,j)$ that the separatrix passing through a saddle point selected at random contains i extrema in one of the loops, and j extrema in the other (Fig. 1). As shown in Appendix 1, the asymptotic form of the probability $P(i,j)$ for $i,j \gg 1$ is as follows:

$$P(i, j) \approx i^{-3/2} j^{-3/2}. \tag{6}$$

Since each contour $\psi = \text{const}$ is included between a pair of closest separatrices, the distribution (6) gives the size distribution function of the contours [see expression (7)].

The relationship of the size of the isolines to the level h of the cross section $\psi = h$ and their fractal properties can be determined from the universality of the indices of continuous and grid percolation for a given dimensionality of the problem.[15] In Appendix 2, the relationship between contours and lattice clusters is obtained by perturbating a periodic function, and this leads to the following results.

The distribution function of the contours $\psi = h (h \ll 1)$ with respect to the transverse size a is

$$f_h(a) \approx a^{-(1-1/\nu)}, \quad 1 < a < a_h, \tag{7}$$

where a_h is the maximum diameter for a given h:

$$a_h = h^{-\nu}, \quad \nu = 4/3, \tag{8}$$

(see Ref. 13). For $a > a_h$, the function $f_h(a)$ decreases exponentially.[10] By $f_h(a)$ is meant the probability that a point for which $\psi(x,y) = h$ belongs to a contour with a cross section of order a (for example, from $a/2$ to a).

The distribution function (7) integrated with respect to h is the size distribution of all the contours

$$F(a) \approx a^{-1}, \tag{9}$$

which is defined as the probability (fraction of area) of hitting a contour with a size of order a.

Each contour of probability 1 is closed. There exists exactly one open contour, and it is located at the level $\psi = 0$.

Long contours ($a \gg 1$) have self-similar fractal properties. They are highly sinuous, and as a result, their length L substantially exceeds the diameter

$$L(a) \approx a^{d_h}, \tag{10}$$

where $d_h = 1 + 1/v = 7/4$ (see Ref. 14) is the fractal dimensionality of long contours. Thanks to the absence of self-intersections and hence, effective self-repulsion, d_h is less than 2, the number of dimensions of a Brownian trajectory permitting self-intersections.[9]

Having obtained an idea of the static picture, let us consider the kinematics of the contours as ψ changes with time. The picture is transformed beyond recognition in a time ω^{-1}, but for each t the distribution (7) remains valid. Since for a given h, short contours are more probable, almost every long contour becomes shorter when it reconnects.

Further discussion requires an estimate of the lifetime τ_h (of the halving of the transverse size) of the contour $\psi = h \ll 1$ with a size of order a_h. Since in some places such a curve comes very close (a distance much shorter than unity) to itself, this lifetime is appreciably shorter than the characteristic time ω^{-1}. As shown in Appendix 3, the following estimate applies:

$$\tau_h \approx h/\omega. \tag{11}$$

2. ESTIMATE OF TURBULENT DIFFUSION

To determine the turbulent diffusion coefficient, to the kinematic picture presented above it is necessary to add dynamic considerations on the nature of particle motion in the flow discussed.

As noted above, for $\omega \ll 1$ the main contribution to the transfer is due to a relatively small number of long streamlines, along which the particles manage to be coherently displaced to a large distance $a_T \gg 1$. This distance (mixing length) is limited by two processes: the finite lifetime τ_h of long trajectories $\psi = h$ and the departure of the particles from the given level h due to the time-dependence of the flow. To determine which of these effects predominates, it is necessary to compare the times of these processes. Let us find the time τ_p at which a particle leaves the level $\psi = h$.

