Mean-field method and phase diagram for a scalar–fermion theory on a lattice

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(Submitted 3 August 1989)
Zh. Eksp. Teor. Fiz. **97**, 409–423 (February 1990)

As a simplified model of the Higgs sector of Weinberg-Salam theory, with a very heavy Higgs particle and a generation of heavy quarks—a model that requires investigation by means other than perturbation theory—the lattice theory of a scalar field and a fermion field with Yukawa interaction is considered. Its phase diagram is investigated in mean-field theory. Four phases are found: a phase with spontaneous symmetry breaking ($\langle\phi\rangle \neq 0$), a symmetry phase ($\langle\phi\rangle = 0$) with massless fermions, a symmetric phase without fermions, and a phase with antiferromagnetic ordering. Approximate formulas are obtained for the phase-transition lines, and also for the quantity $\langle\bar{\psi}\varphi\rangle$.

1. INTRODUCTION

In the Weinberg-Salam theory, which unifies the electromagnetic and weak interactions, a key role is played by the scalar fields which, through the Higgs mechanism, give rise to the masses of the W and Z bosons. Scalar particles (Higgs particles) should then inevitably be present in the theory, but up to now they have not been observed. Their masses are a free parameter of the theory. The question of how heavy these particles can be is of great interest. In the standard $SU(2)\times U(1)$ theory with one complex Higgs doublet Φ, the scalar sector of which has the form

$$\mathscr{L}=\partial_\mu\Phi^+\partial^\mu\Phi+\lambda(2\Phi^+\Phi-\eta^2)^2, \qquad (1)$$

the mass m_H of the Higgs particle is related to the vacuum expectation value η by the relation $m_H = (8\lambda)^{1/2}\eta$. Here, $\eta = (2^{1/2}G_F)^{-1/2} \cong 246$ GeV, where G_F is the Fermi weak-interaction constant.

Thus, a large value of m_H implies a large value of the constant λ. In this limit it is necessary to analyze the dynamics outside the framework of perturbation theory, e.g., with the aid of lattice regularization. It is then possible to obtain an upper bound on λ, and hence on m_H; namely, $m_H \lesssim 700$–800 GeV (Refs. 1-10).

In this problem of heavy Higgs particles there has been only one large coupling constant λ, and this has made it possible to confine oneself to the Higgs sector of the theory and to neglect the interaction of the field Φ with the gauge fields and fermions. This, however, can be done only if the masses of all the fermions are small in comparison with η. Otherwise, at least two large coupling constants arise in the theory: the Higgs interaction λ and a Yukawa-interaction constant Y. In both cases, the interaction with the gauge fields can be neglected in the first approximation, since this interaction is determined by the small electroweak-interaction constants. This leads to the idea of investigating such Higgs–fermion theories on a lattice by nonperturbative methods.[11-19]

Recently, certain simpler lattice scalar–fermion theories with Yukawa interaction (with a one-component scalar field[11,15-19] and with a two-component scalar field[13,14]) have been investigated by numerical simulation. Here, the main attention has been paid to the determination of their phase diagrams.

The simplest lattice scalar–fermion theory (with a one-component scalar field and the simplest discretization of the fermions) has the form[15-17]

$$S = -2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu} + \frac{1}{2}\sum_{x,\mu}\bar\psi_x\gamma_\mu(\psi_{x+\hat\mu}-\psi_{x-\hat\mu}) + \sum_x Y\phi_x\bar\psi_x\psi_x,$$

$$\mu=1,2,3,4. \quad \phi_x=\pm1. \qquad (2)$$

The latter condition on the value of the field ϕ corresponds to the bare constant $\lambda = \infty$. The simple discretization of the fermions lead to the result that the model describes a system with 16 fermions.

The study of the phase diagrams of lattice theories with fermions by the Monte Carlo method is rather laborious. Therefore, it would be useful to have an approximate way of determining the phase-transition lines. It is known that in the case of the Ising model [the model (2) without the fermions] the mean-field method permits one to find the phase-transition point within 10–15% (Refs. 20, 21). In this paper we shall use this method to investigate the phase diagram of the model (2).

In Sec. 2 we review the mean-field method in the Ising model. In Sec. 3 it is generalized to the model (2). In Sec. 4 we discuss the calculation of the fermion determinant and derive formulas for the phase-transition line separating the phase $\langle\phi\rangle = 0$ and $\langle\phi\rangle\neq0$. In Sec. 5 we obtain a simple formula for the fermion condensate $\langle\bar\psi\psi\rangle$. In Sec. 6 we discuss the two phases with $\langle\phi\rangle = 0$, which differ in the properties of the fermions. In Sec. 7 we derive the formula for the line of the phase transition to the phase with antiferromagnetic ordering.

2. MEAN-FIELD METHOD FOR THE ISING MODEL

Since the mean-field method becomes exact for large spatial dimensionality d, when the number of neighbors of each site becomes large, in the following we shall write all formulas for arbitrary d. (In the models with fermions, d is assumed to be even.)

In the Ising model, with action

$$S = -2k\sum_{x,\mu}\phi_x\phi_{x+\mu}, \qquad (3)$$

$$\mu=1,2,\ldots,d; \quad \phi_x=\pm1$$

 0038-5646/90/020228-08$03.00

the partition function in an external field h has the form

$$Z=\sum_{\{\phi_x=\pm1\}}\exp\left(+2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}+h\sum_x\phi_x\right). \quad (4)$$

The mean-field method permits us to carry out an approximate calculation of z as follows (Ref. 20; see also Ref. 21). We rewrite Z in the form

$$Z=Z_H\left\{Z_H^{-1}\sum_{\{\phi_x=\pm1\}}\exp\left[H\sum_x\phi_x+\left(2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}+(h-H)\sum_x\phi_x\right)\right]\right\}, \quad (5)$$

where

$$Z_H=\sum_{\{\phi_x=\pm1\}}\exp\left(H\sum_x\phi_x\right)=(2\,\mathrm{ch}\,H)^N. \quad (6)$$

