Closure of magnetic-field force lines and orthogonality condition

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Attention is called to the misconception that magnetic force lines must be closed. An example of an open planar force line is cited. The connection between the closure of the force lines and the condition of orthogonality of open magnetic traps is discussed.

Unfortunately, the misconception that magnetic force lines are always closed still persists. This false statement can even be found in an encyclopedic physics dictionary and in handbooks.[1,2] I. E. Tamm, discussing the statement that "magnetic force lines can neither originate nor terminate at any point of a field," cited an example in which an unclosed force line fills densely a certain toroidal surface.[3] His example (a field produced by two currents, one (I_1) circular and plane and the other (I_2) directed along the axis of the current I_1, has played an important role in the development of thermonuclear research[4] and has led to the advent of tokamaks.

In Ref. 5 are pointed out three types of force line: those that begin at infinite and go off to infinity, those (closed and open) that remain at all times in a bounded volume, and those that arrive from infinity and become "tangled" in a finite volume. Examples of these three types can be given for three-dimensional force lines. It is interesting that unclosed plane force lines exist. We shall consider just this possibility for axisymmetric open magnetic traps with straight axes.[6] They have found extensive use in thermonuclear experiments. The magnetic field of such a trap is produced by coils having a spatial symmetry most frequently due to the presence of two mutually perpendicular planes (XOZ and YOZ) passing through a magnetic axis z, such that the reflection operation in them superimposes the winding on itself and reverse the direction of the currents in it. These are planes that contain in fact open planar force lines that come from infinity. To verify this, let us obtain the equation for a force line in the XOZ plane for the simplest system of the four infinite straight-line currents I shown in Fig. 1. Introducing in the XOZ plane polar coordinates (r, φ) with center in the conductor, the equation of the force line is

$$\frac{dr}{d\varphi}=2r^2\cos\varphi A\left[1-\frac{r^2}{(2+r\sin\varphi)^2+r^2\cos^2\varphi}-2rA\sin\varphi\right]^{-1}, \tag{1}$$

where $r = r/a$ and

$$A=[1+(r\cos\varphi-2b/a)^2]^{-1}+[(2+r\sin\varphi)^2+r^2\cos^2\varphi]^{-1}.$$

A numerical solution of (1), with a spiral structure of the force line, is shown in Fig. 2 (see the Appendix). Numerical calculations of real nonaxisymmetric open traps always lead to a spiral form of the force lines.

Can a nonaxisymmetric trap have closed force lines? The answer to this question turns out to be closely related to the orthogonality condition.[6,7] This condition on the geometry of the magnetic field of an open trap is the result of the requirement that the charged particles drift strictly over magnetic surfaces. An important property of such traps is the absence of neoclassical transverse processes and of longitudinal plasma currents.[6]

To demonstrate that the force lines are closed in an orthognal magnetic system we introduce Panov's curvilinear coordinates[7]: the transverse coordinate ρ marks the magnetic surface, the angular coordinate θ is measured along the contour $B=$ const on a magnetic surface (B is the modulus of the magnetic field, and s is measured along a force line on the magnetic surface. Recall that in terms of these coordinates the magnetic-field-strength vector **B** has the contravarient components $(0, 0, B^3)$ and that the modulus of B does not depend on θ.

Figure 3 shows schematically a typical situation for a nonclosed force line AB located in the coordinate surface $\varphi=$ const. The intersection of the coordinate surface $\varphi=$ const and the coordinate surface $s=$ const is also shown. We join the ends of the force line AB by two segments, CB on the surface $s=$ const and AC on the surface $\theta=$ const. The resultant closed contour delimits a surface S consisting of two parts, the part S_0 of the surface $\theta=$ const and the part S_1 of the surface $s=$ const.

Using the known Stokes theorem[8]

$$\int_S \mathbf{ds}\,\mathrm{rot}\,\mathbf{F}=\oint_L \mathbf{dlF},$$

where **F** is an arbitrary vector function, we obtain the following relation in terms of the chosen curvilinear coordinates:

$$\int_{S_0}\left(\frac{\partial F_1}{\partial s}-\frac{\partial F_3}{\partial\rho}\right)d\rho\,ds+\int_{S_1}\left(\frac{\partial F_2}{\partial\rho}-\frac{1}{\rho}\frac{\partial F_1}{\partial\theta}\right)\rho\,d\rho\,d\theta$$
$$=\int_{AB}(F_1\,d\rho+F_3\,ds)+\int_{BC}(F_1\,d\rho+\rho F_2\,d\theta)+\int_{CA}(F_1\,d\rho+F_3\,ds). \tag{2}$$

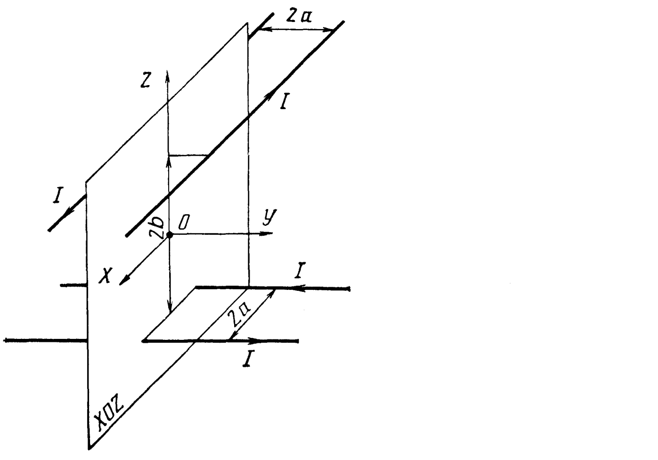

FIG. 1. System of four infinite straight-line currents I. The distances between the current-carrying conductors are $2a$ and $2b$.