For incompressible flow (5), the equations of particle motion are

$$\begin{aligned} \partial x/\partial t &= \partial\psi(x, y, t)/\partial y, \\ \partial y/\partial t &= -\partial\psi/\partial x. \end{aligned} \tag{12}$$

Hence we find the change in ψ for a given particle:

$$d\psi/dt = \partial\psi/\partial t \approx \pm\omega, \tag{13}$$

where the symbols in Eq. (12) emphasize the random alternation of $d\psi/dt$ with the characteristic correlation time $\Delta t = 1$, during which the particle is displaced by the characteristic spatial scale. Equation (12) signifies a diffusive walk of the particle along the ψ coordinate with diffusion coefficient $D_{\psi\psi} \approx \omega^2$. When small bare diffusion $D \ll 1$ is taken into account, an additional departure of the particle arises from a given streamline, and hence, also from the height ψ. Since this process is not coherent with the one discussed above, the diffusion coefficients are summed: $D_{\psi\psi} \approx \omega^2 + D$.

Hence we obtain an estimate for the time of departure of the particle from the level $\psi = h$ (doubling time of the particle level):

$$\tau_p \approx h^2/D_{\psi\psi} \approx h^2/(\omega^2 + D). \tag{14}$$

We now assume that the mixing length is limited by the lifetime τ_h of the streamlines, whereas the loss of particles from these lines is not fundamental (the scope of applicability of this assumption will be determined below). In a time τ_h, the particle travels a path $L \approx v\tau_h = \tau_h$ along the trajectory. If this path does not exceed $L_h = a_h{}^d h$, the particle does not have sufficient time to make a complete revolution; it executes many revolutions along a given streamline. In accordance with Eqs. (10) and (11), we have

$$a \approx \begin{cases} L^{1/d_h} \approx (h/\omega)^{1/d_h}, & \tau_h < h^{-vd_h} \\ a_h, & \tau_h > h^{-vd_h}. \end{cases}$$

Hence, allowing for Eq. (8) we obtain the mixing length

$$a_T = \omega^{-v/(v+2)} \tag{15}$$

and the characteristic level $h = \omega^{1/(v+2)}$.

Allowing for the fraction of particles (9) which are located on the characteristic trajectories, we obtain the following estimate of the turbulent diffusion coefficient:

$$\begin{gathered} D_T \approx F(a_T) a_T{}^2/\tau_h \approx \omega^{1/(v+2)}, \\ D < \omega^{(v+3)/(v+2)} = \omega^{10/13}. \end{gathered} \tag{16}$$

The inequality in expression (16) determining the limits of applicability of the result was obtained from expressions (11) and (14) as the requirement $\tau_h < \tau_p$.

Actually, to estimate D_t it would be necessary to carry out the summation over the trajectories of all the scales. Moreover, thanks to the exponential nature of all the relations, the contributions of small ($a < a_T$) and large ($a > a_T$) streamlines are equal and agree with (16) in order of magnitude.

At higher values of the bare diffusion coefficient, $D > \omega^{10/13}$, the time and length of coherence are determined by the departure of the particles from relatively long-lived long trajectories: $\tau_p < \tau_h$. In this case, the mixing length is determined by the condition $L = \tau_p$, whence, allowing for expressions (10)and (14), we find

$$a_T = D^{-v/(v+3)}, \quad D > \omega^{(v+3)/(v+2)}. \tag{17}$$

Then the turbulent diffusion coefficient is

$$\begin{gathered} D_T \approx F(a_T) a_T{}^2/\tau_p \approx D^{1/(v+3)}, \\ D > \omega^{(v+3)/(v+2)}, \end{gathered} \tag{18}$$

which corresponds to the result of Ref. 3, obtained in a somewhat different manner.