Here N is the number of sites and H is an auxiliary field, which has the meaning of the mean field acting on each spin. Then

$$\begin{aligned}Z&=Z_H\left\langle\exp\left(2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}+(h-H)\sum_x\phi_x\right)\right\rangle_H\\&\geqslant Z_H\exp\left\langle2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}+(h-H)\sum_x\phi_x\right\rangle_H,\end{aligned} \quad (7)$$

where we have made use of the inequality $\langle\exp A\rangle\geqslant\exp\langle A\rangle$, and

$$\langle A\rangle_H=Z_H^{-1}\sum_{\{\phi_x=\pm1\}}\exp\left(H\sum_x\phi_x\right)\cdot A. \quad (8)$$

Since (7) is true for all H, we have

$$Z\geqslant\sup_H(2\mathrm{ch}\,H)^N\exp\left\langle2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}+(h-H)\sum_x\phi_x\right\rangle_H. \quad (9)$$

Taking into account that

$$\begin{aligned}\langle\phi_x\phi_{x+\hat\mu}\rangle_H&=\langle\phi_x\rangle_H\langle\phi_{x+\hat\mu}\rangle_H,\\\langle\phi_x\rangle_H&=\mathrm{th}\,H,\end{aligned} \quad (10)$$

we obtain a bound for the free energy in the external field:

$$W(h)=-\frac{1}{N}\ln Z\leqslant\inf_H W(h,H), \quad (11)$$

where

$$W(h,H)\equiv-\{\ln 2\,\mathrm{ch}\,H+2kd\,\mathrm{th}^2H+(h-H)\,\mathrm{th}\,H\}. \quad (12)$$

The minimum of $W(h,H)$ is reached at

$$H-h=4kd\,\mathrm{th}\,H. \quad (13)$$

Equations (10) and (13) are, obviously, the standard mean-field equations.

In the absence of an external field ($h=0$) the state with unbroken symmetry ($H=0,\langle\phi\rangle=0$) is always an extremum of the function $W(h,H)$. However, it can be stable only if

$$\frac{\partial^2}{\partial H^2}W(h=0,H)\,|_{H=0}>0,$$

which gives $4kd-1<0$, or $k<1/4d$. Thus, the mean-field method predicts a second-order phase transition at $k=1/4d$.

3. MEAN-FIELD METHOD FOR THE MODEL WITH FERMIONS

The above arguments are easily generalized to the case of the model (2). If we integrate in the partition function over the fermion fields, we obtain a boson theory with an effective action containing the contribution of the fermion determinant:

$$Z=\sum_{\{\phi_x\}}\left(\prod_x\int d\psi_x d\bar\psi_x\right)\exp(-S)=\sum_{\{\phi_x\}}\exp(-S_B+\ln\det M), \quad (14)$$

where

$$S_B=-2k\sum_{x,\mu}\phi_x\phi_{x+\hat\mu}, \quad (15)$$

and the matrix M has the form

$$\begin{aligned}M_{xy}&=\frac12\sum_\mu\gamma_\mu(\delta_{y,x+\hat\mu}-\delta_{y,x-\hat\mu})+Y\phi_x\delta_{xy}\\&\equiv K_{xy}+Y\phi_x\delta_{xy}.\end{aligned} \quad (16)$$

(This matrix also has Dirac indices, which are not written out explicitly.)

It is easy to convince oneself that in Eq. (12) an additional contribution arises from $\ln\det M$:

$$W(h,H)=-\left\{\ln 2\,\mathrm{ch}\,H+2kd\,\mathrm{th}^2H+(h-H)\,\mathrm{th}\,H+\frac{1}{N}\langle\ln\det M\rangle_H\right\}. \quad (17)$$

All of this makes sense if $\det M$ is positive for all configurations of the field ϕ.

Ideally, one ought to calculate the $\langle\ln\det M\rangle_H$, using the fact that the values of ϕ_x at different sites are not correlated. However, we have succeeded in calculating this quantity only for small and for large values of Y. This, nevertheless, gives much useful information about the phase diagram.

First of all, we note that for $Y=0$ and $Y=\infty$ the value of $\ln\det M$ reduces to a constant, independent of the field ϕ_x. Therefore, on the (Y,k) plane the phase-transition lines separating the phase $\langle\phi\rangle=0$ and $\langle\phi\rangle\neq0$ should begin at the points $(0,k_c)$ and (∞,k_c), where k_c is the phase-transition point in the Ising model (3).

For $Y\ll1$ we write

$$\begin{aligned}\ln\det M&=\ln\det K+\mathrm{tr}\ln(\delta_{xy}+(K^{-1})_{xy}Y\phi_y)\\&=\ln\det K-\frac12Y^2\,\mathrm{Tr}\sum_{x,y}(K^{-1})_{xy}\phi_y(K^{-1})_{yx}\phi_x+O(Y^4).\end{aligned} \quad (18)$$

Here, tr denotes summation over the space and Dirac indices, and Tr denotes summation over the Dirac indices only.

The expansion of $\ln\det M$ contains only even powers of Y, since the trace of the product of an odd number of γ-matrices is equal to zero, and

$$(K^{-1})_{xy}=i\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\exp\{-ip(x-y)\}$$

$$\times\Big(\sum_\mu \gamma_\mu \sin p_\mu\Big)\Big(\sum_\nu \sin^2 p_\nu\Big)^{-1}. \quad (19)$$

To calculate $\langle \ln\det M\rangle_H$, we note that, the allowance for (10) and (19),

$$\langle\phi_x\phi_y\rangle_H=\begin{cases}\text{th}^2 H, & x\neq y,\\ 1, & x=y,\end{cases} \quad (20)$$

$$(K^{-1})_{xx}=0.$$

We obtain

$$\langle\ln\det M\rangle_H=\ln\det K-\frac{1}{2}Y^2\,\text{th}^2 H\,\text{Tr}\sum_x (K^{-2})_{xx}$$

$$+O(Y^4)=\ln\det K+\frac{1}{2}NDcY^2\,\text{th}^2 H+O(Y^4), \quad (21)$$

where $D=2^{d/2}$ is the number of components of the Dirac fermion[1)] and

$$c\equiv-(K^{-2})_{xx}=\int\frac{d^dp}{(2\pi)^d}\Big(\sum_\mu\sin^2 p_\mu\Big)^{-1}. \quad (22)$$

For $d=4$ we have $c=0.6197...$.