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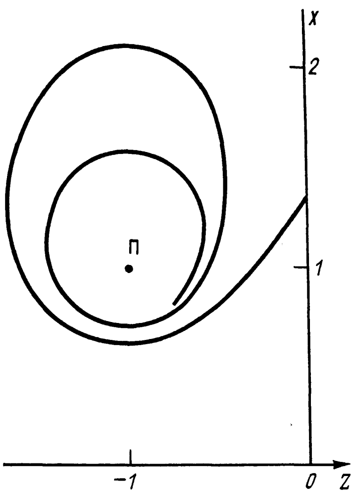

FIG. 2. Force line in the XOZ plane (Π—conductor, $a/b = 1$)

We choose first the covariant components of the vector F in the form $(g^{1/2}B^3, 0, 0)$ and, recognizing that $\rho = \text{const}$ on the force line AB, we obtain from (2)

$$\int_{S_0} \frac{\partial (g^{1/2}B^3)}{\partial s}\, d\rho\, ds - \int_{S_1} \frac{\partial (g^{1/2}B^3)}{\partial \theta}\, d\rho\, d\theta = \int_{BA} g^{1/2}B^3\, d\rho = g^{1/2}B^3 \Delta\rho, \tag{3}$$

where g is the determinant of the metric tensor g_{ik}.

The equation $\operatorname{div} \mathbf{B} = 0$ takes in the coordinates (ρ, θ, s) the form

$$\frac{1}{g^{1/2}} \frac{\partial}{\partial s} (g^{1/2}B^3) = 0$$

and leads to the conclusion that the first term in the left-hand side of (3) is zero. The orthogonality conditions, particularly[7,9]

$$\frac{\partial g_{33}}{\partial \theta} = \frac{\partial g}{\partial \theta} = 0$$

can be used to show that the second term in (3) is also zero. It follows therefore that $\Delta\rho = 0$.

We choose now F in the form $(g^{1/2}B^3, \alpha, 0)$, where $\alpha = \text{const}$. Recognizing that ρ is constant everywhere, we obtain from (2)

$$\int_{S_0} \frac{\partial (g^{1/2}B^3)}{\partial s}\, d\rho\, ds - \int_{S_1} \frac{\partial (g^{1/2}B^3)}{\partial \theta}\, d\rho\, d\theta = \alpha\rho\Delta\theta. \tag{4}$$

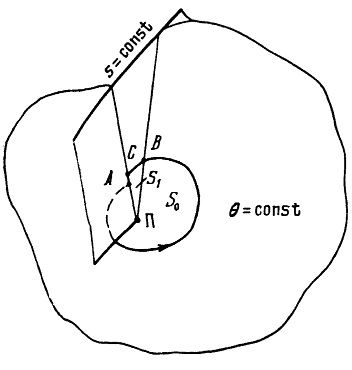

FIG. 3. Coordinate surfaces θ and s (Π—conductor).

Reasoning as above, we conclude that $\Delta\theta = 0$. The force lines are thus closed and the single-valued coordinate surface ρ, θ, s are defined in all of space.

A trivial example of an orthogonal open magnetic trap is the usual "probkotron" mirror machine made up of a system of round turns. It is shown in Ref. 7 that an nonaxisymmetric quadrupole open bottle can be made orthogonal in first order paraxial approximation. It is not clear to this day whether an axisymmetric configuration orthogonal in all of space is possible.

APPENDIX

Consider Eq. (1) in the limit $r \to 0$, i.e., near a conductor. In the expansion of the right-hand side we confine ourselves to terms $\propto r^3$ and obtain

$$\dot r = 2A_0 r^2 \cos\varphi + B_0 r^3 \cos^2\varphi + (4A_0 - 1/2) r^3 \sin\varphi \cos\varphi,$$
$$A_0 = 1/4 + \frac{1}{1+4b^2/a^2}, \quad B_0 = \frac{8b/a}{(1+4b^2/a^2)^2}. \tag{A1}$$

The dot denotes here differentiation with respect to φ.

We seek a perturbative solution of (A1) in the form of the series

$$r = r_0 + r_0^2 f_0(\varphi) + r_0^3 f_1(\varphi), \tag{A2}$$

where r_0 is the initial value at r at $\varphi = 0$. For the functions f_0 and f_1 we obtain the system of equations:

$$\dot f_0 = 2A_0 \cos\varphi,$$
$$\dot f_1 = 4A_0 f_0 \cos\varphi + B_0 \cos^2\varphi + (4A_0 - 1/2) \sin\varphi \cos\varphi. \tag{A3}$$

The solution of this system takes the simple form

$$f_0 = 2A_0 \sin\varphi,$$
$$f_1 = (4A_0^2 + 2A_0 - 1/4) \sin^2\varphi + (B_0/4) \sin 2\varphi + B_0\varphi/2. \tag{A4}$$

It is easy to obtain the force-line disjunction Δr following one complete turn around the conductor

$$\Delta r = r^3 \frac{8\pi b/a}{(1+4b^2/a^2)^2}. \tag{A5}$$

Evidently Δr tends to zero very rapidly as the conductor is approached, but remains always finite.

Note that this behavior of the force line is rigorously possible only in the XOZ (or YOZ) symmetry plane. An arbitrarily small departure from this plane alters radically the form of the force line.

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