Cases (15), (16) and (17), (18) can be combined by common interpolation for the coefficient of 2-D turbulent diffusion for ω, $D \ll 1$:

$$D_T \approx \omega^{1/(v+2)} + D^{1/(v+3)}, \tag{19}$$

and for the turbulent mixing length

$$a_T=[\omega^{\nu/(\nu+2)}+D^{\nu/(\nu+3)}]^{-1}. \quad (20)$$

Changing to dimensional symbols and substituting the numerical value of the indices, we rewrite expressions (19) and (20) in the form

$$D_T\approx\lambda v[(\lambda\omega/v)^{3/10}+(D/\lambda v)^{3/13}], \quad (21)$$

$$a_T\approx\lambda[(\lambda\omega/v)^{4/10}+(D/\lambda v)^{4/13}]^{-1}. \quad (22)$$

3. STOCHASTIC INSTABILITY IN 2-*D* NONSTATIONARY FLOW

Usually, diffusion in Hamiltonian systems is attributed to the stochastic behavior of the phase trajectories.[16] One of the key manifestations of such behavior is the exponential dispersal of close trajectories (stochastic instability), characterized by the Kolmogorov entropy (growth rate)

$$\gamma_s=\lim_{t\to\infty}\lim_{\delta r(0)\to 0} t^{-1}\ln\frac{\langle\delta r(t)\rangle}{\delta r(0)}. \quad (23)$$

In Eq. (23), $\delta r(t)$ represents the distance in phase space at instant t between a pair of points whose coordinates satisfy the equations of motion, and the averaging is carried out over the initial conditions inside the stochasticity region.

In the absence of molecular diffusion ($D=0$), the equations of motion of a passive impurity (12) are Hamiltonian. In the stationary case ($\omega=0$), Hamiltonian motion with one degree of freedom (2-*D* phase plane) is integrable, and the effective diffusion is zero [see (16)]. Time dependence ($\omega\neq 0$) is equivalent to the appearance of an additional degree of freedom,[16] and stochastic behavior[1)] is possible in a 4-*D* phase volume.

For the simplest Hamiltonians in the case of a small perturbation ($\omega\ll v/\lambda$), the stochastic regions are usually concentrated in exponentially narrow layers near the separatrices, where the adiabatic invariant of the particle motion breaks down. (The presence of an adiabatic invariant in the remaining regions, as well as the presence of any other integral of motion, effectively reduces the number of degrees of freedom per unit and accordingly eliminates stochasticity.) In the flow under consideration at a general location, one finds separatrices of arbitrarily large size, along which the reversal time is large, and therefore, the stochastic region is not exponentially small.

The size a_s of the trajectories on which the adiabatic invariant breaks down is given by the equality of the particle reversal time $L(a_s)/v$ and the reconnection time of two neighboring separatrices $t_\delta=[\omega L(a_s)/\lambda]^{-1}$ (see Appendix 3), whence we obtain

$$L(a_s)\approx\lambda(v/\lambda\omega)^{1/2},$$
$$a_s\approx\lambda(v/\lambda\omega)^{\nu/2(\nu+1)}. \quad (24)$$

As shown in Appendix 4, these and longer trajectories are precisely the ones that make the maximum contribution to the stochastic instability growth rate

$$\gamma_s\approx(\omega v/\lambda)^{1/2}\ln(v/\lambda\omega), \quad (25)$$

and after averaging over time, the entire (x,y) plane is the stochastic region.

It is of interest to note that the stochasticity dimension (24) is appreciably smaller than the mixing length (15): $a_s\ll a_T$. Correspondingly, the relative area of the stochastic instability region $F(a_s)$ [see expression (9)] is much greater than the measure of the mixing region $F(a_T)$, and the zone of maximum stochasticity includes the mixing region. It follows from this that the cause of the mixing and of the diffusion walk in the case under consideration is not the internal stochasticity of the Hamiltonian, but the external stochasticity, determined by the richness of its topological structure, which is time-dependent. Thus, turbulent diffusion is not a direct consequence of Hamiltonian stochasticity. Rather, both effects are consequences of the time-dependence of the flow.

With inclusion of an arbitrarily small molecular diffusion D, the impurity density gradients, which grow exponentially with time, will be "smeared" through several of the growth rates (25), but this in no way affects the effective transfer of the trace material when

$$D\ll\lambda v(\lambda\omega/v)^{10/13}.$$

From this point of view, the hypothesis of negative interference of molecular and turbulent diffusion (i.e., the decrease in turbulent diffusion D_T for a small $D\neq 0$ in comparison to the case $D=0$) (Ref. 17) appears to be improbable.