Thus, we obtain the expression

$$W(h,H)=-\{\ln 2\,\text{ch}\,H+2kd\,\text{th}^2 H+(h-H)\,\text{th}\,H$$

$$+{}^1/_2 DcY^2\,\text{th}^2 H$$

$$+O(Y^4)\}+\text{const}. \quad (23)$$

The mean-field equation now has the form [to within terms $O(Y^4)$]

$$H-h=(4kd+DcY^2)\,\text{th}\,H. \quad (24)$$

The condition for stability of the solution $H=\langle\phi\rangle=0$ has the form

$$\frac{\partial^2}{\partial H^2}W(h=0,H)=1-4kd-DcY^2>0. \quad (25)$$

Thus, the curve of the second-order phase transition ($\langle\phi\rangle=0\rightarrow\langle\phi\rangle\neq0$) on the (Y,k) plane for $Y\ll1$ is described by the equation

$$k=\frac{1}{4d}(1-DcY^2)+O(Y^4). \quad (26)$$

For $Y\gg1$ we write

$$\ln\det M=\ln\det(Y\phi_x\delta_{xy})+\text{tr}\ln\Big(\delta_{xy}+\frac{1}{Y}K_{xy}\phi_y\Big)$$

$$=N\ln Y^D-\frac{1}{2Y^2}\text{Tr}\sum_{x,y}K_{xy}K_{yx}\phi_x\phi_y+O(1/Y^4). \quad (27)$$

In the first term we use $(\phi_x)^D=1$. Terms with odd powers of $1/Y$ are absent. Taking into account $K_{xx}=0$, and also (20), we obtain

$$\langle\ln\det M\rangle_H=N\ln Y^D-\frac{1}{2Y^2}\text{th}^2 H\,\text{Tr}\sum_x (K^2)_{xx}+O(1/Y^4). \quad (28)$$

Using the explicit form of K_{xy} [see (16)], we find

$$\text{Tr}\sum_x (K^2)_{xx}=-\frac{1}{2}dDN,$$

whence

$$W(h,H)=-\Big\{\ln 2\,\text{ch}\,H+2kd\,\text{th}^2 H+(h-H)\,\text{th}\,H$$

$$+\frac{dD}{4Y^2}\text{th}^2 H+O(1/Y^4)\Big\}+\text{const}. \quad (29)$$

The corresponding condition for stability of the phase with $\langle\phi\rangle=0$ has, to order $O(1/Y^4)$, the form

$$1-4kd-\frac{dD}{2Y^2}>0, \quad (30)$$

whence the equation of the phase-transition line is

$$k=\frac{1}{4d}\Big(1-\frac{dD}{2Y^2}\Big)+O(1/Y^4). \quad (31)$$

4. CALCULATION OF THE FERMION DETERMINANT IN LEADING ORDER IN $1/d$

The calculation of the phase-transition lines can be improved if we bear in mind the following two circumstances.

First, the stability of the solution $\langle\phi\rangle=0$ is determined by the sign of

$$\frac{\partial^2}{\partial H^2}W(h=0,H)\,|_{H=0}.$$

Therefore, to study the stability it is sufficient to calculate $W(h=0,H)$ to terms of order H^2, or, equivalently, to terms of order σ^2, where

$$\sigma\equiv\langle\phi\rangle_H=\text{th}\,H. \quad (32)$$

Second, the mean-field method, which is, generally speaking, approximate, becomes exact as $d\rightarrow\infty$. It gives the correct answers for all quantities in the leading order of the expansion in $1/d$. If in each order of the expansion in $1/Y$ we retain the leading term in $1/d$ (just as, in the $1/N$ expansion, in each order of perturbation theory in the coupling constant one takes the leading term in $1/N$), it is possible to sum the resulting series:

$$\langle\ln\det M\rangle_H=N\ln Y^D+\Big\langle\text{tr}\ln\Big(\delta_{xy}+\frac{1}{Y}K_{xy}\phi_y\Big)\Big\rangle_H$$

$$=N\ln Y^D-\frac{1}{2}\frac{1}{Y^2}\text{Tr}\sum_{x,y}K_{xy}K_{yx}\langle\phi_x\phi_y\rangle_H-\ldots$$

$$-\frac{1}{n}\frac{1}{Y^n}\text{Tr}\sum_{x,y_1,\ldots,y_{n-1}}K_{xy_1}K_{y_1y_2}\ldots K_{y_{n-1}x}\langle\phi_{y_1}\phi_{y_2}\ldots\phi_x\rangle_H-\ldots \quad (33)$$

To each term in this expansion there corresponds a diagram in the form of a closed broken line with vertices at the points $x,y_1,y_2,\ldots,y_{n-1}$. Since $K_{xy}\neq0$ only if x and y are nearest neighbors, the links of the broken line should join neighboring sites. Typical diagrams in order $1/Y^6$ are shown in Fig. 1. Each lattice site through which the broken line passes an odd number of times gives a factor σ. Sites through which it passes an even number of times give a factor 1.

It can be seen that various types of diagram contribute to the terms of order σ^2. Each of them appears with a factor

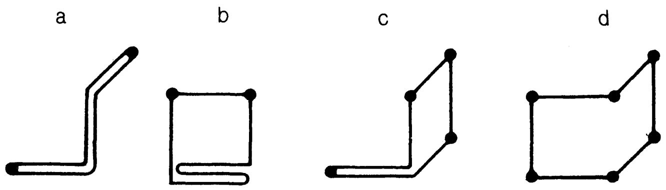

FIG. 1. Typical diagrams in order $1/Y^6$ in the expansion (33). The black circles denote unpaired vertices; each of them gives a factor σ. The diagrams a and b have order σ^2, diagram c has order σ^4, and diagram d has order σ^6.

equal to the number of ways of placing the diagram on the lattice. Diagrams of the type in Fig. 1a, i.e., all possible "doubled broken lines" with noncoinciding ends, are especially interesting. In leading order in d there are $\frac{1}{2}(2d)^{n/2}N$ of them. It seems highly plausible that all the other diagrams giving a contribution proportional to σ^2 appear with a factor with a lower power of d. Indeed, for a fixed n and $d \gg n$ the leading contribution in d, proportional to $(2d)^{n/2}$, is exhausted by the diagrams whose first $n/2$ steps are in $n/2$ different directions. Of these, only doubled broken lines all $n/2$ links of which go in $n/2$ different directions give a contribution of order σ^2.

In the framework of this assumption it is easy to calculate the part of $\langle \ln \det M\rangle_H$ proportional to σ^2. In each order in $1/Y$ the leading term in $1/d$ is equal to

$$-\frac{1}{n}\frac{1}{Y^n}D\sigma^2\left(\frac{1}{2}\right)^n(-1)^{n/2}\frac{n}{2}(2d)^{n/2}N, \qquad (34)$$

where the factor $(1/2)^n(-1)^{n/2}$ is the product of the matrix elements K_{xy}, and the factor n takes into account the different ways of labeling the vertices. Thus, in this approximation the contribution to $N^{-1}\langle \ln \det M\rangle_M$ proportional to σ^2 is equal to

$$-\sigma^2 D\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n}Y^{-n}\left(\frac{1}{2}\right)^n(-1)^{n/2}\frac{n}{2}(2d)^{n/2}$$

$$=\frac{\sigma^2}{2}D\frac{d}{d+2Y^2}, \qquad (35)$$

and the equation for the phase-transition line takes the form

$$\frac{\partial^2}{\partial H^2}W(h=0,H)\,|_{H=0}=1-4kd-\frac{Dd}{d+2Y^2}=0,$$

$$k=\frac{1}{4d}\left(1-\frac{Dd}{d+2Y^2}\right). \qquad (36)$$

What is the region of applicability of this formula? Of course, it agrees with (31), but it is clearly not correct for small Y. It is reasonable to assume that the region in which it is applicable is bounded by the radius of convergence of the series, i.e., $Y^2 > d/2$ (see Fig. 4, curve BB', below).

In the region of small Y the quantity $\langle \ln \det M\rangle_H$ can be expanded in a series in Y:

$$\langle \ln \det M\rangle_H=\ln\det K+\langle \operatorname{tr}\ln(\delta_{xy}+YK_{xy}^{-1}\phi_y)\rangle_H$$

$$=\ln\det K-\frac{1}{2}Y^2\operatorname{Tr}\sum_{x,y}K_{xy}^{-1}K_{yx}^{-1}\langle\phi_y\phi_x\rangle_H-\ldots$$

$$-\frac{1}{n}Y^n\operatorname{Tr}\sum_{x,y_1,\ldots,y_{n-1}}K_{xy_1}^{-1}K_{y_1y_2}^{-1}\ldots K_{y_{n-1}x}^{-1}\langle\phi_{y_1}\phi_{y_2}\ldots\phi_x\rangle_H-\ldots \qquad (37)$$

The matrices K^{-1} are nonlocal, and therefore the diagrams do not have such a simple form as in the expansion in $1/Y$. However, as $d\to\infty$ the matrix K^{-1} becomes more and more local. Its matrix elements have the form

$$K_{0y}^{-1}=i\int\frac{d^dp}{(2\pi)^d}e^{ipy}\sum_\mu\gamma_\mu\sin p_\mu\left(\sum_\nu\sin^2p_\nu\right)^{-1}$$

$$=\sum_\mu\frac{1}{2}\gamma_\mu(G^{-1}_{0,y+\hat\mu}-G^{-1}_{0,y-\hat\mu}), \qquad (38)$$

where

$$G_{0y}^{-1}=\int\frac{d^dp}{(2\pi)^d}e^{ipy}\left(\sum_\nu\sin^2p_\nu\right)^{-1}$$

$$=\int_0^\infty d\alpha\int\frac{d^dp}{(2\pi)^d}\exp\left(-\alpha\sum_\nu\sin^2p_\nu+ipy\right). \qquad (39)$$

If at least one coordinate y_ν is odd, then $G_{0y}^{-1}=0$. If all the y_ν are even, then

$$G_{0y}^{-1}=\int_0^\infty d\alpha\exp\left(-\frac{\alpha d}{2}\right)\prod_\nu I_{|y_\nu|/2}(\alpha/2), \qquad (40)$$

where $I_n(x)$ is a modified Bessel function. Thus, K_{0y}^{-1} is nonzero only if precisely one coordinate y_{μ_0} of the point y is odd and the others are even. It is easy to convince oneself that, for large d,

$$K_{0y}^{-1}\propto\begin{cases}\gamma_{\mu_0}(1/d)^{1/2+1/2\sum_\nu|y_\nu|} & \text{for } \sum_\nu|y_\nu|\ll d^{1/2},\\ \sum_\mu y_\mu\gamma_\mu\Big/\left(\sum_\nu y_\nu^2\right)^{d/2} & \text{for } \sum_\nu y_\nu^2\gg d.\end{cases} \qquad (41)$$