CONCLUSION

Thus, turbulent diffusion in the limit of low frequencies of the change in velocity field $\omega\ll v/\lambda$ and large Péclet numbers $\lambda v/D\gg 1$ is sensitive to the dimensionality of the problem and characterized by the presence of long-range percolation-type correlations. Such a problem cannot be solved on the basis of conventional methods of perturbation theory and requires an analysis of the topological properties of random flows by means of methods of percolation theory.

In the 2-*D* case, there is a simple relationship between random incompressible flows and the lattice clusters of percolation theory. This, as well as the presence of exact results in the theory of 2-*D* percolation, makes it possible to find the appropriate exponents in the asymptotic form of the turbulent diffusion coefficient (21) and mixing length (22).

In the stationary case, the estimate (21) was checked numerically in Ref. 3, which obtained a satisfactory agreement between the calculations and the analytic theory. It is difficult to carry out analogous calculations for a nonstationary velocity field because of the appearance, caused by $\omega\neq 0$, of stochastic instability of the trajectories in Eq. (1).

In the 3-*D* case, there is no direct relationship between the topology of the flows and the sets of a level, and therefore, the methods discussed in our work do not extend to this case. In the remark, which continues to hold, on the importance of long-range correlations, the question of the low-frequency asymptotic forms of 3-*D* turbulent diffusion remains open.

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APPENDIX 1

STATISTICS OF RANDOM FUNCTION SEPARATRICES

The values of ψ at two different hyperbolic singular points (saddle points) differ with a probability of 1. Therefore, the separatrix closes on itself, dividing the plane into three parts — two bounded and one unbounded (Fig. 1). We introduce the index (i,j) of the saddle point, as the numbers of extrema in the two bounded parts of its separatrix. We distinguish the order of i and j, assuming, for example, that the first index corresponds to the part located farther on the left. Let $P(i,j)$ be the fractions of saddle points with the corresponding indices in a given realization of the field $\psi(x,y)$. These fractions are not independent. Indeed, each separatrix with the index (i,j) is surrounded by a separatrix with the index $(i+j, l)$ or $(l, i+j)$, so that

$$\sum_{i=1}^{k-1} P(i, k-i) = 2\sum_{l=1}^{\infty} P(k, l). \tag{A1.1}$$

Introducing the characteristic function

$$f(x, y) = \sum_{i,j=1}^{\infty} P(i, j) x^i y^j,$$

we rewrite Eq. (A1.1) in the form

$$f(x, x) = 2f(x, 1) - x. \tag{A1.2}$$

The normalization condition $f(1,1) = 1$ was taken into account in Eq. (A1.2). The relation (A1.2) is of course insufficient to determine f. It may be assumed that for a random function the parts of the separatrices are independent, i.e., $F(i,j) = P_i P_j$. Then f can also be factored: $f(x,y) = f(x)f(y)$, and Eq. (A1.2) gives

$$f(x) = 1 - (1-x)^{1/2} = \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} x^n. \tag{A1.3}$$

Expansion of Eq. (A1.3) gives

$$P(i, j) = \frac{(2i-3)!!(2j-3)!!}{(2i)!!(2j)!!} \approx \frac{1}{4\pi} i^{-3/2} j^{-3/2}. \tag{A1.4}$$

It is clear that, in fact, the parts of the separatrices are not completely independent, since a large i means that the value of ψ at the (i,j) saddle point is close to zero (see Appendix 2). This raises the chances of j also turning out to be large. However, the assumption that

$$P(i, j) \approx i^{-\alpha} j^{-\beta} \tag{A1.5}$$

for $i > j$, where $\alpha \geqslant \beta$, with allowance for the dependence indicated above, leads first to $\alpha = 3/2$, and then, as is evident from Appendix 3, also to $\beta = 3/2$. It will be shown here that $\alpha = 3/2$.