It can be seen that for $d\to\infty$ the matrix element K_{xy}^{-1} (where x and y are nearest neighbors) becomes the leading one. Calculating it, we obtain

$$K_{xy}^{-1}=\frac{1}{d}\sum_\mu\gamma_\mu(\delta_{x,y+\hat\mu}-\delta_{x,y-\hat\mu})+O\left(\frac{1}{d^2}\right)$$

$$=-\frac{2}{d}K_{xy}+O\left(\frac{1}{d^2}\right). \qquad (42)$$

We adopt the hypothesis that in the calculation of the series (37) in leading order in d one can neglect the corrections $O(1/d^2)$ in (42). For example, in order Y^2 an exact calculation gives [see (21) and (22)]

$$\langle \ln\det M\rangle_H=\text{const}+{}^1/_2NDcY^2\sigma^2, \qquad (43)$$

where

$$c=\int\frac{d^dp}{(2\pi)^d}\left(\sum_\mu\sin^2p_\mu\right)^{-1}$$

$$=\int_0^\infty d\alpha\exp(-\alpha d/2)\,(I_0(\alpha/2))^d=2/d+O(1/d^2). \qquad (44)$$

On the other hand, substituting $K^{-1}=-(2/d)K$ into (37) gives

$$\langle \ln\det M\rangle_H=\text{const}+\frac{1}{2}NDY^2\frac{2}{d}, \qquad (45)$$

i.e., gives the correct answer in leading order in d.

The substitution $K^{-1} = -(2/d)K$ reduces the series (37) to the series (33), if in it we make the replacement $Y \to d/2Y$. As a result, the equation for the phase-transition line takes the form [cf. (36)]

$$\frac{\partial^2}{\partial H^2} W(h=0, H)\,|_{H=0} = 1-4kd-Dd/[d+2(d/2Y)^2]=0,$$

$$k=\frac{1}{4d}\left(1-\frac{2DY^2}{d+2Y^2}\right). \qquad (46)$$

The situation with the region of applicability of this formula is analogous to the case of large Y: It gives an explicitly incorrect result for $Y\to\infty$, and so it is logical to assume that it is applicable where the series (37) converges, i.e., for $Y^2 < d/2$ (see Fig. 4, curve AA', below).

We note that the regions of applicability of the formulas (36) and (46) join at $Y^2 = d/2$, where they both give the same result[2)]:

$$k=\frac{1}{4d}\left(1-\frac{D}{2}\right).$$

5. THE FERMION CONDENSATE

In the preceding section we have studied the phase structure of the model (2), using as the order parameter the vacuum expectation value $\langle\phi\rangle$ of the scalar field. This, however, does not give information about the properties of the fermions in the given theory. In Refs. 15–17 it was noted that the fermions behave substantially differently for small and for large values of Y. Therefore, it is of interest to study quantities involving the fermion propagator. For this, we introduce into the action (2) an extra parameter m (the bare mass of the fermion):

$$Z(m) \equiv \exp(-NW(m))$$

$$=\sum_{\{\phi_x\}}\left(\prod_x \int d\psi_x\, d\bar\psi_x\right)\exp\left(-S-m\sum_x \bar\psi_x\psi_x\right)$$

$$=\sum_{\{\phi_x\}}\exp(-S_B+\ln\det\tilde M),\qquad \tilde M_{xy}=K_{xy}+Y\phi_x\delta_{xy}+m\delta_{xy}. \qquad (47)$$

The value $Z(m)$ can be calculated by the mean-field method, as above. Equation (17) takes the form

$$W(h,H,m)=-\Big\{\ln 2\,\mathrm{ch}\, H+2kd\,\mathrm{th}^2 H+(h-H)\,\mathrm{th}\, H$$
$$+\frac{1}{N}\langle\ln\det\tilde M\rangle_H\Big\}. \qquad (48)$$

Above, in essence, we calculated $\langle\ln\det\tilde M\rangle_H$ to terms of order σ^2, for $m=0$. the terms of order σm and m^2 can be calculated in an analogous manner. First we make use of the expansion in $1/Y$:

$$\langle\ln\det\tilde M\rangle_H = N\ln Y^D + \left\langle \mathrm{tr}\ln\left[\delta_{xy}+\frac{1}{Y}(m\delta_{xy}+K_{xy})\phi_y\right]\right\rangle_H$$

$$=N\ln Y^D+\frac{1}{Y}\mathrm{Tr}\sum_x (m\delta_{xx}+K_{xx})\langle\phi\rangle_H$$

$$-\frac{1}{2Y^2}\mathrm{Tr}\sum_{x,y}(m\delta_{xy}+K_{xy})(m\delta_{yx}+K_{yx})\langle\phi_y\phi_x\rangle_H$$

$$\ldots+\frac{(-1)^{n+1}}{nY^n}\mathrm{Tr}\sum_{x,y_1,\ldots,y_{n-1}}(m\delta_{xy_1}+K_{xy_1})(m\delta_{y_1y_2}+K_{y_1y_2})$$

$$\ldots(m\delta_{y_{n-1}x}+K_{y_{n-1}x})\langle\phi_{y_1}\phi_{y_2}\ldots\phi_x\rangle_H+\ldots \qquad (49)$$

The contribution of each term of this expansion to the terms proportional to m^2 can be depicted by diagrams analogous to those considered above. The vertices of the closed broken line are the points $x, y_1, \ldots, y_{n-1}$, and among these are two pairs of coinciding points (corresponding to the two terms $m\delta_{xy}$). Typical diagrams in order $1/Y^8$ are shown in Fig. 2.

As before, we assume that in the calculation of the contribution of order m^2 in leading order in d it is sufficient to take into account only diagrams of the "doubled broken line" type (Fig. 2a). The contribution of these diagrams in order $1/Y^n$ (here n is even) amounts to

$$-\frac{1}{nY^n}Dm^2\frac{n}{2}(2d)^{n/2-1}(-1)^{n/2-1}\left(\frac{1}{2}\right)^{n-2}N. \qquad (50)$$

Summing over all even n, we find that the contribution of order m^2 to $N^{-1}\langle\ln\det\tilde M\rangle_H$ in this approximation is equal to

$$-m^2\frac{D}{d+2Y^2}. \qquad (51)$$

The contribution of order $m\sigma$ in the expansion (49) is described by the same diagrams as the contribution of order m^2, but instead of one of the vertices m we have an unpaired ϕ_x (see Figs. 2a and 2b; these diagrams now arise in order $1/Y^7$). The contribution of diagrams of the "doubled broken line" type in order $1/Y^n$ (here n is odd) is equal to

$$\frac{1}{nY^n}Dm\sigma n(2d)^{(n-1)/2}(-1)^{(n-1)/2}\left(\frac{1}{2}\right)^{n-1}N. \qquad (52)$$

Summing, we obtain the contribution to $N^{-1}\langle\ln\det\tilde M\rangle_H$ of orders $m\sigma$:

$$m\sigma D\frac{2Y}{d+2Y^2}. \qquad (53)$$

In the region of small Y we use the expansion of $\langle\ln\det\tilde M\rangle_H$ in powers of Y. With the same assumptions as were made in Sec. 4, we find the contributions top $N^{-1}\langle\ln\det\tilde M\rangle_H$ of order m^2 and $m\sigma$:

$$\frac{m^2D}{d+2Y^2}+\frac{2m\sigma DY}{d+2Y^2}. \qquad (54)$$

Combining (51), (53) for $Y^2 > d/2$ and (54) for $Y^2 < d/2$, we have

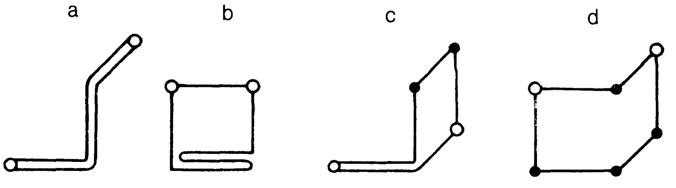

FIG. 2. Typical diagrams in order $1/Y^8$ in the expansion (49), giving a contribution proportional to m^2. The light circles are vertices giving a factor m, and the black circles are unpaired vertices giving a factor σ.

$$\frac{\partial^2}{\partial m^2} W(h{=}0, H, m)\,|_{m=0} = \operatorname{sgn}(2Y^2-d)\frac{2D}{d+2Y^2} + O(\sigma^2), \tag{55}$$

$$\frac{\partial}{\partial m} W(h{=}0, H, m)\,|_{m=0} = -D\frac{2Y}{d+2Y^2}\sigma + O(\sigma^3). \tag{56}$$

Taking into account that

$$\langle\bar\psi\psi\rangle = -\frac{1}{N}\frac{1}{Z}\frac{\partial Z}{\partial m} = \frac{\partial W(m)}{\partial m}, \tag{57}$$

we can calculate the fermion condensate (more precisely, the lattice fermion propagator at coinciding points):

$$\langle\bar\psi\psi\rangle\,|_{m=0} = -\frac{2DY}{d+2Y^2}\sigma + O(\sigma^3). \tag{58}$$

In Ref. 17, the fermion condensate $\langle\psi_x^\alpha\bar\psi_x^\alpha\rangle$ (there is no summation over x and the Dirac indices α) was calculated in the approximation of nondynamical fermions (the quenched approximation) in the model (2) on a $8^3\times 16$ lattice for $k = 0.08$. It is easy to convince oneself that in this problem neglect of correlations of the field ϕ at different points in the approximation of a large d leads to the same formula (58).

In the approximation of nondynamical fermions the fermion condensate is equal to (the index q denotes "quenched")

$$\langle\psi\bar\psi\rangle_q \equiv \langle\psi_x^\alpha\bar\psi_x^\alpha\rangle_q = \langle (M^{-1})_{xx}^{\alpha\alpha}\rangle_{S_B} = \frac{1}{ND}\langle \operatorname{tr} M^{-1}\rangle_{S_B}, \tag{59}$$

where

$$\langle A\rangle_S \equiv \sum_{\{\varphi_x\}} \exp(-S_B)\cdot A \Big/ \sum_{\{\varphi_x\}} \exp(-S_B). \tag{60}$$

If we neglect correlations of the field ϕ_x, we obtain

$$\langle\psi\bar\psi\rangle_q \approx \frac{1}{ND}\langle \operatorname{tr} M^{-1}\rangle_H = \frac{\partial}{\partial m}\frac{1}{ND}\langle \ln\det\tilde M\rangle_H\,|_{m=0}$$

$$= \frac{2Y\sigma}{d+2Y^2} + O(\sigma^3), \tag{61}$$

where $\langle\ldots\rangle_H$ is defined in (8), H is chosen from the condition $\langle\phi\rangle_{S_B} = \langle\phi\rangle_H \equiv \sigma$, and we have made use of Eqs. (53) and (54). A calculation by the Monte Carlo method for

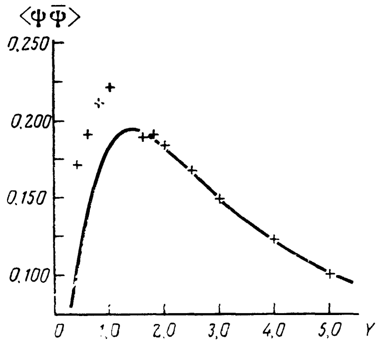

FIG. 3. The fermion condensate in the approximation of nondynamical fermions. The curve corresponds to formula (61) for $\sigma = 0.55$, and the points are the result of a calculation by the Monte Carlo method[17] for $k = 0.08$ on an $8^3\times 16$ lattice.

$k = 0.08$ on an $8^3\times 16$ lattice gives $\langle\phi\rangle_{S_B} \equiv 0.55$. Substituting $\sigma = 0.55$ into (61), we obtain the curve depicted in Fig. 3.