The probability that a point chosen at random will lie on a contour ψ with a size of order a [see expression (9)] is $F(a) \approx a^{-1}$. Since the area enclosed between two successive separatrices is of order 1, this probability can be estimated as the probability that a randomly chosen separatrix contains saddle points of order $N = a^2$:

$$F(a) = \sum_{\substack{i>j \\ i \approx N}} P(i, j) \approx a^{-2(\alpha-1)},$$

and hence,

$$\alpha = 3/2. \tag{A1.6}$$

APPENDIX 2

RELATIONSHIP BETWEEN RANDOM FUNCTION LEVEL LINES AND THE PERCOLATION PROBLEM

In this section, we shall discuss in detail a method for reducing the problem of continuous flow to a lattice problem, briefly mentioned in an earlier paper.[3]

We shall consider the function $\psi_0(x,y) = \sin x \sin y$ whose separatrices constitute a (π,π) = periodic square lattice at the nodes of which there are saddle points, the size of all the nonsingular contours of this lattice being bounded from above. The idea of the method consists in modeling the random function $\psi(x,y)$ by a small perturbation of the function ψ_0 that eliminates the existing degeneracies (periodicity and coincidence of the level of all saddle points):

$$\psi(x, y) = \psi_0(x, y) + \varepsilon\psi_1(x, y), \tag{A2.1}$$

where $\psi_1 \approx 1$, $\varepsilon \ll 1$, and the sign and magnitude of ψ_1 at the nodes of the unperturbed lattice are random. Perturbation changes the topology of the isolines $\psi = h$ in only a small range of heights $|h| < \varepsilon$, and the direction of uncoupling of the unperturbed node is uniquely determined by the sign of the expression $\varepsilon\psi_1 - h$. In particular, for $h \to 0$, the bifurcation direction is random (see Fig. 2).

Let us consider a square lattice, dual to the initial one, whose nodes are located at points where $\psi_0(x,y) = \text{sign } h$. We make the convention to connect a pair of nearest-neighbor nodes if at the corresponding saddle point of the initial lattice $h(\varepsilon\psi_1 - h) > 0$, i.e., if the relationship acts as a "mirror" for the level line $\psi = h$ (see Fig. 2). Otherwise, no connection is drawn. Accordingly, the probability of establishing a connection on the dual lattice is

$$p(h) = 1/2 - |h|/\varepsilon, \tag{A2.2}$$

and the topology of the contours $\psi = h$ turns out to be uniquely related to the clusters of the percolation problem of the connections on the dual lattice with probability (A2.2).

As we know, the critical probability of percolation of the connections on a 2-D square lattice is $p_c = 1/2$ (Ref. 10). For $p = p_c$ on a plane, there is exactly one infinite cluster, and hence, among the contours $\psi = 0$ there is one infinite one, which is the outer envelope of an infinite cluster on the dual lattice. "Flooding" considerations lead to a similar conclusion.[8,3]

When $h \neq 0$ $(p < p_c)$, the dual clusters do not exceed the transverse correlation size

$$a(p) = (p_c - p)^{-\nu} \approx (|h|/\varepsilon)^{-\nu}, \quad \nu = 4/3 \tag{A2.3}$$

(see Ref. 13), in the sense that the probability of the opposite is exponentially small.[10] Thus, the same holds true for the corresponding contours.

It follows from the method of construction of the dual lattice and of establishment of connections on it that the corresponding level is precisely the envelope of a dual cluster, for which, in the case of its large size in the theory of 2-D percolation, the fractal (self-similar) structure is known, and there is an exact result for the fractal dimensionality $d = 1 + 1/\nu = 7/4$ (Ref. 13), relating the length and cross section of the shell [see Eq. (10)].

The results obtained are accurate for $|h| < \varepsilon \ll 1$, and the estimates (A2.3) and (9) remain valid at the limit of applicability $\varepsilon = 1$, when the perturbation is not small, the specific properties of the unperturbed function $\psi_0(x,y)$ are omitted, and only the essential features of a function of general position remain.