It can be seen that for $Y < (d/2)^{1/2} \cong 1.4$ the formula is in very good agreement with the results obtained by the Monte Carlo method. For $Y < 1.4$ the agreement becomes substantially worse. This happens, in our opinion, for the following two reasons. First, in this region the applicability of the approximations used the calculation of $\langle \ln\det\tilde M\rangle_H$ can become worse. Second, the fermions become light, and in a Monte Carlo calculation on a $8^3\times 16$ lattice strong finite-size effects can arise.

6. THE TWO DIFFERENT PHASES WITH $\langle\phi\rangle = 0$

We return to our model. We note that

$$\frac{\partial^2}{\partial m^2} W(m)\,|_{m=0} = \frac{\partial}{\partial m}\langle\bar\psi_x\psi_x\rangle\,|_{m=0}$$

$$= -\sum_y \left(\langle\bar\psi_x\psi_x\bar\psi_y\psi_y\rangle - \langle\bar\psi_x\psi_x\rangle^2\right). \tag{62}$$

The fact that, according to (55), in the region $\sigma = 0$ this four-fermion correlator changes discontinuously in passing through the boundary $Y = (d/2)^{1/2}$ implies a first-order phase transition at $Y = (d/2)^{1/2}$. [In the calculation of (62) it is necessary to take into account the dependence of H on m by virtue of the relation

$$\partial W(h{=}0, H, m)/\partial H = 0.$$

One adds to (55) the term

$$-(\partial^2 W/\partial H^2)^{-1}(\partial^2 W/\partial m\,\partial H)^2,$$

which is continuous at $Y = (d/2)^{1/2}$. It is easy to see that this term corresponds to the contribution to the correlator (62) from an intermediate state containing one scalar ϕ.] Thus, the region $\sigma = 0$ is not just a single phase in which there is no spontaneous symmetry breaking, but is divided into two regions, characterized by different signs of $\partial^2 W(h = 0, H, m)/\partial m^2$ at $m = 0$.

The calculation $\partial^2 W/\partial m^2$ in the theory of massive free fermions (i.e., for $Y = 0$) in leading order in d gives

$$\frac{\partial^2}{\partial m^2} W(m) = -2D\frac{d-2m^2}{(d+2m^2)^2}. \tag{63}$$

It can be seen that a positive value of this quantity corresponds to a very large fermion mass $m > (d/2)^{1/2}$, i.e., m is greater than the inverse lattice constant.

Thus, for $Y^2 < d/2$ and the theory contains the field ϕ and massless fermions, while for $Y^2 > d/2$ it contains the field ϕ and very heavy fermions, i.e., the fermions cease to exist as long-wavelength degrees of freedom.

We have calculated the quantity (55) for $\sigma = 0$. Since it has a continuous dependence on σ, the phase-transition line on which it experiences a discontinuity extends a certain distance into the region of spontaneously broken symmetry, where $\sigma \neq 0$ holds (see Fig. 4). This agrees with the calculations in the approximation of nondynamical fermions in Ref. 17, in which, in the region $\sigma = 0.4$–0.8, a discontinuous increase of the fermion masses was observed at $Y \cong 1.4$.

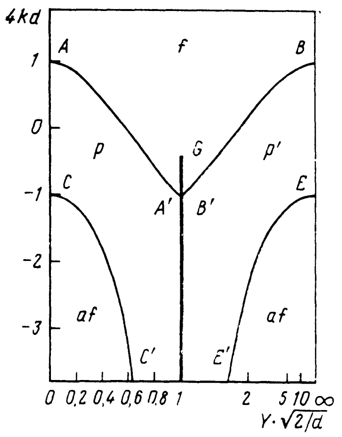

FIG. 4. Schematic illustration of the phase diagram of the model (2). The curves for ($D = 4$) AA', BB', CC', and EE' are second-order transition lines, described by Eqs. (46), (36), and (66). The thick vertical line is a first-order phase transition; we do not know the exact position of the point G. The ferromagnetic phase is indicated by f, the paramagnetic phases by p and p', and the antiferromagnetic phase by af.

7. THE ANTIFERROMAGNETIC PHASE

Up to now, we have considered the phase transition from paramagnet ($\langle\phi\rangle = 0$) to ferromagnet ($\langle\phi\rangle \neq 0$). Ferromagnetic order corresponds to the configuration $\phi_x = 1$. In the Ising model (3), however, for negative values of k antiferromagnetic order becomes favored: $\phi_x = \zeta_x$, where

$$\zeta_x \equiv (-1)^{\sum_\mu x_\mu}.$$

The corresponding phase transition is easy to study if in the original theory we make the replacement

$$k \to -k, \qquad \phi_x \to \zeta_x \phi_x. \tag{64}$$

It is easy to see that the action (3) has the same form in the new variables. Ferromagnetic order $\phi_x = 1$ in the new variables corresponding to antiferromagnetic order in the original variables. Thus, in the Ising model a paramagnetic-antiferromagnetic phase transition arises at $k = -k_c$.

On the phase diagram of the model (2) in the (Y,k) plane the phase transition under consideration should occur on certain lines, starting at the points $(0, -k_c)$ and $(\infty, -k_c)$. These lines can be calculated by means of our approximations.

It is easily verified that the action of the model (2) does not change its form if in it we make the replacements

$$\begin{gathered} k \to -k, \qquad \phi_x \to \zeta_x \phi_x, \\ \psi_x \to \exp\left(i\zeta_x \frac{\pi}{4}\right)\psi_x, \qquad \bar\psi_x \to \exp\left(i\zeta_x \frac{\pi}{4}\right)\bar\psi_x, \\ Y \to -iY. \end{gathered} \tag{65}$$

Now, in order to obtain the equation of the paramagnet-antiferromagnet phase-transition line in our approximations it is sufficient to make the replacements (65) in Eqs. (36) and (46), as a result of which we obtain

$$k = \begin{cases} -\dfrac{1}{4d}\left(1 + \dfrac{2DY^2}{d - 2Y^2}\right), & Y^2 < d/2, \\ -\dfrac{1}{4d}\left(1 - \dfrac{Dd}{d - 2Y^2}\right), & Y^2 > d/2 \end{cases} \tag{66}$$

(see Fig. 4, curves CC' and EE').

8. CONCLUSION

We have investigated the very simple lattice model (2) with scalar fields and fermion fields in Euclidean space by the mean-field method. To calculate the fermion determinant we used the fact that in this approximation correlations of the scalar field at different lattice sites are absent, and expanded in $1/d$, where d is the dimensionality of space.

By investigating the stability of the symmetric phase ($\langle\phi_k\rangle = 0$) we obtained an approximate phase diagram for the model (Fig. 4). We found a ferromagnetic phase ($\langle\phi_x\rangle \neq 0$), lying above the lines AA' and BB', an antiferromagnetic phase, lying below the lines CC' and EE', and two paramagnetic ($\langle\phi_x\rangle = 0$) phases p and p', differing in that in phase p massless fermions are present while in phase p' they are not. The latter two phases are separated by a first-order phase transition, which occurs at $Y = (d/2)^{1/2}$ and is characterized by a discontinuity of the quantity (62). The line of this phase transition can extend a certain distance into the region of the ferromagnetic phase. The presence of this transition is consistent with the results of calculations by the Monte Carlo method in the approximation of nondynamical fermions.[15-17] Inasmuch as we studied only the local stability of the symmetric phase, it is possible, in principle, that other first-order phase-transition lines are present on the phase diagram.

Using our approximate methods, we obtained a simple formula (58) for the fermion condensate; this formula is in good agreement with the results of calculations by the Monte Carlo method[17] (see Fig. 3).

The phase diagram of Fig. 4 was obtained to lowest order in the expansion of $1/d$. For $d = 4$ certain differences can be observed. For example, the points A' and B' may be noncoincident, with slightly different ordinates. it is unlikely that the lines CC' and EE' pass into the region $k \to -\infty$; it is more likely that they lie against the line $Y \cong (d/2)^{1/2} \cong 1.4$.

Our calculations show that the most interesting phenomena in lattice Higgs–fermion theories occur in the region $k < 0$, and, therefore, it would be especially interesting to investigate this region by the Monte Carlo method.

It would also be interesting to extend the mean-field method to models with $\lambda \neq \infty$, to more-realistic many-component theories, to models with Wilson fermions, etc. Our preliminary calculations in a model with a two-component field ϕ give good agreement with the phase-transition lines found by Stephenson and Thornton[13,14] by the Monte Carlo method.

We are grateful to I. V. Andreev, S. M. Apenko, and M. B. Voloshin, who drew our attention to this problem, and to Ya. I. Kogan and V. A. Faĭnberg for useful discussions.

APPENDIX

We can calculate $\det M$ for the configuration

$$\phi_x = \begin{cases} 1, & x \neq x_0, \\ -1, & x = x_0, \end{cases} \quad \text{where} \quad x_0 = (0, 0, \ldots, 0). \tag{A1}$$

The Fourier transform of the matrix M(16) for this configuration has the form

$$\begin{aligned} \hat M_{pq} &= \frac{1}{N}\sum_{xy} \exp(ipx - iqy) M_{xy} \\ &= \left(-i\sum_\mu \gamma_\mu \sin p_\mu + Y\right)\delta_{pq} - \frac{2}{N} Y. \end{aligned} \tag{A2}$$

It is convenient to normalize det M by the determinant of the matrix M_0 corresponding to the configuration $\phi_x \equiv 1$. It is obvious that

$$\det M/\det M_0 = \det \hat{M}/\det \hat{M}_0 = \exp \operatorname{tr} \ln \hat{M}\hat{M}_0^{-1}, \tag{A3}$$

where $\hat{M}_0$ (the Fourier transform of M_0) coincides with the first term in the right-hand side of (A2). The expression for $\operatorname{tr} \ln \hat{M}\hat{M}_0^{-1}$ is easily calculated by expanding it in a series. As a result, we obtain

$$\det M = \det M_0 [1 - 2Y^2 F(Y)]^D, \tag{A4}$$

where, for $N \to \infty$,

$$F(Y) = \int \frac{d^d p}{(2\pi)^d} \left(\sum_\mu \sin^2 p_\mu + Y^2 \right)^{-1} = \int_0^\infty d\alpha \exp\left[-\alpha\left(\frac{d}{2} + Y^2\right)\right] I_0^d(\alpha/2). \tag{A5}$$

For large d, we have

$$F(Y) \approx \left(\frac{d}{2} + Y^2\right)^{-1}, \tag{A6}$$

and therefore

$$\det M \approx (\det M_0) [(d - 2Y^2)^D/(d + 2Y^2)^D] \tag{A7}$$

vanishes when $Y^2 = d/2$. For $d = 4$ we find that $\det M = 0$ for $Y = 1.3647...$.

[1)] The fermion part of the action (2) can be diagonalized in the Dirac indices by a unitary transformation of the fields $\bar{\psi}$ and $\bar{\varphi}$ (Ref. 22). The action then decomposes into a sum of $D = 2^{d/2}$ terms, each of which the action of a Kogut–Susskind (KS) fermion. The results of our paper can be generalized trivially to the theory containing an arbitrary (even) number D of KS fermions.

[2)] The value $Y^2 = d/2$ is also remarkable in that, for this value, the fermion determinant vanishes for certain configurations of the field ϕ (see the Appendix). It is evident that the poor convergence of the algorithms for inverting the matrix M for $d = 4$ and $1.2 < Y < 1.6$ (Refs. 16, 17) is connected with this.

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Translated by P. J. Shepherd