To substantiate the estimates (7) and (9) given in the text, let us note that thanks to the power law dependence (8), the width of a bundle isolines differing in the transverse

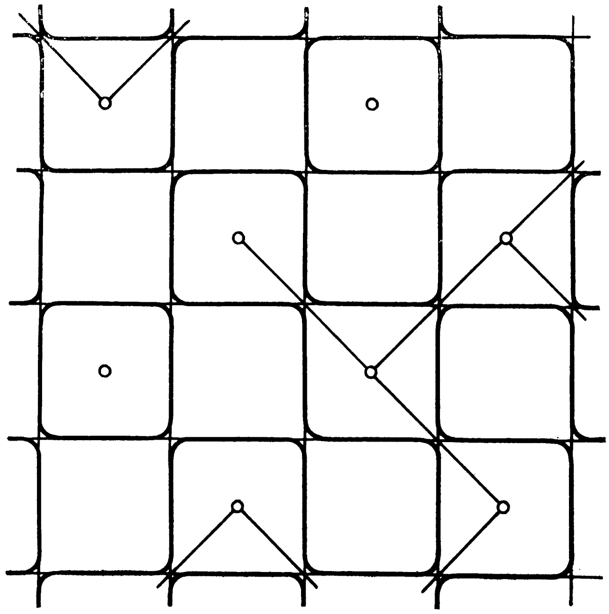

FIG. 2. Level lines $\psi = h \leqslant \varepsilon$ of the function (A2.1) and clusters corresponding to them. ○ — nodes of dual lattice.

direction by a factor $\leqslant 2$ is of order h, whence we obtain the fraction of area

$$F(a) = L(a_h)\,h/a_h^2 \approx a_h^{-1} \qquad \text{(A2.4)}$$

in accordance with expression (9). Referring the probability (A2.4) to a unit interval of change in h, we obtain the distribution function (7).

APPENDIX 3

ESTIMATE OF THE LIFETIME OF NONSTATIONARY LEVEL LINES

We shall consider a contour $\psi = h \ll 1$ with a diameter on the order of the maximum one: $a \approx a_h$. The distance δ from this line to the nearest saddle point is determined by the fact that the density of the saddle points on a plane is of order unity: $\delta L(a) \approx 1$. By virtue of the unit value of the characteristic gradient ψ, the same quantity δ determines the difference of the level of the nearest saddle point from h. As ψ changes with time, the level of the saddle point changes at a rate of order ω, and hence, it intersects the level h of the line under consideration in a time $t_\delta \approx (L(a_h)\omega)^{-1}$. For such an intersection, the contour $\psi = h$ loses (although it can also acquire with lower probability) a j loop (i.e., a loop containing j saddle points) with probability $P(i,j)$ [see (A1.5), (A1.6)], where $i + j = N \approx a_h^2$.

Let us now estimate the probability that in one such bifurcation, the contour will immediately be shortened by approximately one-half:

$$P_{1/2} \approx \Bigl[\sum_{\substack{j\approx N/2\\ i+j=N}} P(i,j)\Bigr]\Bigl[\sum_{i+j=N} P(i,j)\Bigr]^{-1} \approx N^{-(\beta-1)}, \quad \beta \leqslant 3/2. \qquad \text{(A3.1)}$$

Furthermore, the probability that such bifurcation will take place in a time $t > t_\delta$ is $(t/t_\delta)P_{1/2}$. Equating the latter to unity, we obtain an upper estimate, for the desired lifetime τ_h since the loss of length could also accumulate sooner, through less catastrophic reconnections with $j \ll N$:

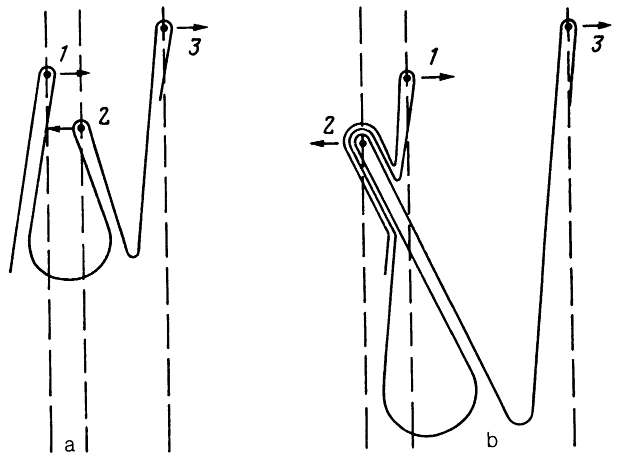

FIG. 3. Stretching of fluid curve. Dashes denote the separatrices. Fractal lines are straightened for clarity.

$$\tau_h \leqslant t_\delta / P_{1/2} = a_h^{2(\beta-1)} / (L(a_h)\,\omega) \leqslant a_h / (L(a_h)\,\omega) = h/\omega. \qquad \text{(A3.2)}$$

The final estimate for τ_h can be obtained by comparing the contours with lattice clusters. Supplementing the picture presented in Appendix 2 with the time dependence, we find that the nonstationary contours $\psi = h$ can be compared to the nonstationary problem of percolation with a fixed probability $p(h) = 1/2 - |h|$, but the probability tests themselves should be conducted again after a time ω^{-1} has elapsed. This corresponds to the fact that the number of connections on the dual lattice does not change, but in a time $t < \omega^{-1}$ the fraction ωt of randomly chosen connections is transferred to a different, free, randomly chosen location. If the process of reconnection is replaced by removal, to which corresponds the time-dependent probability

$$p = {}^1/_2 - |h| - \omega t, \qquad \text{(A3.3)}$$

then such a trial solution only accelerates the degradation of the clusters and their envelopes, i.e., $\psi = h$ contours. Thus from (A2.3) and (A3.3) we find the lower estimate for the lifetime of a contour:

$$\tau_h \geqslant h/\omega. \qquad \text{(A3.4)}$$

Comparing (A3.2) and (A3.4), we arrive at the result (11). We also note that the presence of equality in (A3.2) has been proven at the same time, i.e., $\beta = 3/2$, which confirms the hypothesis that the large loops of separatrices are independent and the estimate (6) resulting from it.

APPENDIX 4

STOCHASTIC INSTABILITY GROWTH RATE

The average rate of divergence of close trajectories (23) can be calculated as the rate of elongation of a Lagrangian (fluid) curve, since any curve consists of close points. An arbitrary initial location of the curve guarantees the validity of the averaging.

A characteristic feature of the evolution of the fluid curve is the fact that it catches the saddle points of the flow (see Fig. 3). Let us consider in more detail the process of stretching of portions of the fluid curve along the long streamlines ($L \gg \lambda$).

The time-dependence of the flow is manifested in two respects. First, the fluid elements do not retain their initial value of ψ [see Eq. (13)]. Second, the saddle points themselves move at a velocity of order $\lambda\omega$. It is evident that the second process takes place considerably faster than the first, and therefore, it may be assumed that the fluid elements retain their value of ψ, but on the other hand, the separatrices move and reconnect in a characteristic time $t_\delta = \lambda/(\omega L)$ (see Appendix 3).

The process is shown schematically in Fig. 3, where the points denote the saddle points ψ together (as a rule) with small separatrix loops, the dashed lines denote the separatrices, and an L-periodic boundary condition along the vertical direction is assumed. The stretching of the fluid curve is governed by the intersection of its saddle point (Fig. 3 b), which takes place after a time t_δ has elapsed. The length of the curve in the channel considered between adjacent separatrices increases by a factor of $t_\delta v/L$. Obviously, the most effective elongation is reached in channels for which $t_\delta v/L \approx 2$, i.e., on streamlines defined by the estimate (24), and the contribution of streamlines of smaller scale can soon be neglected altogether.[2)] The length L_f of the fluid curve varies exponentially with time:

$$L_f = L_{f0} \exp\left[(\omega v/\lambda)^{1/2} t\right],$$

until the streamlines of characteristic size a_s, which comprise the beam of relevant channels of the same scale are destroyed. After the lifetime $\tau_h \approx h_s/\omega \equiv (\lambda/a_s)^{1/v}/\omega \gg t_\delta$ has elapsed, we obtain the length of the curve

$$L_f = L_{f0} (v/\lambda\omega)^{v/2(v+1)} \exp\left[(\omega v/\lambda)^{1/2} \tau_h\right],$$

where the preceding result was multiplied by the number of channels $(a_s/\lambda)^2 F(a_s)$ in the bundle. Then, instead of the disrupted bundle, at a distance of the order of a_s from it, a new one is formed, and further evolution of each portion of the now twisted curve is repeated in similar fashion, whence we obtain the desired growth rate (25).

[1)] In the literature, the phase space is occasionally constricted to a 3-D manifold of a constant Hamiltonian; this is associated with the so-called $1\frac{1}{2}$ degrees of freedom.

[2)] As for large-size channels, their contribution to the stretching is of the same order of magnitude as that of channels of size a_s. Because of the rapid intersection of the fluid curve by the saddle points, the latter manages to stretch along each channel only up to the scale of a_s, i.e., as before, expression (24) remains the characteristic scale of stochasticity.

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, 2 vols., ed. J. L. Lumley, MIT Press, Cambridge (1975).

[2] R. H. Kraichnan, J. Fluid Mech. **77**, 733 (1976).

[3] M. B. Isichenko, Ya. L. Kalda, E. B.Tatarinova *et al.*, Zh. Eksp. Teor. Fiz. **96**, 913 (1989) [Sov. Phys. JETP **69**, 517 (1989)].

[4] J. H. Misguich, R. Balescu, A. L. Pecseli *et al.*, Plasm Phys. Contr. Fusion **29**, 825 (1987).

[5] C. W. Horton, in *Basic Plasma Physics*, Vol. 2, A. A. Galeev and R. Sudan, Eds. North-Holland, Amsterdam (1984).

[6] B. B. Kadomtsev and O. P. Pogutse, in *Plasma Physics and Controlled Nuclear Fusion Research. Proc. 7th Int. Conf.*, Innsbruck, 1978, Vol. 1. Vienna: IAEA, 1979, p. 549.

[7] Yu. A. Dreizin and A. M. Dykhne, Zh. Eksp. Teor. Fiz. **63**, 242 (1972) [Sov. Phys. JETP **36**, 127 (1973).

[8] A. L. Éfros, *Physics and Geometry of Disorder*, Mir, Moscow (1986).

[9] I. M. Sokolov, Usp. Fiz. Nauk **156**, 221 (1986) [Sov. Phys. Usp. **29**, 924 (1986)].

[10] H. Kesten, *Percolation Theory for Mathematicians*, Birkhauser, Cambridge (1982).

[11] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York (1978).

[12] A. A. Chernikov, B. A. Petrovichev, A. V. Rogal'sky *et al.*, *Anomalous Transport of Stream Lines Due to the Chaos and the Space Topology. Proc. Int. Workshop on Nonlinear and Turbulent Processes in Physics*, Kiev, 9–22 October 1989.

[13] A. Z. Dolginov and N. A. Silant'ev, Zh. Eksp. Teor. Fiz. **93**, 159 (1987) [Sov. Phys. JETP **66**, 90 (1987)].

[14] H. Saleur and B. Duplantier, Phys. Rev. Lett. **58**, 2325 (1987).

[15] B. I. Shklovskiĭ and A. L. Éfros, *Electronic Properties of Doped Semiconductors*, Springer, New York (1984).

[16] A. Lichtenberg and M. Lieberman, *Regular and Stochastic Motion*, Springer, New York (1983).

[17] H. K. Moffatt, J. Fluid Mech. **106**, 22 (1981).

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